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# Guaranteed Reconstruction for Image Super-resolution

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**Abstract**—This paper presents a new reconstruction operator to be used in a super-resolution scheme. Here, by reconstruction in super-resolution, we mean the back-projection operation, i.e. the way  $K$  low resolution (LR) images are aggregated to obtain a smooth high resolution (HR) image. Within this method, we replace the usual reconstruction procedure by a non-additive reconstruction operation based on the nice properties of fuzzy partitions. This non-additive reconstruction operator represents a convex family of usual additive reconstruction operators. The obtained reconstructed image is thus a convex family of usual reconstructed images. It allows the super-resolution method to be less sensitive to the choice of the reconstruction method. To make the reading of this method easier, it is presented with  $1D$  signals. We present some experiments to illustrate the proved properties of this new operator.

**Keywords**—*Super-resolution, imprecise guaranteed reconstruction, Choquet integral, capacities.*

## I. INTRODUCTION

Images with a high spatial resolution (HR) can be required when only low spatial resolution imagers are available. HR images can e.g. improve the performances of a pattern recognition algorithm or enhance relevant details in the context of image based medical diagnosis.

Super-resolution image reconstruction is a technique that recovers a HR image from a sequence of  $K$  aliased, blurred and noisy low resolution (LR) images acquired from the same scene by one or several image sensors. The reconstructed high resolution image must be a disaliased version of the low resolution ones and should contain more details. It is often expressed as an optimization process leading to reconstruct the image that should have been obtained by using the same imager but with a sharper point spread function (i.e. the response of an imaging system to an impulse signal) and with a higher spatial resolution. Such reconstruction need the LR images to be acquired with sub-pixel projected motions. Thus an accurate knowledge of the motion between images is also required.

Most of the super-resolution techniques are based on two dual operators called projection and back-projection. Projection models how the LR images can be derived from the HR image. Back-projection consists in aggregating the LR images in order to reconstruct a smoothed version of the sought after HR image. Super-resolution techniques are usually very sensitive to the choice of those two operators.

In this paper, we propose an interval-valued back-projection operator that ensures a kind of guarantee in the back-projected image. In fact, the interval-valued back-projected image is the convex family of any back-projected image that would have been obtained by a reconstruction technique using a wide range of positive monomodal probabilistic kernels. This work is a first step towards the definition of a new super-resolution imaging technique that should be robust w.r.t. a choice of both projection and back-projection kernels.

In order to ease the understanding of this new technique, we restrict its presentation to  $1D$  signals (i.e. reconstructing the lines of the image). The  $2D$  extension of the presented concepts is straightforward.

Section II presents the up-to-date literature about super-resolution techniques. The projection and back-projection models are also mentioned here. The following part of this paper, Section III, presents operators based on imprecise probability theories (capacity and possibility theory) that extend the usual kernel based convolution. These operators allow convoluting a signal with a convex family of usual kernels. In Section IV, we present the guaranteed reconstruction and back-projection operators which are the main results of this paper. The guaranty property is clarified and proved. Section V contains experiments that illustrate the guaranty property of the reconstruction and back-projection operators.

## II. SUPER-RESOLUTION

### A. State of art

The first work on reconstructing a HR image from a sequence of LR images was published in 1984 [1] and the term "Super-resolution" itself appeared at around 1990 [2].

An extensive literature exists about super-resolution techniques that has been published in the last two decades. A recent survey on super-resolution imaging techniques is available in [3]. [4] also provides a snapshot of methods and techniques to improve images and video beyond the capabilities of the cameras.

These techniques differ with respect to modeling and algorithmic aspects. They are usually divided into four broad categories:

- 1) Frequency domain-based techniques,
- 2) Iterative back-projection techniques,

- 3) Optimization techniques and
- 4) Projection onto convex sets (POCS) techniques.

1) *Frequency domain-based techniques*: Tsai and Huang [1] at first proposed a frequency domain approach for solving the problem of super-resolution image reconstruction in 1984. This approach is based on assuming the original high resolution image to be band-limited. It exploits the translational property of the Fourier Transform by using the aliasing relationship that exists between the Continuous Fourier Transform (CFT) of the original real scene and the Discrete Fourier Transform (DFT) of the acquired low resolution images. This early approach deals neither with blur nor noise. It has been further extended by Kim *et al* [5] by introducing a weighted recursive least square algorithm that combines filtering and reconstruction in order to account for noise, then by Kim and Su [6] to account for blur. However, the obtained method is not stable due to the fact that super-resolution is an ill-posed inverse problem. They thus proposed a recursive algorithm [5] that includes a regularization term.

2) *Iterative back-projection techniques*: Iterative back-projection techniques work in the spatial domain. It consists in reconstructing a HR image with  $K$  LR images. It is based on two dual operators: projection and back-projection. Projection<sup>1</sup> consists in estimating  $K$  LR images based on a HR image. Back-projection consists in restoring a smoothed HR image based on the  $K$  LR images. Classically, the projection is modeled by a linear operator corrupted by an additive noise:

$$I^k = A^k \hat{I} + \beta, \quad \forall k = 1, \dots, K \quad (1)$$

where  $I^k$  is the  $k^{th}$   $N$ -dimensional vector which is the lexicographically ordered version of the  $k^{th}$  LR image and  $\hat{I}$  is the  $M$ -dimensional vector which is the lexicographically ordered version of the HR image.  $\beta$  is a  $N$ -dimensional additive noise and  $A^k$  is a  $N \times M$  matrix.

$A^k$  can be decomposed into a down sampling operator  $D^k$  ( $N \times M$  matrix), a blurring operator  $B^k$  ( $M \times M$  matrix) which expresses the point spread function (PSF) of the imager and a warping operator  $W^k$  ( $M \times M$  matrix) modeling the motion between  $I^k$  and  $\hat{I}$ .  $A^k$  is usually computed that way:

$$A^k = D^k B^k W^k, \quad \forall k = 1, \dots, K. \quad (2)$$

It is also convenient to represent the projection as one linear equation:

$$I = A \hat{I} + \beta \quad (3)$$

Moreover, the back-projection is based on a  $M \times NK$  matrix  $R$  that performs an aggregation of the  $K$  low resolution images into a smooth HR image:

$$\hat{I} = RI. \quad (4)$$

In many papers,  $R$  is defined as being  $A^{bp}$  the dual operator of  $A$ .

Iterative back-projection techniques generally use an iterative algorithm to solve the super-resolution issue [7]. Starting from an initial guess  $\hat{I}^0$ , this algorithm recursively update the current guess  $\hat{I}^n$  based on reducing the error between the low

resolution images  $\{A^k \hat{I}^n\}_{k=1 \dots K}$  obtained by projecting this current guess and the measured LR images  $\{I^k\}_{k=1 \dots K}$ . A good example of such a process, based on the Schultz iterative method, can be written as:

$$\hat{I}^{n+1} = \hat{I}^n + \lambda R(I - A \hat{I}^n), \quad (5)$$

where  $\lambda$  is a factor that ensures the convergence of the algorithm. Some other models have been proposed that consider occlusion and transparency [8] or specific models for the back-projection operator [9].

3) *Optimization techniques*: Optimization techniques try to solve Equation (3) in an optimization framework. It is based on defining an objective function composed of two terms. The first term expresses how the HR image fits the LR images by mean of the back-projection model. The second term discards inappropriate solutions, preventing over-fitting. A very common expression of this kind of algorithm, where the fitting term is based on quadratic error, can be written as:

$$\epsilon(\hat{I}) = \|I - A \hat{I}\|^2 + \lambda \|Q \hat{I}\|^2 \quad (6)$$

where  $Q$  is the regularisation matrix and  $\lambda > 0$  is a regularisation parameter used to control the regularization level of the solution. Note that optimizing the unregularized Equation (6) leads to the recursive Equation (5).

Iterative back-projection techniques can be seen as a particular case of optimization techniques. Both techniques make use of a projection and a back-projection operator to alternatively go from HR space to LR space and from LR space to HR space.

4) *Projection onto convex sets techniques*: The Projection Onto Convex Sets (POCS) techniques aim at solving the super-resolution problem on the constraint satisfaction problem framework. The POCS method has been introduced in [10] [11] in 1982. In [12] [13], Stark explains the general technique for applying POCS for image restoration. He applied this concept to the super-resolution problem in [14]. These techniques also extensively use projection and back-projection operators.

Recently POCS has been used for estimating a high resolution image from multi-camera low resolution surveillance imaging [15].

## B. Projection and Back-projection

In order to simplify the presentation of the developed guaranteed reconstruction operators, we restrict this problem to one dimension. Instead of considering an image  $I$ , we work with a 1D signal  $S$ . The extension in 2D of the proposed concepts is straightforward.

Reconstruction is involved in the projection and back-projection model underlying most super-resolution methods. Indeed, a projection is, in some sense, a modified resampling method for retrieving a LR signal from a HR signal. This is a modified resampling since it allows displacements whose norm is not sampling step multiplicative.

The reconstruction of a signal  $S = (S_n)_{n=1, \dots, N}$  is conditioned by the Nyquist Shannon sampling theorem and, in most cases, a sampled version of the Sine Cardinal kernel is convoluted with the sampled image to achieve its reconstruction.

<sup>1</sup>This operator is often referred to *observation model* in the literature

It is often better [16], [17] to consider a reconstruction which involves a band limited discrete kernel  $\eta^\omega = (\eta_n^\omega)_{n=1, \dots, N}$ , which is not translation invariant since its shape depends on the reconstruction location  $\omega$ . Reconstruction, in that way, of the sampled signal  $S$  at position  $\omega \in \Omega$  is written as:

$$\tilde{S}(\omega) = \sum_{n=1}^N S_n \eta_n^\omega. \quad (7)$$

In super-resolution, the projection describes the way a LR signal  $S^k$  is obtained from the HR signal  $\hat{S}$ . The global description of this model is the following sequence I/ reconstruction of the continuous signal, II/ transformation of the obtained continuous signal and III/ LR sampling. We thus recognize a transformed down-sampling procedure.

$$\hat{S} \xrightarrow{I} \tilde{S} \xrightarrow{II} \tilde{S}_k \xrightarrow{III} S^k$$

The formal mathematical description follows here and can be understood by taking the previous scheme in the reverse order. Thus, for any  $n = 1, \dots, N$ ,

$$S_n^k = \tilde{S}_k(\omega_n) = \tilde{S}(t_k^{-1}(\omega_n)) = \sum_{m=1}^M \hat{S}_m \eta_m^{t_k^{-1}(\omega_n)}. \quad (8)$$

Where  $t_k^{-1}$  is a geometric transformation that we consider here to be a translation. From this projection model, the transformation and down-sampling operations clearly appear.

While projection is some kind of transformed down-sampling, the back-projection model that we propose here is a transformed up-sampling model. This model will be used and extended in the next sections.

Within this approach, we obtain either  $K$  back-projected HR signals  $\hat{S}^k$  from  $K$  LR signals  $S^k$  or one fused back-projected HR signal  $\hat{S}$ .

First, let us describe how we obtain the  $K$  back-projected HR signals. Back-projecting the  $k^{th}$  LR signal involves I/ reconstructing the LR signal in order to obtain a continuous signal, II/ transforming the obtained continuous signal, III/ HR sampling this signal. We thus recognize a transformed up-sampling procedure. The formal mathematical description follows here.

$$S^k \xrightarrow{I} \tilde{S}_k \xrightarrow{II} \tilde{S} \xrightarrow{III} \hat{S}^k$$

Once again, the following formal mathematical description should be read from point III to I in the previous scheme. Thus, for any  $m = 1, \dots, M$ ,

$$\hat{S}_m^k = \tilde{S}(\omega_m) = \tilde{S}_k(t_k(\omega_m)) = \sum_{n=1}^N S_n^k \eta_n^{t_k(\omega_m)}. \quad (9)$$

Then, how the fused back-projected HR signal is computed is described here. Let us consider the  $m^{th}$  pixel in the HR space. The  $n^{th}$  pixel of the  $k^{th}$  LR image can be seen as an information with a weight  $\eta_n^{t_k(\omega_m)}$  in the reconstruction process of the  $m^{th}$  pixel value. Thus, a straightforward way to fuse all the information provided by the  $K$  LR images should be:

$$\hat{S}_m = \gamma \sum_{k=1}^K \sum_{n=1}^N S_n^k \eta_n^{t_k(\omega_m)}, \quad (10)$$

$\gamma$  being a normalizing factor such that, if  $\forall k \in \{1, \dots, K\}$ ,  $\forall n \in \{1, \dots, N\}$ ,  $S_n^k = C$  ( $C$  being a constant value), then  $\forall m \in \{1, \dots, M\}$ ,  $\hat{S}_m = C$ . This normalizing factor  $\gamma$  equals  $\frac{1}{K}$  due to the fact that  $\eta^{t_k(\omega_m)}$  is summative:

$$\begin{aligned} C &= \gamma \sum_{k=1}^K \sum_{n=1}^N C \eta_n^{t_k(\omega_m)} \\ \implies \frac{1}{\gamma} &= \sum_{k=1}^K \sum_{n=1}^N \eta_n^{t_k(\omega_m)} = \sum_{k=1}^K 1 = K. \end{aligned}$$

Thus, Equation (10) is nothing else but the usual average of the  $K$  reconstructed signals  $\hat{S}^k$ :

$$\hat{S}_m = \frac{1}{K} \sum_{k=1}^K \hat{S}_m^k. \quad (11)$$

### III. SIGNAL CONVOLUTION WITH A CAPACITY

#### A. Capacity, possibility measure and imprecise expectation

Let  $\Theta$  be any discrete or continuous space. A capacity  $\nu$  on  $\Theta$  is a monotone confidence measure, i.e. a set-valued function defined on the power set  $\mathcal{P}(\Theta)$  such that  $\nu(\emptyset) = 0$ ,  $\nu(\Theta) = 1$  and if  $A \subseteq B \subseteq \Theta$ , then  $\nu(A) \leq \nu(B)$ .

A probability measure is a particular case of capacity which respects the additivity axiom. Another particular capacity which is of interest for us is the concave capacity: this is a capacity which is 2-alternating, i.e. for any  $A$  and  $B \subseteq \Theta$ , then

$$\nu(A \cap B) + \nu(A \cup B) \leq \nu(A) + \nu(B).$$

From a concave capacity, a dual confidence measure  $\nu^c$ , which is convex (opposite inequality to concave), is computed in this way:

$$\forall A \subseteq \Theta, \nu^c(A) = 1 - \nu(A^c). \quad (12)$$

The two measures,  $\nu^c$  and  $\nu$ , encode a family of probability measures, denoted by  $\mathcal{M}(\nu)$ , and defined by:

$$\mathcal{M}(\nu) = \{P \mid \forall A \subseteq \Theta, \nu^c(A) \leq P(A) \leq \nu(A)\}.$$

This encoding property is due to the sensitivity analysis interpretation [18] of concave capacities.

A possibility measure is a particular case of concave capacity [19]. Similarly to probability theory, a possibility measure  $\Pi$  is equivalently represented by its distribution function: the possibility distribution  $\pi$ . In probability theory, the link between distribution and measure is summative. In possibility theory, this link is maxitive: for any  $A \subseteq \Omega$ ,  $\Pi(A) = \max_{\omega \in A} \pi(\omega)$ .

Since a concave capacity measure is non-additive, the conventional expectation operator cannot be used. The expectation operator must be replaced by its generalization, called the Choquet integral (denoted  $\mathbb{C}$ ) [20]. Using a Choquet integral to compute the expectation of any bounded (discrete or continuous) function  $f$  leads to an interval-valued operator whose bounds are given by:

$$\overline{E}_\nu(f) = [\underline{E}_\nu(f), \overline{E}_\nu(f)] = [\mathbb{C}_{\nu^c}(f), \mathbb{C}_\nu(f)]. \quad (13)$$

The key point of this approach is that the interval-valued expectation obtained by means of a concave capacity measure is



the set of all the single-valued expectations obtained by using all the convolution kernels encoded by the considered concave capacity. This fundamental property comes from the work of Denneberg [21] linking precise and imprecise expectations.

*Theorem 1:* Let  $f : \Theta \rightarrow \mathbb{R}$  be a (discrete or continuous) bounded function and let  $\nu$  be a concave capacity defined on  $\Theta$ ,  $\forall P \in \mathcal{M}(\nu)$ ,  $\mathbb{E}_P(f) \in \overline{\mathbb{E}}_\nu(f)$  and  $\forall y \in \overline{\mathbb{E}}_\nu(f)$ ,  $\exists P \in \mathcal{M}(\nu)$  such that  $y = \mathbb{E}_P(f)$ .

### B. Signal convolution with a capacity

In most convolution-based signal applications (like low-pass filtering, sampling or reconstruction), the used convolution kernels are positive and have a unitary gain. These kernels are called *summative kernels*. For instance, the super resolution involves a reconstruction (7) kernel that is summative:

$$\sum_{n=1}^N \eta_n^\omega = 1.$$

In that case, the convolution kernel can be seen as a probability distribution or a probability density function that induces a probability measure, computed in this way:

$$\forall A \subseteq \{1, \dots, N\}, P_{\eta^\omega}(A) = \sum_{n \in A} \eta_n^\omega.$$

From this remark, we can show that reconstruction, which is a convolution operation can be written as an expectation operator:

$$\tilde{S}(\omega) = \mathbb{E}_{\eta^\omega}(S). \quad (14)$$

We propose to rewrite these operations with the expectation operator since it allows dealing with a family of convolution kernels by switching from the usual probability theory to imprecise probability theory. Since a probability measure is in one to one correspondence with a probability distribution or a probability density function (and thus with summative convolution kernels) we can claim that a concave capacity  $\nu$  encodes a family of summative convolution kernels equivalent to  $\mathcal{M}(\nu)$ .

All the necessary definitions were given in Sections III-A to directly define the imprecise convolution based upon a capacity neighborhood.

*Definition 1:* Let  $S : \Omega \rightarrow \mathbb{R}$  be a (discrete or continuous) signal and let  $\nu$  be a concave capacity defined on  $\Omega$ , The convolution of  $S$  by  $\nu$  is defined by:

$$S \overline{\ast} \nu = \overline{\mathbb{E}}_\nu(S) = [\mathbb{C}_{\nu^c}(S), \mathbb{C}_\nu(S)] \quad (15)$$

$S \overline{\ast} \nu$  represents all the convolutions we would have obtained with the set of convolution kernels encoded by  $\nu$ .

Some particular applications of this operator have already been proposed: imprecise linear filtering and noise level estimation [22], [23], guaranteed image rigid transformation [24] or link between fuzzy morphology and imprecise filtering [25].

## IV. GUARANTEED IMPRECISE BACK-PROJECTION

The back-projection method is particularly sensitive to the reconstruction kernel choice: see its expression (9). In this section we propose a particular case of the precise reconstruction involved in the back-projection. This operator is based on a reconstruction kernel which is the convolution of a summative kernel  $\kappa$  with a fuzzy partition of the LR signal domain. This operator is interesting for two reasons: 1/ it allows to transfer to the reconstructed signal the smoothness of the chosen fuzzy partition and 2/ it allows an extension to a guaranteed imprecise reconstruction operator and then to a guaranteed back-projection operator, which is the main object of this article.

### A. Fuzzy partition based precise reconstruction

The reconstruction kernel  $\eta^\omega$  that we propose to use is constructed from a fuzzy partition on the signal domain. Let  $\{C_n\}_{n=1, \dots, N}$  be the  $N$  atoms of a fuzzy partition à la Ruspini of  $\Omega$ , i.e. a set of unimodal symmetrical fuzzy intervals  $\{\mu_n\}_{n=1, \dots, N}$  complying with [26], [27]:  $\forall \omega \in \Omega$ ,

- $\sum_{n=1}^M \mu_n(\omega) = 1$ ,
- $\exists! n$ , such that  $\mu_n(\omega) > 0$  and  $\mu_{n+1}(\omega) \geq 0$ ,
- $\mu_n$  is continuous.

A useful tool for our developments is the definition of the union of fuzzy subsets in the fuzzy partition.

*Definition 2:* Let  $A \subseteq \{1, \dots, N\}$  be a subset of nodes of the à la Ruspini fuzzy partition  $\{\mu_n\}_{n=1, \dots, N}$ . Let  $\perp A$  be the notation for  $\bigcup_{n \in A} C_n$  where the membership function of this union of fuzzy subsets is the **Lukasiewicz** T-conorm defined by:

$$\forall \omega \in \Omega, \mu_{\perp A}(\omega) = \sum_{n \in A} \mu_n(\omega).$$

Note that  $\forall A \subseteq \{1, \dots, N\}$ ,  $\forall \omega \in \Omega$ ,  $\min(1, \sum_{n \in A} \mu_n(\omega)) = \sum_{n \in A} \mu_n(\omega)$ , since  $\sum_{n \in A} \mu_n(\omega) \leq 1$ .

Such operator fulfills an important property: for any sets of partition nodes  $A, B \subseteq \{1, \dots, N\}$ ,

$$\forall \omega \in \Omega, \mu_{\perp A \cup \perp B}(\omega) + \mu_{\perp A \cap \perp B}(\omega) = \mu_{\perp A}(\omega) + \mu_{\perp B}(\omega). \quad (16)$$

Equation (16) is directly deduced from the following general equation which is true for any sequence  $(u_n)_{n=1, \dots, N}$  of real numbers:  $\sum_{n \in A \cup B} u_n = \sum_{n \in A} u_n + \sum_{n \in B} u_n - \sum_{n \in A \cap B} u_n$ .

Usually a reconstruction operator (7) involves, for any  $\omega \in \Omega$ , a summative kernel  $\eta^\omega = (\eta_n^\omega)_{n=1, \dots, N}$ . In our approach, we propose to work with a summative kernel  $\eta^\omega$  which is the convolution of another summative kernel  $\kappa$  with the fuzzy partition  $\mu$ . Thus, we propose to consider a reconstruction kernel defined for any  $\omega$  of the image domain  $\Omega$  by:

$$\forall n = 1, \dots, N, \eta_n^\omega = (\kappa * \mu_n)(\omega) = \mathbb{E}_{\kappa^\omega}(\mu_n). \quad (17)$$

In that case, the reconstruction of a signal can be written as

$$\tilde{S}_\kappa(\omega) = \sum_{n=1}^N S_n(\kappa * \mu_n)(\omega). \quad (18)$$

It can be shown that the set of weights  $((\kappa * \mu_n)(\omega))_{n=1, \dots, N}$  is a summative kernel. Indeed,  $\sum_{n=1}^N \eta_n^\omega = \sum_{n=1}^N \int_{\Omega} \kappa(u - \omega) \mu_n(u) du = \int_{\Omega} \kappa(u - \omega) \sum_{n=1}^N \mu_n(u) du = \int_{\Omega} \kappa(u - \omega) du = 1$ .

The interest of this reconstruction is its stability due to the smoothness of the fuzzy partition basis function. Indeed, convolution is a type of averaging: it tends to be a smoothing operation. Generally speaking, a convolution of two functions inherits the “best” properties of both its operands. For instance, if  $\mu$  is continuous on  $\Omega$ , then, for any  $n$ ,  $\omega \mapsto \eta_n^\omega = (\kappa * \mu_n)(\omega)$  is continuous and thus is  $\tilde{S}_\kappa$  since it is a weighted sum of these  $N$  functions, see (18). Fixed smoothness could be passed to the reconstructed signal when the smoothness of the fuzzy partition is fixed.

Now we can show that the probability measure associated to the summative kernel  $\eta^\omega$  (17), which is a probability distribution, can be written, for any coalition of sampling positions (or fuzzy partition nodes)  $A$ , by

$$P_{\eta^\omega}(A) = \mathbb{E}_{\kappa^\omega}(\mu_{\perp A}). \quad (19)$$

Indeed,  $P_{\eta^\omega}(A) = \sum_{n \in A} \eta_n^\omega = \sum_{n \in A} \int_{\Omega} \kappa(u - \omega) \mu_n(u) du = \int_{\Omega} \kappa(u - \omega) \sum_{n \in A} \mu_n(u) du = \int_{\Omega} \kappa(u - \omega) \mu_{\perp A}(u) du = \mathbb{E}_{\kappa^\omega}(\mu_{\perp A})$ . Finally, the precise reconstruction operator is

$$\tilde{S}_\kappa(\omega) = \mathbb{E}_{P_{\eta^\omega}}(S). \quad (20)$$

## B. Guaranteed reconstruction

Proposition 1 shows the stability of the expectation (precise or imprecise) operator when applied to coalitions of fuzzy subsets of a fuzzy partition à la Ruspini.

*Proposition 1:* Let  $\{C_n\}_{n=1, \dots, N}$  be a fuzzy partition à la Ruspini of  $\Omega$ .

- 1) Let  $P$  be a continuous probability measure on  $\Omega$ . Then  $P_\mu$ , defined, for any subset of nodes  $A \subset \{1, \dots, N\}$ , by:  $P_\mu(A) = \mathbb{E}_P(\mu_{\perp A})$  is a discrete probability measure.
- 2) Let  $\nu$  be a continuous concave capacity, then  $\nu_\mu$ , for any subset of nodes  $A \subset \{1, \dots, N\}$ , by:  $\nu_\mu(A) = \mathbb{E}_\nu(\mu_{\perp A}) = \mathbb{C}_\nu(\mu_{\perp A})$  is a discrete concave capacity.
- 3) if  $P \in \mathcal{M}(\nu)$  then  $P_\mu \in \mathcal{M}(\nu_\mu)$ .

*Proof:*

1) This point is directly due to the linearity of the expectation operator and to property (16).

2) First it should be noted that for any concave capacity  $\nu$ , for any  $f$  and  $g$  in  $L_1(\Omega)$ ,  $\mathbb{C}_\nu(f + g) \leq \mathbb{C}_\nu(f) + \mathbb{C}_\nu(g)$ . If  $f$  and  $g$  are comonotonic on  $\Omega$ , then  $\mathbb{C}_\nu(f + g) = \mathbb{C}_\nu(f) + \mathbb{C}_\nu(g)$ .

$\mu_{\perp A \cup B}$  and  $\mu_{\perp A \cap B}$  are comonotonic on  $\Omega$ . Indeed, due to the Lukasiewicz T-conorm definition, (i) if  $\mu_{\perp A \cap B} = 0$  and is flat, then whatever the monotonicity of  $\mu_{\perp A \cup B}$ , they are comonotonic; (ii) if  $\mu_{\perp A \cap B} > 0$  then  $\mu_{\perp A \cap B} = \mu_{\perp A \cup B}$  or  $\mu_{\perp A \cup B} = 1$  and thus  $\mu_{\perp A \cup B}$  and  $\mu_{\perp A \cap B}$  are comonotonic.

Therefore  $\mathbb{C}_\nu(\mu_{\perp A \cup B} + \mu_{\perp A \cap B}) = \mathbb{C}_\nu(\mu_{\perp A \cup B}) + \mathbb{C}_\nu(\mu_{\perp A \cap B})$ .

From expression (16), we also have that  $\mathbb{C}_\nu(\mu_{\perp A \cup B} + \mu_{\perp A \cap B}) = \mathbb{C}_\nu(\mu_{\perp A} + \mu_{\perp B})$ , which is  $\leq \mathbb{C}_\nu(\mu_{\perp A}) + \mathbb{C}_\nu(\mu_{\perp B})$ , because  $\nu$  is concave. Thus,

$$\nu_\mu(A \cup B) \leq \nu_\mu(A) + \nu_\mu(B) - \nu_\mu(A \cap B),$$

$\nu_\mu$  is a concave capacity.

3) If  $P_\mu(A) = \mathbb{E}_P(\mu_{\perp A})$  is a probability measure and  $\nu_\mu(A) = \mathbb{C}_\nu(\mu_{\perp A})$  is a concave capacity and  $P \in \mathcal{M}(\nu)$ , then a direct implication of Theorem 1 is that  $P_\mu \in \mathcal{M}(\nu_\mu)$ . ■

From these stability results, we construct an imprecise reconstruction operator which is guaranteed to contain a set of reconstructors that we can tune. This new operator employs a fuzzy representation of the involved reconstruction kernel  $\kappa$ . Instead of considering usual summative reconstruction kernel  $\kappa$  as it is done in the reconstruction operator (18), we use a possibility distribution  $\pi$ . It allows explicitly working with families of usual reconstruction kernels.

Within this modeling, we propose to use the discrete concave capacity defined, for any coalition of sampling points  $A$ , by:

$$\nu_{\pi^\omega}(A) = \mathbb{C}_{\Pi^\omega}(\mu_{\perp A}) \quad (21)$$

The imprecise reconstruction operator is obtained from this concave capacity constructed from the fuzzy partition and the fuzzy (or possibilistic) reconstruction kernel  $\pi$  by the Choquet integral operator.

*Definition 3 (Guaranteed reconstruction operator):* Let  $S = (S_n)_{n=1, \dots, N}$  be a sampled signal. Let  $\pi$  be a fuzzy neighborhood modeling the ill-known reconstruction kernel of  $S$ . Let  $\mu$  be a fuzzy partition on the reconstruction domain.

$$[\tilde{S}_\pi(\omega), \overline{\tilde{S}}_\pi(\omega)] = [\mathbb{C}_{\nu_{\pi^\omega}^c}(S), \mathbb{C}_{\nu_{\pi^\omega}}(S)], \quad (22)$$

is the guaranteed reconstructed signal of  $S$ , where  $\nu_{\pi^\omega}$  is defined by (21).

Our main result is that the imprecise reconstruction obtained with a fuzzy reconstruction modeling  $\pi$  that we propose is guaranteed to include the set all the precise reconstruction operators obtained with the set reconstruction models of the family  $\mathcal{M}(\pi)$  represented by  $\pi$ .

*Theorem 2 (Guaranteed reconstruction theorem):* Let  $\mu$  be a fuzzy partition on the reconstruction domain. Let  $\pi$  be a fuzzy subset representing a family of summative reconstruction kernels  $\mathcal{M}(\pi)$ . Then, for any summative reconstruction kernel  $\kappa$  of  $\mathcal{M}(\pi)$ ,

$$\forall \omega \in \Omega, \tilde{S}_\kappa(\omega) \in [\tilde{S}_\pi(\omega), \overline{\tilde{S}}_\pi(\omega)]. \quad (23)$$

*Proof:* From the definitions of  $\tilde{S}_\kappa$  and  $[\tilde{S}_\pi, \overline{\tilde{S}}_\pi]$  (respectively (20) and (22)), from the measure constructions (19) and (21) as well as from Proposition 1, it is directly proved. ■

Moreover, the smoothness of the set of reconstructed signals is guaranteed to be at least the same as the fuzzy partition one.

### C. Guaranteed back-projection

The usual back-projection can thus be extended to a guaranteed back-projection. The first step is the direct application of the previous imprecise guaranteed reconstruction to all the LR signals  $S^k$  for the transformed sampling points:  $t_k(\omega_m)$  (cf. expression (9)). Thus, for any  $k = 1, \dots, K$ , we have

$$\forall m = 1, \dots, M, [\hat{S}_m^k, \overline{\hat{S}_m^k}] = [\tilde{S}_\pi(t_k(\omega_m)), \overline{\tilde{S}_\pi(t_k(\omega_m))}], \quad (24)$$

which contains all the HR reconstructed signals we would have obtained with all the summative reconstruction kernels of  $\mathcal{M}(\pi)$ .

The guaranteed back-projection is the fusion of these guaranteed reconstructed HR signals with the same fusion operator than in the precise case. Thus,

$$\forall m = 1, \dots, M, [\hat{S}_m, \overline{\hat{S}_m}] = \left[ \frac{1}{K} \sum_{k=1}^K \hat{S}_m^k, \frac{1}{K} \sum_{k=1}^K \overline{\hat{S}_m^k} \right]. \quad (25)$$

**Theorem 3 (Guaranteed back-projection theorem):** Let  $\mu$  be a fuzzy partition on the reconstruction domain. Let  $\pi$  be a fuzzy subset representing a family of summative reconstruction kernels  $\mathcal{M}(\pi)$ . All the back-projection operators obtained with the summative kernels  $\kappa \in \mathcal{M}(\pi)$  are included in the guaranteed intervalist back-projection operator (25) obtained with  $\pi$ .

*Proof:* Summing intervals does not alter the guaranty. Therefore Theorem 3 is proved. ■

## V. EXPERIMENTS

In this experimental section, we illustrate the properties of this new back-projection operator that have been proved in Section IV. This section is decomposed in three parts. In the first part, we explain and illustrate how a sub-pixel projected translation modifies the measured illumination distribution in the acquired image. The second and third parts are respectively dedicated to the illustration of Theorem 2 and Theorem 3. In the second part, we consider reconstructing a single LR image. In the third part, we perform a back-projection of a HR image with 20 LR images. The LR images are simulated by subsampling a very-high resolution ( $22610 \times 22279$ ) image of the painting "The Ambassadors" (1533) by Hans Holbein the Younger, according to the projection model (8). To provide an easier reading, the results are given in  $1D$ , i.e. the considered signals are lines of the images. Extending this method in  $2D$  can be achieved in two ways. The first way consists in defining a  $2D$  capacity as in [25]. A second and easier technique can be used that consists in considering the family of separable kernels. In this case, performing the  $2D$  reconstruction can be simply performed by performing the reconstruction row-wise then column-wise.

### A. Translational sampling effects

As mentioned in the introduction, achieving super-resolution involves LR input images which are related by sub-pixel motions. The high resolution arises from the fact that the LR images contain different information. The LR images are obtained by sub-sampling a very high resolution image by using the projection model (8) with different translation values.

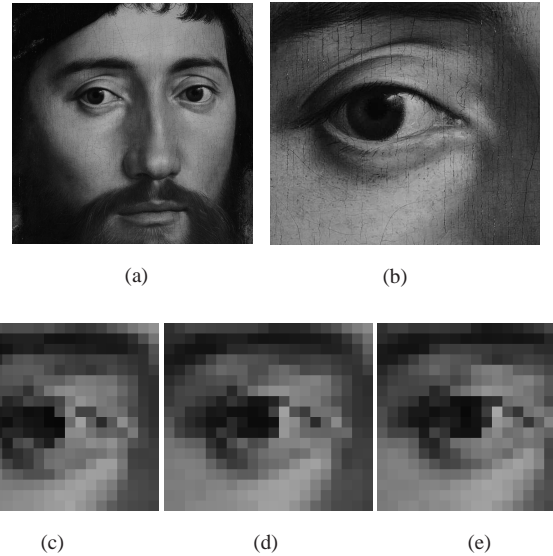


Figure 1: (a) HR image, (b) Zoom on a part of the HR image, (c), (d), (e) zoom on parts of three LR images generated from the image (a) with different translation values.

Figure 1(c)(d)(e) shows details of three LR images resulting from the HR image (Figure 1(a), detail on (b)). It clearly appears in Figure 1 that the three low resolution images carry different information related to the original image: have a close look, for example, at the white spot in the eye or the shadow under the eyebrow.

### B. Single-input reconstruction

This part aims at illustrating that any precise reconstructed signal obtained with Expression (18) is included in the interval-valued reconstructed signal obtained by considering the concave capacity  $\nu_\pi$  defined by Expression 21, if  $\kappa$  is included in the core of  $\pi$ .

Within this experiment, we consider the triangular maxitive kernel whose spread  $\Delta$  equals five times the spread of the sought after HR sampling in order to consider a wide range of unimodal symmetric kernels [28]. The set of summative kernels used in this experiment includes Epanechnikov, truncated Gaussian, cubic splines, uniform, and randomly generated kernels having a spread lower or equal to  $\Delta$ . The considered LR image is obtained by down-sampling each line of the original HR image by 25, using a cubic spline kernel whose spread equals 25.

Figure 2 plots a part of one of the original image lines. Figure 3 is the LR signal obtained by down-sampling the original line. Figure 4 plots the imprecise reconstructed signal obtained by using the concave capacity  $\nu_\pi$  (upper bound in blue, lower bound in red) superimposed with ten precise reconstructions (in light blue) obtained with summative kernels constructed as proposed in Expression (9). Figure 4(b) clearly shows that all the precise reconstructed signals belong to the imprecise reconstructed signal.

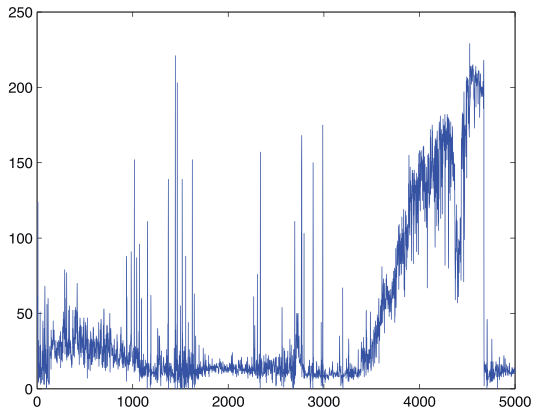


Figure 2: Original HR signal.

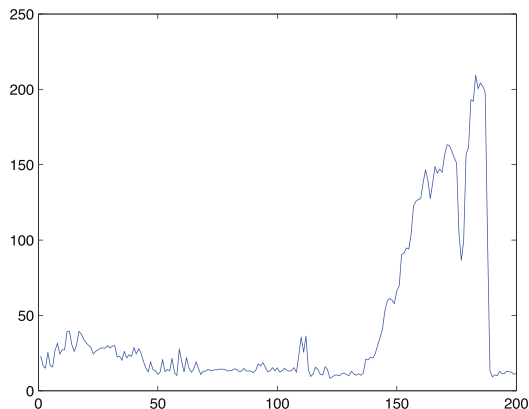
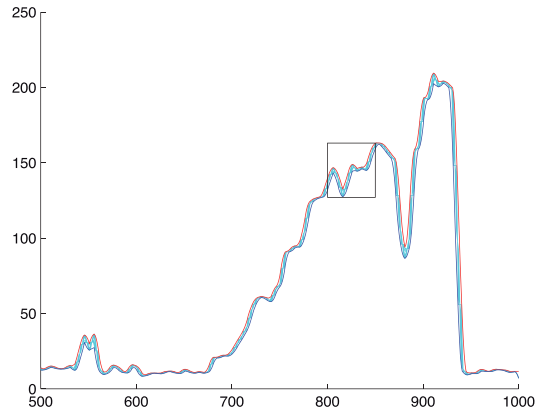
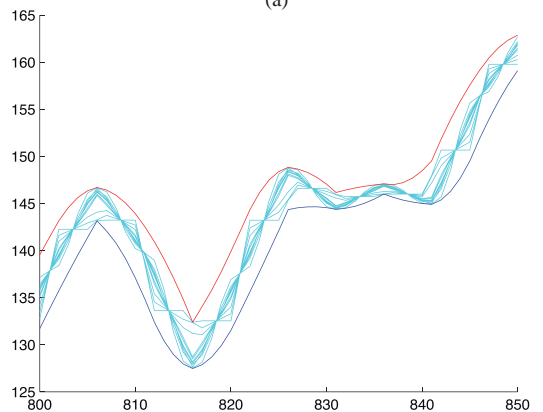


Figure 3: LR generated signal.



(a)



(b)

Figure 4: (a): 10 precise-reconstruction signals with a single LR input signal (light blue) and upper (resp. lower) bound of the imprecise reconstruction in red (resp. blue). (b): enlargement of the portion framed in black in (a)

### C. Multiple-input reconstruction

The aim of this part is to illustrate that the interval-valued reconstruction, obtained using several LR signals related by sub-pixel translations (according to Expression (25)) and the concave capacity  $\nu_\pi$ , includes all precise reconstructions obtained using the same LR signals (according to Expression (10)) and by considering a summative kernel obtained by convoluting a summative kernel  $\kappa$  (included in the core of  $\pi$ ) with a fuzzy partition of the LR space.

20 LR signals have been generated according to the translational projection equation (8). The translation values are integers in the original image space but not in the low resolution space. The other settings remain the same as the settings of Section V-B.

For each of the 20 LR signals, an interval-valued reconstruction is computed taking into account the known translation values and using the same maxitive kernel  $\pi$  than in the previous experiment. These 20 interval-valued reconstructions are fused using a mean operator, i.e. the lower (resp. upper) boundary is the mean of all lower (resp. upper) boundaries according to Expression (25). The upper (resp. lower) bound of this interval-valued signal is plotted in blue (resp. in red) on Figure 5(a). As in the previous experiment, we have considered 10 different reconstructions according to Expression (10) by considering 10 different summative reconstruction kernels

generated by convoluting a fuzzy partition in the LR space with 10 summative kernels belonging to the core of  $\pi$ .

On Figure 5(a), those 10 precise reconstructed signals are plotted in light blue. Figure 5(b) clearly shows that the interval-valued reconstructed signal includes every precise reconstructed signals. This property is true on the whole domain of the signal.

## VI. CONCLUSION

In this paper, we proposed an original non-additive interval-valued reconstruction operator in a super resolution scheme. This solution allows coping with an a priori choice of reconstruction kernels which is a crucial point in super resolution techniques.

Our approach consists on shifting from a precise to an imprecise representation of the reconstruction kernel. This imprecise representation is based on constructing a concave capacity, whose core contains a wide range of usual reconstruction kernels, by exploiting the nice properties of fuzzy partitions. The obtained reconstructed image (here lines of the image) is interval-valued. We prove that this interval contains every precise reconstructions obtained by using a reconstruction kernel belonging to the core of the considered capacity.



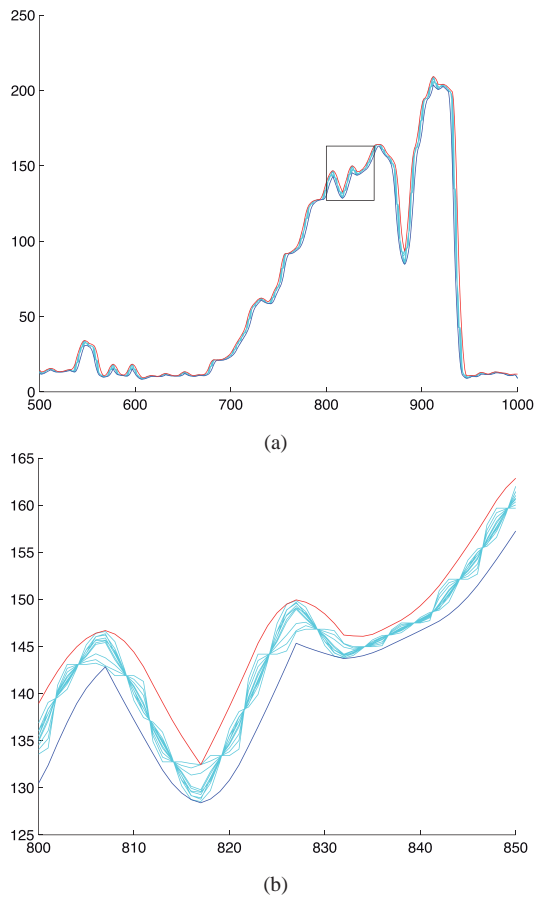


Figure 5: (a): 10 precise-reconstruction signals with 20 LR input signals in light blue and upper (resp. lower) bound of the intervallistic reconstruction in red (resp. blue). (b): enlargement of the portion framed in black in (a)

Future work will study the specificity of the obtained interval-valued reconstructed signal (is it the most specific interval having this property), the extension of this approach in two dimension and its use in an interval valued super-resolution reconstruction of the original image. This last part will need an interval-valued extension of the reconstruction procedures described in Section II-A.

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