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Constraint Acquisition via Partial Queries*

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Abstract

We learn constraint networks by asking the user partial queries. That is, we ask the user to classify assignments to subsets of the variables as positive or negative. We provide an algorithm that, given a negative example, focuses onto a constraint of the target network in a number of queries logarithmic in the size of the example. We give information theoretic lower bounds for learning some simple classes of constraint networks and show that our generic algorithm is optimal in some cases. Finally we evaluate our algorithm on some benchmarks.

1 Introduction

A major bottleneck in the use of constraint solvers is modelling. How does the user write down the constraints of a problem? Several techniques have been proposed to tackle this bottleneck. For example, the matchmaker agent [Freuder and Wallace, 1998] interactively asks the user to provide one of the constraints of the target problem each time the system proposes an incorrect solution. In Conacq\textsuperscript{1} [Bessiere et al., 2004; 2005], the user provides examples of solutions and non-solutions. Based on these examples, the system learns a set of constraints that correctly classifies all examples given so far. This is a form passive learning. In [Lallouet et al., 2010], a system based on inductive logic programming uses background knowledge on the structure of the problem to learn a representation of the problem correctly classifying the examples. A last passive learner is ModelSeeker [Beldiceanu and Simonis, 2012]. Positive examples are provided by the user to the system which arranges each of them as a matrix and identifies constraints in the global constraints catalog that are satisfied by rows or columns of all examples. One weakness of passive learning is that the user needs to provide diverse examples for the target set of constraints to be learned.

By contrast, in an active learner like Conacq\textsuperscript{2} [Bessiere et al., 2007], the system proposes examples to the user to classify as solutions or non solutions. Such questions are called membership queries [Angluin, 1987]. Such active learning has several advantages. It can decrease the number of examples necessary to converge to the target set of constraints. Another advantage is that the user need not be a human. It might be a previous system developed to solve the problem. For instance, the Normind company has recently hired a constraint programming specialist to transform their expert system for detecting failures in electric circuits in Airbus airplanes in to a constraint model in order to make it more efficient and easier to maintain. As another example, active learning is used to build a constraint model that encodes non-atomic actions of a robot (e.g., catch a ball) by asking queries of the simulator of the robot in [Paulin et al., 2008]. Such active learning introduces two computational challenges. First, how does the system generate a useful query? Second, how many queries are needed for the system to converge to the target set of constraints? It has been shown that the number of membership queries required to converge to the target set of constraints can be exponentially large [Bessiere and Koriche, 2012].

In this paper, we propose QUACQ (for QuickAcquisition), an active learner that asks the user to classify partial queries. Given a negative example, QUACQ is able to learn a constraint of the target constraint network in a number of queries logarithmic in the number of variables. In fact, we identify information theoretic lower bounds on the complexity of learning constraint networks which show that QUACQ is optimal on some simple languages. We demonstrate the promise of this approach in practice with some experimental results.

One application for QUACQ would be to learn a general purpose model. In constraint programming, a distinction is made between model and data. For example, in a Sudoku puzzle, the model contains generic constraints like each sub-square contains a permutation of the numbers. The data, on the other hand, gives the pre-filled squares for a specific puzzle. As a second example, in a time-tabling problem, the model specifies generic constraints like no teacher can teach multiple classes at the same time. The data, on the other hand, specifies particular room sizes, and teacher availability for a particular time-tabling problem instance. The cost of learning the model can then be amortized over the lifetime of the model. Another advantage of this approach is that it provides less of a burden on the user. First, it often converges quicker than other methods. Second, partial queries will be easier to answer than complete queries. Third, as opposed to existing techniques, the user does not need to give positive examples. This might be useful if the problem has not yet been solved.

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so there are no examples of past solutions.

2 Background

The learner and the user need to share some common knowledge to communicate. We suppose this common knowledge, called the vocabulary, is a (finite) set of \( n \) variables \( X \) and a domain \( D = \{ D(X_i) \}_{X_i \in X} \), where \( D(X_i) \subseteq \mathbb{Z} \) is the finite set of values for \( X_i \). Given a sequence of variables \( S \subseteq X \), a constraint is a pair \((c,S)\) (also written \( c_S \)), where \( c \) is a relation over \( \mathbb{Z} \) specifying which sequences of \( |S| \) values are allowed for the variables \( S \). \( S \) is called the scope of \( c_S \). A constraint network is a set \( C \) of constraints on the vocabulary \((X,D)\). An assignment \( e_X \) on a set of variables \( Y \subseteq X \) is rejected by a constraint \( c_S \) if \( S \subseteq Y \) and the projection \( e_Y[S] \) of \( e \) on the variables in \( S \) is not in \( c \). An assignment on \( X \) is a solution of \( C \) if it is not rejected by any constraint in \( C \). We write \( \text{sol} (C) \) for the set of solutions of \( C \), and \( C[Y] \) for the set of constraints from \( C \) whose scope is included in \( Y \).

Adapting terms from machine learning, the constraint bias, denoted by \( B \), is a set of constraints built from the constraint language \( \Gamma \) on the vocabulary \((X,D)\) from which the learner builds the constraint network. A concept is a Boolean function over \( D^X = \Pi_{X_i \in X} D(X_i) \), that is, a map that assigns to each \( e \in D^X \) a value in \( \{0,1\} \). A target concept is a concept \( f_T \) that returns 1 for \( e \) if and only if \( e \) is a solution of the problem the user has in mind. A membership query \( \text{ASK} (e) \) is a classification question asked of the user whether \( e \) is a complete assignment in \( D^X \). The answer to \( \text{ASK}(e) \) is “yes” if and only if \( f_T(e) = 1 \).

To be able to use partial queries, we have an extra condition on the capabilities of the user. Even if she is not able to articulate the constraints of her problem, she is able to decide if partial assignments of \( X \) violate some requirements or not. The target model or target constraint network is a network \( C_T \) such that \( \text{sol}(C_T) = \{ e \in D^X \mid f_T(e) = 1 \} \). A partial query \( \text{ASK} (e_Y) \), with \( Y \subseteq X \), is a classification question asked of the user, where \( e_Y \) is a partial assignment in \( D^Y = \Pi_{X_i \in Y} D(X_i) \). A set of constraints \( C \) accepts a partial query \( e_Y \) if and only if there does not exist any constraint \( c_S \) in \( C \) rejecting \( e_Y[S] \). The answer to \( \text{ASK}(e_Y) \) is “yes” if and only if \( C_T \) accepts \( e_Y \). For any assignment \( e_Y \) on \( Y \), \( \kappa_B(e_Y) \) denotes the set of all constraints in \( B \) rejecting \( e_Y \). A classified assignment \( e_Y \) is called positive or negative example depending on whether \( \text{ASK}(e_Y) \) is “yes” or “no”.

We now define convergence, which is the constraint acquisition problem we are interested in. We are given a set \( E \) of (partial) examples labelled by the user 0 or 1. We say that a constraint network \( C \) agrees with \( E \) if \( C \) accepts all examples labelled 1 in \( E \) and does not accept those labelled 0. The learning process has converged on the network \( C_L \subseteq B \) if \( C_L \) agrees with \( E \) and for every other network \( C' \subseteq B \) agreeing with \( E \), we have \( \text{sol}(C') = \text{sol}(C_L) \). If there does not exist any \( C_L \subseteq B \) such that \( C_L \) agrees with \( E \), we say that we have collapsed. This happens when \( C_T \not\subseteq B \).

3 Generic Constraint Acquisition Algorithm

We propose \textsc{QuAcq}, a novel active learning algorithm. \textsc{QuAcq} takes as input a bias \( B \) on a vocabulary \((X,D)\). It asks partial queries of the user until it has converged on a constraint network \( C_L \) equivalent to the target network \( C_T \), or collapses. When a query is answered \( \text{yes} \), constraints rejecting it are removed from \( B \). When a query is answered \( \text{no} \), \textsc{QuAcq} enters a loop (functions \text{FindScope} and \text{FindC}) that will end by the addition of a constraint to \( C_L \).

3.1 Description of \textsc{QuAcq}

\textsc{QuAcq} (see Algorithm 1) initializes the network \( C_L \) it will learn to the empty set (line 1). If \( C_L \) is unsatisfiable (line 3), the space of possible networks collapses because there does not exist any subset of the given bias \( B \) that is able to correctly classify the examples already asked of the user. In line 4, \textsc{QuAcq} computes a complete assignment \( e \) satisfying \( C_L \) but violating at least one constraint from \( B \). If such an example does not exist (line 5), then all constraints in \( B \) are implied by \( C_L \), and we have converged. If we have not converged, we propose the example \( e \) to the user, who will answer by \( \text{yes} \) or \( \text{no} \). If the answer is \( \text{yes} \), we can remove from \( B \) the set \( \kappa_B(e) \) of all constraints in \( B \) that reject \( e \) (line 6). If the answer is \( \text{no} \), we are sure that \( e \) violates at least one constraint of the target network \( C_T \). We then call the function \text{FindScope} to discover the scope of one of these violated constraints. \text{FindC} will select which one with the given scope is violated by \( e \) (line 8). If no constraint is returned (line 9), this is again a condition for collapsing as we could not find in \( B \) a constraint rejecting one of the negative examples. Otherwise, the constraint returned by \text{FindC} is added to the learned network \( C_L \) (line 10).

<table>
<thead>
<tr>
<th>Algorithm 1: \textsc{QuAcq}: Acquiring a constraint network ( C_T ) with partial queries</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( C_L \leftarrow \emptyset );</td>
</tr>
<tr>
<td>2 while true do</td>
</tr>
<tr>
<td>3 ( \text{if } \text{sol}(C_L) = \emptyset \text{ then return } \text{“collapse”;} )</td>
</tr>
<tr>
<td>4 ( \text{choose } e \in D^X \text{ accepted by } C_L \text{ and rejected by } B );</td>
</tr>
<tr>
<td>5 ( \text{if } e = \text{nil then return } \text{“convergence on } C_L \text{”;} )</td>
</tr>
<tr>
<td>6 ( \text{if } \text{ASK}(e) = \text{yes then } B \leftarrow B \setminus \kappa_B(e) );</td>
</tr>
<tr>
<td>7 ( \text{else} )</td>
</tr>
<tr>
<td>8 ( c \leftarrow \text{FindC}(e, \text{FindScope}(e, \emptyset, X, \text{false})); )</td>
</tr>
<tr>
<td>9 ( \text{if } c = \text{nil then return } \text{“collapse”;} )</td>
</tr>
<tr>
<td>10 ( \text{else } C_L \leftarrow C_L \cup {c} );</td>
</tr>
</tbody>
</table>

The recursive function \text{FindScope} takes as parameters an example \( e \), two sets \( R \) and \( Y \) of variables, and a Boolean \text{ask}\_\text{query}. An invariant of \text{FindScope} is that \( e \) violates at least one constraint whose scope is a subset of \( R \cup Y \). When \text{FindScope} is called with \text{ask}\_\text{query} = \text{false}, we already know whether \( R \) contains the scope of a constraint that rejects \( e \) (line 1). If \text{ask}\_\text{query} = \text{true} we ask the user whether \( e[R] \) is positive or not (line 2). If \text{yes}, we can remove all the constraints that reject \( e[R] \) from the bias, otherwise we return the empty set (line 3). We reach line 4 only in case \( e[R] \) does not violate any constraint. We know that \( e[R \cup Y] \) violates a constraint. Hence, as \( Y \) is a singleton, the variable it contains necessarily belongs to the scope of a constraint that violates
### 3.2 Example

We illustrate the behavior of QUACQ on a simple example. Consider the set of variables $X_1, \ldots, X_5$ with domains $\{1, 5\}$, a language $\Gamma = \{=, \neq\}$, a bias $B = \{=_{ij}, =_{ij}\} \cup \{=_{ij}, =_{ij}\}$ and a target network $C_T = \{=_{15}, \neq_{14}\}$. The first example generated in line 4 of QUACQ is $e_1 = (1, 1, 1, 1, 1)$. The trace of the execution of FindScope($e_1, \emptyset, X_1, \ldots, X_5$, false) is in the table below. Each line corresponds to a call to FindScope. Queries are always on the variables in $R$, "x" in the column ASK means that the previous call returned $\emptyset$, so the question is skipped. The queries in lines 1 and 2 in the table permit FindScope to remove the constraints $\neq_{12}, \neq_{13}, \neq_{23}$ and $\neq_{14}, \neq_{24}$ from $B$. Once the scope $(X_3, X_4)$ is returned, FindC requires a single example to return $\neq_{14}$ from $B$. Suppose the next example generated by QUACQ is $e_2 = (1, 2, 3, 4, 5)$.

<table>
<thead>
<tr>
<th>call</th>
<th>$R$</th>
<th>$Y$</th>
<th>ASK</th>
<th>return</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\emptyset$</td>
<td>$X_1, X_2, X_3, X_4, X_5$</td>
<td>$x$</td>
<td>$X_3, X_4$</td>
</tr>
<tr>
<td>1</td>
<td>$X_1, X_2, X_3$</td>
<td>$X_4, X_5$</td>
<td>yes</td>
<td>$X_4$</td>
</tr>
<tr>
<td>1.1</td>
<td>$X_1, X_2, X_3, X_4$</td>
<td>$X_5$</td>
<td>no</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>1.2</td>
<td>$X_1, X_2, X_3$</td>
<td>$X_4$</td>
<td>$x$</td>
<td>$X_4$</td>
</tr>
<tr>
<td>2</td>
<td>$X_4$</td>
<td>$X_1, X_2, X_3$</td>
<td>yes</td>
<td>$X_3$</td>
</tr>
<tr>
<td>2.1</td>
<td>$X_4, X_1, X_2$</td>
<td>$X_3$</td>
<td>yes</td>
<td>$X_3$</td>
</tr>
<tr>
<td>2.2</td>
<td>$X_4, X_3$</td>
<td>$X_1, X_2$</td>
<td>no</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

### 3.3 Analysis

We analyse the complexity of QUACQ in terms of the number of queries it can ask of the user. Queries are proposed to the user in lines 6 of QUACQ, 2 of FindScope and 9 of FindC.

**Proposition 1.** Given a bias $B$ built from a language $\Gamma$, a target network $C_T$, a scope $Y$, FindC uses $O(|\Gamma|)$ queries to return a constraint $c_Y$ from $C_T$ if it exists.

**Proof.** Each time FindC asks a query, whatever the answer of the user, the size of $\Delta$ strictly decreases. Thus the total number of queries asked in FindC is bounded above by $|\Delta|$, which itself, by construction in line 2, is bounded above by the number of constraints from $\Gamma$ of arity $|Y|$. \qed

**Proposition 2.** Given a bias $B$, a target network $C_T$, an example $e \in D^X \setminus \text{sol}(C_T)$, FindScope uses $O(|S| \cdot \log |X|)$ queries to return the scope $S$ of one of the constraints of $C_T$ violated by $e$.

**Proof.** FindScope is a recursive algorithm that asks at most one query per call (line 2). Hence, the number of queries is bounded above by the number of nodes of the tree of recursive calls to FindScope. We will show that a leaf node is
either on a branch that leads to the elucidation of a variable in the scope \( S \) that will be returned, or is a child of a node of such a branch. When a branch does not lead to the elucidation of a variable in the scope \( S \) that will be returned, that branch necessarily only leads to leaves that correspond to calls to \texttt{FindScope} that returned the empty set. The only way for a leaf call to \texttt{FindScope} to return the empty set is to have received a no answer to its query (line 3). Let \( R_{\text{child}}, Y_{\text{child}} \) be the values of the parameters \( R \) and \( Y \) for a leaf call with a no answer, and \( R_{\text{parent}}, Y_{\text{parent}} \) be the values of the parameters \( R \) and \( Y \) for its parent call in the recursive tree. From the no answer to the query \( \text{ASK}(e[R_{\text{child}}], e[R_{\text{parent}}]) \), we know that \( S \subseteq R_{\text{child}} \) but \( S \not\subseteq R_{\text{parent}} \) because the parent call received a yes answer. Consider first the case where the leaf is the left child of the parent node. By construction, \( R_{\text{parent}} \not\subseteq R_{\text{child}} \subseteq R_{\text{parent}} \cup Y_{\text{parent}} \). As a result, \( Y_{\text{parent}} \) intersects \( S \), and the parent node is on a branch that leads to the elucidation of a variable in \( S \). Consider now the case where the leaf is the right child of the parent node. As we are on a leaf, if the \texttt{ask}_\text{query} Boolean is false, we have necessarily exited from \texttt{FindScope} through line 4, which means that this node is the end of a branch leading to a variable in \( S \). Thus, we are guaranteed that the \texttt{ask}_\text{query} Boolean is true, which means that the left child of the parent node returned a non empty set and that the parent node is on a branch to a leaf that elucidates a variable in \( S \).

We have proved that every leaf is either on a branch that elucidates a variable in \( S \) or is a child of a node on such a branch. Hence the number of nodes in the tree is at most twice the number of nodes in branches that lead to the elucidation of a variable from \( S \). Branches can be at most \( \log |X| \) long. Therefore the total number of queries \texttt{FindScope} asks is at most \( 2 \cdot |S| \cdot \log |X| \), which is in \( O(|S| \cdot \log |X|) \).

**Theorem 1.** Given a bias \( B \) built from a language \( \Gamma \) of bounded arity constraints, and a target network \( C_T \), \texttt{QUACQ} uses \( O(|C_T| \cdot (\log |X| + |\Gamma|)) \) queries to find the target network or to collapse and \( O(|B|) \) queries to prove convergence.

**Proof.** Each time line 6 of \texttt{QUACQ} classifies an example as negative, the scope of a constraint \( c_B \) from \( C_T \) is found in at most \( |S| \cdot \log |X| \) queries (Proposition 2). As \( \Gamma \) only contains constraints of bounded arity, either \( |S| \) is bounded and \( c_B \) is found in \( O(|\Gamma|) \) or we collapse (Proposition 1). Hence, the number of queries necessary for finding \( C_T \) or collapsing is in \( O(|C_T| \cdot (\log |X| + |\Gamma|)) \). Convergence is obtained once \( B \) is wiped out thanks to the examples that are classified positive in line 6 of \texttt{QUACQ}. Each of these examples necessarily leads to at least one constraint removal from \( B \) because of the way the example is built in line 4. This gives a total in \( O(|B|) \). □

## 4 Learning Simple Languages

In order to gain a theoretical insight into the “efficiency” of \texttt{QUACQ}, we look at some simple languages, and analyze the number of queries required to learn networks on these languages. In some cases, we show that \texttt{QUACQ} will learn problems of a given language with an asymptotically optimal number of queries. However, for some other languages, a suboptimal number of queries can be necessary in the worst case. Our analysis assumes that when generating a complete example in line 4 of \texttt{QUACQ}, the solution of \( C_L \) maximizing the number of violated constraints in the bias \( B \) is chosen.

### 4.1 Languages for which \texttt{QUACQ} is optimal

**Theorem 2.** \texttt{QUACQ} learns Boolean networks on the language \( \{=,\neq\} \) in an asymptotically optimal number of queries.

**Proof.** (Sketch.) First, we give a lower bound on the number of queries required to learn a constraint network in this language. Consider the restriction to equalities only. In an instance of this language, all variables of a connected component must be equal. This is isomorphic to the set of partitions of \( n \) objects, whose size is given by Bell’s Number:

\[
C(n + 1) = \left\{ \begin{array}{ll}
1 & \text{if } n = 0 \\
\sum_{i=1}^{n-1} \binom{n}{i} C(n-i) & \text{if } n > 0
\end{array} \right.
\]

By an information theoretic argument, at least \( \log C(n) \) queries are required to learn such a problem. This entails a lower bound of \( \Omega(n \log n) \) since \( \log C(n) \in \Omega(n \log n) \) (see [De Bruijn, 1970] for the proof). The language \( \{=,\neq\} \) is richer and thus requires at least as many queries.

Second, we consider the query submitted to the user in line 6 of \texttt{QUACQ} and count how many times it can receive the answer yes and no. The key observation is that an instance of this language contains at most \( O(n) \) non-redundant constraints. For each no answer in line 6 of \texttt{QUACQ}, a new constraint will eventually be added to \( C_L \). Only non-redundant constraints are discovered in this way because the query must satisfy \( C_L \). It follows that at most \( O(n) \) such queries are answered no, each one entailing \( O(\log n) \) more queries through the procedure \texttt{FindScope}.

Now we bound the number of yes answers in line 6 of \texttt{QUACQ}. The same observation on the structure of this language is useful here as well. We show in the complete proof that a query maximizing the number of violations of constraints in the bias \( B \) while satisfying the constraints in \( C_L \) violates at least \( \lceil |B|/2 \rceil \) constraints in \( B \). Thus, each query answered yes at least halves the number of constraints in \( B \). It follows that the query submitted in line 6 of \texttt{QUACQ} cannot receive more than \( O(\log n) \) yes answers. The total number of queries is therefore bounded by \( O(n \log n) \).

The same argument holds for simpler languages (\{=\} and \{\neq\} on Boolean domains). Moreover, this is still true for \{=\} on arbitrary domains.

**Corollary 1.** \texttt{QUACQ} can learn constraint networks with unbounded domains on the language \( \{=\} \) in an asymptotically optimal number of queries.

### 4.2 Languages for which \texttt{QUACQ} is not optimal

First, we show that a Boolean constraint network on the language \( \{<\} \) can be learnt with \( O(n) \) queries. Then, we show that \texttt{QUACQ} requires \( \Omega(n \log n) \) queries.

**Theorem 3.** Boolean constraint networks on the language \( \{<\} \) can be learned in \( O(n) \) queries.
Proof. Observe that in order to describe such a problem, the variables can be partitioned into three sets, one for variables that must take the value 0 (i.e., on the left side of a < constraint), a second for variables that must take the value 1 (i.e., on the right side of a < constraint), and the third for unconstrained variables. In the first phase, we greedily partition variables into three sets, \( L, R, U \) initially empty and standing respectively for Left, Right and Unknown. During this phase, we have three invariants:

1. There is no \( x, y \in U \) such that \( x < y \) belongs to the target network
2. \( x \in L \) iff there exists \( y \in U \) and a constraint \( x < y \) in the target network
3. \( x \in R \) iff there exists \( y \in U \) and a constraint \( y < x \) in the target network

We go through all variables of the problem, one at a time. Let \( x \) be the last variable picked. We query the user with an assignment where \( x \), as well as all variables in \( U \) are set to 0, and all variables in \( R \) are set to 1 (variables in \( L \) are left unassigned). If the answer is yes, then there is no constraints between \( x \) and any variable in \( y \in U \), hence we add \( x \) to the set of undecided variables \( U \) without breaking any invariant. Otherwise we know that \( x \) is either in a constraint \( y < x \) with \( y \in U \), or a constraint \( x < y \) with \( y \in U \). In order to decide which way is correct, we make a second query, where the value of \( x \) is flipped to 1 and all other variables are left unchanged. If this second query receives a yes answer, then the former hypothesis is true and we add \( x \) to \( R \); otherwise, we add it to \( L \). Here again, the invariants still hold.

At the end of the first phase, we therefore know that variables in \( U \) have no constraints between them. However, they might be involved in constraints with variables in \( L \) or in \( R \). In the second phase, we go over each undecided variable \( x \in U \), and query the user with an assignment where all variables in \( L \) are set to 0, all variables in \( R \) are set to 1 and \( x \) is set to 0. If the answer is no, we conclude that there is a constraint \( y < x \) with \( y \in L \) and therefore \( x \) is added to \( R \) (and removed from \( U \)). Otherwise, we ask the same query, but with the value of \( x \) flipped to 1. If the answer is no, there must exists \( y \in R \) such that \( x < y \) belongs to the network, hence \( x \) is added to \( R \) (and removed from \( U \)). Last, if both queries get the answer yes, we conclude that \( x \) is not constrained. When every variable has been examined in this way, variables remaining in \( U \) are not constrained.

\[ \square \]

**Theorem 4.** QUACQ does not learn Boolean networks on the language \( \{<\} \) with a minimal number of queries.

**Proof.** By Theorem 3, we know that these networks can be learned in \( O(n) \) queries. Such networks can contain up to \( n - 1 \) non-redundant constraints. QUACQ learns constraints one at a time, and each call to \( \text{FindScope} \) takes \( \Omega(\log n) \) queries. Therefore, QUACQ requires \( \Omega(n \log n) \) queries. \[ \square \]

## 5 Experimental Evaluation

To test the practicality of such active learning, we ran some experiments with QUACQ, on an Intel Xeon E5462 @ 2.80GHz with 16 Gb of RAM. We used several benchmarks.
vation is that max-1 generally requires less queries than sol to reach convergence. This is especially true for rand_50_10_12, which is very sparse, and Golomb-8, which contains many redundant constraints. If we have a closer look, these differences are mainly explained by the fact that max-1 requires significantly less complete positive queries than sol to prune $B$ totally and prove convergence (22 complete positive queries for max-1 and 121 for sol on rand_50_10_12). But in general, sol is not as bad as we could have expected. The reason is that, except on very sparse networks, the number of constraints from $B$ violated 'by chance' with sol is large enough. The second observation is that when the network contains a lot of redundancies, max-1 converges on a smaller network than sol. We observed this on Golomb-8, and other problems not reported here. The third observation is that the average size of queries is always significantly smaller than the number of variables in the problem. A last observation is that sol is very fast for all its queries (see the time column). We can expect it to be usable on even larger problems.

As a second experiment we evaluated the effect of the size of the bias on the number of queries. On the zebra problem we fed QUACQ with biases of different sizes and stored the number of queries for each run. Figure 1 shows that when $|B|$ grows, the number of queries follows a logarithmic scale. This is very good news as it means that learning problems with expressive biases will scale well.

QUACQ has two main advantages over learning with membership queries, as in CONACQ. One is the small average size of queries $\bar{q}$, which are probably easier to answer by the user. The second advantage is the time to generate queries. Conacq.2 needs to find examples that violate exactly one constraint of the bias to make progress towards convergence. This can be expensive to compute, preventing the use of Conacq.2 on large problems. QUACQ, on the other hand, can use cheap heuristics like max-1 and sol to generate queries.

5.2 QUACQ as a solver
Conacq.2 and ModelSeeker need complete positive examples to learn a constraint network. By comparison, QUACQ can learn without any complete positive example. This extends its use to solving a single instance of a problem never solved before. We simply need to exit QUACQ as soon as a complete example is classified yes by the user. We assessed this feature by solving a sequence of 5 instances of Sudoku, that

<table>
<thead>
<tr>
<th>$C_L$</th>
<th>$#q$</th>
<th>$#q_r$</th>
<th>$q$</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>rand_50_10_12</td>
<td>max-1</td>
<td>12</td>
<td>196</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>sol</td>
<td>12</td>
<td>286</td>
<td>133</td>
</tr>
<tr>
<td>rand_50_10_122</td>
<td>max-1</td>
<td>86</td>
<td>1074</td>
<td>94</td>
</tr>
<tr>
<td></td>
<td>sol</td>
<td>83</td>
<td>1062</td>
<td>120</td>
</tr>
<tr>
<td>Golomb-8</td>
<td>max-1</td>
<td>91</td>
<td>488</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>sol</td>
<td>138</td>
<td>709</td>
<td>153</td>
</tr>
<tr>
<td>Zebra</td>
<td>max-1</td>
<td>62</td>
<td>656</td>
<td>63</td>
</tr>
<tr>
<td></td>
<td>sol</td>
<td>62</td>
<td>641</td>
<td>63</td>
</tr>
<tr>
<td>Sudoku 9 x 9</td>
<td>max-1</td>
<td>810</td>
<td>8645</td>
<td>821</td>
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<tr>
<td></td>
<td>sol</td>
<td>810</td>
<td>9593</td>
<td>815</td>
</tr>
</tbody>
</table>

Figure 1: QUACQ behavior on different bias sizes for Zebra is. Sudoku grids with their pre-filled cells. For each of them, QUACQ exits when the solution is found. As the goal is not to converge we replaced max-1 by a min-1 heuristic, that is, a heuristic that tries to satisfy as much as possible the constraints in $B$, with a cutoff at 1 second. Each run takes as input the $C_L$ and $B$ in the state they were at the end of the previous run because the partially learned network is a valid starting point for further learning. The number of queries required to solve each of the 5 instances in the sequence was 3859, 1521, 687, 135, and 34 respectively. The size of $C_L$ after each run was 340, 482, 547, 558, and 561, respectively. We see that for the first grid, where QUACQ starts with a complete bias, we find the solution in only 44% of the queries needed to QUACQ to converge (See Table 1). On each run, QUACQ needs fewer queries to find a solution as $C_L$ becomes closer to the target network.

6 Conclusion
We have proposed QUACQ, an algorithm that learns constraint networks by asking the user to classify partial assignments as positive or negative. Each time it receives a negative example, the algorithm converges on a constraint of the target network in a logarithmic number of queries. We have shown that QUACQ is optimal on certain constraint languages. Our approach has several advantages over existing work. First, it always converges on the target constraint network in a polynomial number of queries. Second, the queries are often much shorter than membership queries, so are easier to handle for the user. Third, as opposed to existing techniques, the user does not need to provide positive examples to converge. This last feature can be very useful when the problem has not been previously solved. Our experimental evaluation shows that generating good queries in QUACQ is not computationally difficult and that when the bias increases in size, the increase in number of queries follows a logarithmic shape. These results are promising for the use of QUACQ on real problems. However, problems with dense constraint networks (as Sudoku) require a number of queries that could be too large for a human user. An interesting direction would be to combine ModelSeeker and QUACQ to quickly learn global constraints and use QUACQ to finalize the model.
References


