A linear kernel for planar red-blue dominating set

Valentin Garnero, Ignasi Sau, Dimitrios M. Thilikos

A linear kernel for planar
red-blue dominating set *

Valentin Garnero[1], Ignasi Sau[1], and Dimitrios M. Thilikos[1,2]

[1]AlGCo project-team, CNRS, LIRMM, Montpellier, France.
[2]Department of Mathematics, National & Kapodistrian University of Athens, Greece.

In the RED-BLUE DOMINATING SET problem, we are given a bipartite graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has a subset $D \subseteq V_B$ of at most k 'blue' vertices such that each 'red' vertex from V_R is adjacent to a vertex in D. We provide the first explicit linear kernel for this problem on planar graphs.

Keywords: parameterized complexity, planar graphs, linear kernels, domination.

1 Introduction

The field of parameterized complexity (see [4]) deals with algorithms for decision problems whose instances consist of a pair (x, k), where k is known as the *parameter*. A fundamental concept in this area is that of *kernelization*. A kernelization algorithm, or *kernel*, for a parameterized problem takes an instance (x, k) of the problem and, in time polynomial in $|x| + k$, outputs an equivalent instance (x', k') such that $|x'|, k' \leq g(k)$ for some function g. The function g is called the *size* of the kernel and may be viewed as a measure of the "compressibility" of a problem using polynomial-time preprocessing rules. A natural problem in this context is to find polynomial or linear kernels for problems that admit such kernelization algorithms.

A celebrated result in this area is the linear kernel for DOMINATING SET on planar graphs by Alber *et al.* [2], which gave rise to an explosion of (meta-)results on linear kernels on planar graphs [8] and other sparse graph classes [3, 5, 9]. Although of great theoretical importance, these meta-theorems have two important drawbacks from a practical point of view. On the one hand, these results rely on a problem property called *Finite Integer Index*, which guarantees the *existence* of a linear kernel, but it is still not yet clear how and when such a kernel can be effectively *constructed*. On the other hand, at the price of generality one cannot hope that general results of this type may directly provide explicit reduction rules and small constants for particular graph problems.

In this article we follow this research avenue and focus on the RED-BLUE DOMINATING SET problem (RBDS for short) on planar graphs. In the RED-BLUE DOMINATING SET problem, we are given a bipartite graph $G = (V_B \cup V_R, E)$ and an integer k, and asked whether G has

---

a subset $D \subseteq V_B$ of at most k 'blue' vertices such that each 'red' vertex from V_R is adjacent to a vertex in D. From a (classical) complexity point of view, finding a red-blue dominating set of minimum size is NP-complete on planar graphs [1]. From a parameterized complexity perspective, RBDS parameterized by the size of the solution is $W[2]$-complete on general graphs and FPT on planar graphs [4].

The fact that RBDS involves a *coloring* of the vertices of the input graph makes it unclear how to make the problem fit into the general frameworks of [3, 5, 8, 9]. In this article we provide the first explicit (and quite simple) polynomial-time data reduction rules for RED-BLUE DOMINATING SET on planar graphs, which lead to a linear kernel for the problem.

Theorem 1. RED-BLUE DOMINATING SET *parameterized by the solution size has a linear kernel on planar graphs. More precisely, there exists a poly-time algorithm that for each positive planar instance* (G, k) *returns an equivalence instance* (G', k) *such that* $|V(G')| \leq 48 \cdot k$.

This result complements several explicit linear kernels on planar graphs for other domination problems such as DOMINATING SET [2], EDGE DOMINATING SET [8], EFFICIENT DOMINATING SET [8], CONNECTED DOMINATING SET [7, 11], or TOTAL DOMINATING SET [6]. We stress that our constant is considerable smaller than most of the constants provided by these results. Since one can easily reduce the FACE COVER problem on a planar graph to RBDS (without changing the parameter)[1], the result of Theorem 1 also provides a linear *bikernel* for FACE COVER (i.e., a polynomial-time algorithm that given an input of FACE COVER, outputs an equivalent instance of RBDS with a graph whose size is linear in k). To the best of our knowledge, the best existing kernel for FACE COVER is quadratic [10]. Our techniques are much inspired from those of Alber *et al.* [2] for DOMINATING SET, although our reduction rules and analysis are slightly simpler.

2 A linear kernel for planar red-blue dominating set

We first propose several reduction rules and then we analyze the size of the obtained graph.

Reduction rules. We start with an elementary rule that turns out to be helpful in simplifying the instance, and then we present the rules for a single vertex and a pair of vertices. For simplicity, we will use the shorthand rbds to denote a red-blue dominating set in a graph.

Rule 1. *Iteratively remove blue vertices whose neighborhood is included into the neighborhood of another blue vertex. Similarly, remove red vertices whose neighborhood includes the neighborhood of another red vertex.*

Definition 1. *Let* $G = (V_B \cup V_R, E)$ *be a graph. The* neighborhood *of a vertex* $v \in V_B \cup V_R$ *is the set* $N(v) = \{u : \{v, u\} \in E\}$. *The* private neighborhood *of a blue vertex* b *is the set* $P(b) = \{r \in N(b) : N(N(r)) \subseteq N(b)\}$.

Rule 2. *Let* $b \in V_B$ *be a blue vertex. If* $|P(b)| > 1$, *remove* $P(b)$ *from* G *and add a new red vertex* r *and the edge* $\{b, r\}$.

Definition 2. *Let* $G = (V_B \cup V_R, E)$ *be a graph. The* neighborhood *of a blue pair of vertices* $b, c \in V_B$ *is the set* $N(b, c) = N(b) \cup N(c)$. *The* private neighborhood *of a blue pair of vertices* $b, c \in V_B$ *is the set* $P(b, c) = \{r \in N(b, c) : N(N(r)) \subseteq N(b, c)\}$.

---

[1] Just consider the *radial graph* corresponding to the input graph G and its dual G^*, and color the vertices of G (resp. G^*) as red (resp. blue).

Rule 3. *Let b, c be two distinct blue vertices. If $|P(b, c)| > 2$ and there is no blue vertex $d \neq b, c$ which dominates $P(b, c)$:*

1. *if $P(b, c) \not\subseteq N(b)$ and $P(b, c) \not\subseteq N(c)$:*
   - *remove $P(b, c)$ from G,*
   - *add two new red vertices r_b, r_c and the edges $\{b, r_b\}, \{c, r_c\}$;*

2. *if $P(b, c) \subseteq N(b)$ and $P(b, c) \subseteq N(c)$:*
   - *remove $P(b, c)$ from G,*
   - *add a new red vertex r and the edges $\{b, r\}, \{c, r\}$;*

3. *if $P(b, c) \subseteq N(b)$ and $P(b, c) \not\subseteq N(c)$:*
   - *remove $P(b, c)$ from G,*
   - *add a new red vertex r and the edge $\{b, r\}$;*

4. *if $P(b, c) \not\subseteq N(b)$ and $P(b, c) \subseteq N(c)$:*
   - *symmetrically to Case 3.*

Lemma 1. $[\star]^2$ *Let $G = (V_B \cup V_R, E)$ be a graph. If G' is the graph obtained from G by the application of Rules 1, 2, or 3, then there is a rbds in G of size at most k if and only if there is one in G'.*

Analysis of the kernel size. We will show that a graph *reduced* under our rules (i.e., a graph for which none of the rules can be applied anymore) has size linear in $|D|$, the size of a solution. To this aim we assume that the graph is *plane* (that is, given with a fixed embedding) and we will define a notion of region adapted to our definition of neighborhood. Then we will show that, given a solution D, there is a maximal region decomposition $\Re$ such that:

- $\Re$ has $O(|D|)$ regions,
- $\Re$ covers all vertices but $O(|D|)$ of them,
- each region of $\Re$ has size $O(1)$.

The three following propositions treat respectively each of the above claims.

Definition 3. *Let $G = (V_B \cup V_R, E)$ be a plane graph and let $v, w \in V_B$. A region $R(v, w)$ between v and w is a closed subset of the plane such that:*

- *the boundary of $R(v, w)$ is formed by two simple paths connecting v and w, each of them having at most 4 edges;*
- *all vertices (strictly) inside $R(v, w)$ belong to $N(v, w)$ or $N(N(v, w))$.*

Definition 4. *Let $G = (V_B \cup V_R, E)$ be a plane graph and let $D \subseteq V_B$. A D-decomposition of G is a set of regions $\Re$ between pairs of vertices in D such that:*

- *any region between v, w does not contain vertices in $D \setminus \{v, w\}$;*
- *any two regions have only the boundary in common.*

We note $V(\Re) = \bigcup_{R \in \Re} V(R)$. A D-decomposition is maximal *if there is no region $R \notin \Re$ such that $\Re \cup \{R\}$ is a D-decomposition with $V(\Re) \subsetneq V(\Re \cup \{R\})$.*

---

[2] The proofs of the results marked with '[$\star$]' are omitted in this extended abstract.

Proposition 1. [⋆] *Let G be a reduced plane graph and let D be a rbds in G. There is a maximal D-decomposition of G such that $|\Re| \leq 3 \cdot |D| - 6$.*

Proposition 2. [⋆] *Let $G = (V_B \cup V_R, E)$ be a reduced plane graph and let D be a rbds in G. If $\Re$ is a maximal D-decomposition, then $|V \setminus (V(\Re) \cup D)| \leq 2 \cdot |D|$.*

Proposition 3. [⋆] *Let $G = (V_B \cup V_R, E)$ be a reduced plane graph, let D be a rbds in G, and let $v, w \in D$. A region R between v and w contains at most 15 vertices distinct from v, w.*

We are finally ready to piece everything together and prove Theorem 1.

Proof of Theorem 1. Let G be the plane input graph and let G' be the reduced graph obtained from G. According to Lemma 1, G admits a rbds with size at most k if and only if G' admits one. It is easy to see that the same time analysis of [2] implies that our reduction rules can be applied in time $O(|V(G)|^3)$. According to Propositions 1, 2, and 3, if G' admits a rbds with size at most k, then G' has size at most $k + 15 \cdot (3k - 6) + 2k \leq 48k$. □

References

[1] J. Alber, H. Bodlaender, H. Fernau, and R. Niedermeier. Fixed parameter algorithms for planar dominating set and related problems. In *Proc. of the 7th Scandinavian Workshop on Algorithm Theory (SWAT)*, volume 1851 of *LNCS*, pages 97–110, 2000.

[2] J. Alber, M. Fellows, and R. Niedermeier. Polynomial-Time Data Reduction for Dominating Set. *Journal of the ACM*, 51(3):363–384, 2004.

[3] H. L. Bodlaender, F. V. Fomin, D. Lokshtanov, E. Penninkx, S. Saurabh, and D. M. Thilikos. (Meta) Kernelization. In *Proc. of the 50th IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 629–638. IEEE Computer Society, 2009.

[4] R. G. Downey and M. R. Fellows. *Parameterized Complexity.* Springer, 1999.

[5] F. V. Fomin, D. Lokshtanov, S. Saurabh, and D. M. Thilikos. Bidimensionality and kernels. In *Proc. of the 21st ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 503–510. SIAM, 2010.

[6] V. Garnero and I. Sau. A linear kernel for planar total dominating set. Manuscript available at `arxiv.org/abs/1211.0978`, 2012.

[7] Q. Gu and N. Imani. Connectivity is not a limit for kernelization: Planar connected dominating set. In *Proc. of the 9th Latin American Symposium on Theoretical Informatics (LATIN)*, volume 6034 of *LNCS*, pages 26–37, 2010.

[8] J. Guo and R. Niedermeier. Linear problem kernels for NP-hard problems on planar graphs. In *Proc. of the 34th International Colloquium on Automata, Languages and Programming (ICALP)*, volume 4596 of *LNCS*, pages 375–386, 2007.

[9] E. J. Kim, A. Langer, C. Paul, F. Reidl, P. Rossmanith, I. Sau, and S. Sikdar. Linear kernels and single-exponential algorithms via protrusion decompositions. Manuscript available at `arxiv.org/abs/1207.0835`, to appear in *Proc. of ICALP'13*, 2012.

[10] T. Kloks, C.-M. Lee, and J. Liu. New Algorithms for k-Face Cover, k-Feedback Vertex Set, and k Disjoint Cycles on Plane and Planar Graphs. In *Proc. of 28th International Workshop on Graph-Theoretic Concepts in Computer Science (WG)*, volume 2573 of *LNCS*, pages 282–295, 2002.

[11] D. Lokshtanov, M. Mnich, and S. Saurabh. A linear kernel for planar connected dominating set. *Theoretical Computer Science*, 23(412):2536–2543, 2011.