Natural Language Semantics in Biproduct Dagger Categories

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Abstract

Biproduct dagger categories serve as models for natural language. In particular, the biproduct dagger category of finite dimensional vector spaces over the field of real numbers accommodates both the extensional models of predicate calculus and the intensional models of quantum logic. The morphisms representing the extensional meanings of a grammatical string are translated to morphisms representing the intensional meanings such that truth is preserved. Pregroup grammars serve as the tool that transforms a grammatical string into a morphism. The chosen linguistic examples concern negation, relative noun phrases, comprehension and quantifiers.

Keywords: Compositional semantics, biproduct bagger categories, compact closed categories, quantum logic, pregroup grammars, compact bilinear logic, proof graphs, two-sorted logic

1 Introduction

Biproduct dagger categories have been studied extensively in quantum logic, [Selinger, 2007], [Abramsky and Coecke, 2004], [Heunen and Jacobs, 2010]. They also constitute a natural candidate as a foundation of natural language semantics, because they formalize count words (biproduct) and relative pronouns (dagger), two logical abstractions present in natural language with a few exceptions - both are absent in the Amazonian Pirahã, [Everett, 2005].

These two operations are powerful enough to comprehend the structure of a compact closed category and with it the representation of morphisms by graphs that represent information flow. Information flow along morphisms handles among other things the grammatical notions of dependency and control. The noun phrase *birds who fly* in 5.2 illustrates how the graphical representation of meanings recuperates the dependency links. An example concerning control can be found in 3.6 of [Preller and Prince, 2008].

The biproduct and dagger also make it possible to imbed predicate logic. In particular, the logical content of words like *and*, *or*, *not*, *all*, *no*, *who*, *some* can be captured by an explicit definition and the notions of truth and logical

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consequence can be introduced. The property of the morphisms interpreting *some*, however, does not restrict to a unique morphism. Each occurrence of the word *some* in the text may by interpreted differently.

The biproduct and dagger also provide an abstract definition of the inner product and therefore a geometrical representation of the linguistic notion of similarity by the inner product (cosine) and of the logical notion of negation by orthogonality. Thus, the geometrical operators on subspaces and projectors proposed in [Rijsbergen, 2004] and [Widdows, 2004] can be generalised from $\mathbf{FdVect}_{\mathbb{R}}$, the finite dimensional vector spaces over the field of real numbers, to biproduct dagger categories.

The *semantic categories* considered here are biproduct dagger categories with a generating object. Typical semantic categories are the category $\mathbf{FdVect}_{\mathbb{R}}$ and the category $\mathit{2}\mathcal{SF}$ of finite sets and two-sorted functions. The latter are models of two-sorted first order predicate logic, which according to [Benthem and Doets, 1983] is equivalent to second order logic with general models.

The mathematical tool for recognising grammatical strings of words and computing their meanings is that of pregroup grammars.

The syntactical analysis is carried out in the free quasi-pregroup $\mathcal{C}(\mathcal{B})$ generated by a partially ordered set $\mathcal{B}$ of *basic types* introduced in [Lambek, 1999]. It is a compact bicategory, that is to say a compact closed category. The pregroup dictionaries of [Lambek, 2008], which list pairs *word* : T consisting of a word and a type, lack semantics. Therefore, the pregroup lexicons proposed here consist of triples *word* : $T :: \overline{\texttt{word}}$, where $\overline{\texttt{word}}$ is a formal expression in the language of compact closed categories. These formal morphisms play a role similar to that of lambda-terms in categorial grammars.

The meanings of grammatical strings are computed in the *lexical category* $\mathcal{C}(\mathcal{B} \cup \mathcal{L})$, the free compact closed category generated by the partially ordered set $\mathcal{B}$ and the set of basic morphisms $\mathcal{L}$ given by the pregroup lexicon. The interpretation of the grammatical string is mediated by a functor from the lexical category into an arbitrary biproduct dagger category with a generating object. If the functor preserves the compact closed structure, it guarantees compositionality, but it must satisfy supplementary conditions to become a model for natural language.

The compositional semantics of [Clark et al., 2008] and [Kartsaklis et al., 2013] for vector space models in $\mathbf{FdVect}_{\mathbb{R}}$ also use pregroup grammars and a 'functorial method'. This claim has to be taken with a caveat. The suggested functor is partial, it is not defined for strings containing logical words, relative, determiners and so on. In fact, such a partial functor cannot be extended to the logical words expressing negation and implication, neither for classical logic nor for quantum logic, [Preller, 2013].

The categorical semantics in biproduct dagger categories proposed here is more general. One can not only define vector space models in the abstract setting of an arbitrary semantic category, but also functors that are defined on all of the lexical category and that also simulate truth and logical consequence. These are the 'truth-theoretical' models. For every truth-theoretical model there is a 'canonical' vector space model and an isomorphism from the lattice of predicates in the former to the lattice of projectors introduced by quantum logic in the latter. This is due to the fact that the projectors corresponding to words have a common basis of eigenvectors.

The material of this article is organised as follows. Section 2 presents the basic properties of biproduct dagger categories with an emphasis on the class of projectors called 'intrinsic', because their matrix representation is the same in any biproduct dagger category. They include the morphisms arising from grammatical strings. Section 3 concentrates on biproduct dagger categories with a generating object and their important property of 'explicit definitions'. Section 4 establishes the equivalence between the quantum logic of intrinsic projectors and the logic of predicates. The essential characteristic of a predicate is that it assigns truth values both to individuals and sets of individuals. Section 5 starts with a cut free axiomatisation of compact bilinear logic, [Lambek, 1993] and [Buszkowski, 2002], and shows how pregroup grammars construct syntactical analysis and semantical representation in the lexical categories based on the proof graphs of [Preller and Lambek, 2007]. The section concludes with a few linguistic examples linking relative noun phrases and comprehension as well as quantifiers and negation.

2 Basic properties

This section recalls definitions and properties frequently intervening in quantum logic, see for example [Abramsky and Coecke, 2004], [Heunen and Jacobs, 2010], [Selinger, 2007]. Only the emphasis on 'intrinsic' morphisms is new.

2.1 Biproduct dagger categories

A *dagger category* is a category $\mathcal{C}$ together with a contravariant involution functor *dagger* $\dagger : \mathcal{C} \to \mathcal{C}$ that is the identity on objects. This means that the following equalities hold for any object V and morphisms $f : V \to W$, $g : W \to U$

$$\begin{aligned} V^\dagger &= V \\ 1_V^\dagger &= 1_V \\ (g \circ f)^\dagger &= f^\dagger \circ g^\dagger : U \to V \\ f^{\dagger\dagger} &= f : V \to W\,. \end{aligned}$$

Call $f^\dagger$ the *adjoint* of f.

In a dagger category any coproduct of V and W with canonical injections q_1 and q_2 is also a product of V and W with canonical projections $q_1^\dagger$ and $q_2^\dagger$ and vice versa. Indeed, the dagger inverts the diagram expressing the universal property. Hence coproducts are biproducts in a dagger category. Similarly, an initial object 0 of a dagger category is also a terminal object. Indeed, if $0_V : 0 \to V$ is the unique morphism from 0 to V then $0_V^\dagger : V \to 0$ is the unique morphism from V to 0. Hence 0 is a zero object where $0_{VW} = 0_W^\dagger \circ 0_V : V \to W$ is the unique morphism that factors through 0 . The subscripts may be dropped, context permitting.

Definition 1. *A* biproduct dagger category *is a dagger category $\mathcal{C}$ equipped with an initial object* 0 *and binary coproducts such that the canonical injections* $q_1 : V \to V \oplus W$ *and* $q_2 : W \to V \oplus W$ *satisfy*

$$q_i^\dagger \circ q_i = 1, q_j^\dagger \circ q_i = 0 \textit{ for } i, j = 1, 2,\ i \neq j\,. \tag{1}$$

Note that $V \oplus 0 \simeq V$. Indeed, $q_1 : V \to V \oplus 0$ and $q_1^\dagger : V \oplus 0 \to V$ are inverse of each other, because $(q_1 \circ q_1^\dagger) \circ q_1 = q_1 = 1_{V \oplus 0} \circ q_1$ and $(q_1 \circ q_1^\dagger) \circ q_2 = 0 = 1_{V \oplus 0} \circ q_2$. Therefore $q_1 \circ q_1^\dagger = 1_{V \oplus 0}$.

Given $g_j : U \to V_j$, denote $\langle g_1, g_2 \rangle : U \to V_1 \oplus V_2$ the unique morphism satisfying

$$q_j^\dagger \circ \langle g_1, g_2 \rangle = g_j \text{ for } j = 1, 2\,.$$

Similarly, for $h_i : W_i \to E$ denote $[h_1, h_2] : W_1 \oplus W_2 \to E$ the morphism determined by

$$[h_1, h_2] \circ q_i = h_i \text{ for } i = 1, 2\,.$$

Finally, for $f_i : V_i \to W_i$, denote $f_1 \oplus f_2 : V_1 \oplus V_2 \to W_1 \oplus W_2$ the unique morphism such that

$$q_i^\dagger \circ (f_1 \oplus f_2) \circ q_i = f_i \text{ and } q_i^\dagger \circ (f_1 \oplus f_2) \circ q_j = 0_{V_j W_i}, \text{ for } i, j = 1, 2, i \neq j\,.$$

We have for any $g : U' \to U$ and $h : E \to E'$

$$\begin{aligned} \langle g_1, g_2 \rangle \circ g &= \langle g_1 \circ g, g_2 \circ g \rangle, \\ h \circ [h_1, h_2] &= [h \circ h_1, h \circ h_2] \\ (f_1 \oplus f_2) \circ \langle g_1, g_2 \rangle &= \langle f_1 \circ g_1, f_2 \circ g_2 \rangle \\ [h_1, h_2] \circ (f_1 \oplus f_2) &= [h_1 \circ f_1, h_2 \circ f_2] \end{aligned}$$

Any morphism $f : V_1 \oplus V_2 \to W_1 \oplus W_2$ is uniquely determined by the four morphisms $q_i^\dagger \circ f \circ q_j$, for $i, j = 1, 2$. These four morphisms may be displayed in the form of a matrix

$$M_f = \begin{pmatrix} q_1^\dagger \circ f \circ q_1 & q_1^\dagger \circ f \circ q_2 \\ q_2^\dagger \circ f \circ q_1 & q_2^\dagger \circ f \circ q_2 \end{pmatrix}\,.$$

Proposition 1. *The following equalities hold in a biproduct dagger category*

$$\begin{aligned} 0_{VW}^\dagger &= 0_{WV} \\ \langle f_1, f_2 \rangle^\dagger &= [f_1^\dagger, f_2^\dagger] \\ (f_1 \oplus f_2)^\dagger &= f_1^\dagger \oplus f_2^\dagger\,. \end{aligned} \tag{2}$$

Any biproduct category $\mathcal{C}$ is enriched over abelian monoids, i.e. the binary operation defined on each hom-set $\mathcal{C}(V, W)$ by

$$f_1 + f_2 = [1_W, 1_W] \circ (f_1 \oplus f_2) \circ \langle 1_V, 1_V \rangle,\ \textit{for } f_1, f_2 : V \to W$$

is associative and commutative with unit 0_{VW}.

Moreover, addition is bilinear

$$h \circ (f_1 + f_2) \circ g = h \circ f_1 \circ g + h \circ f_2 \circ g, \ \textit{for } g : V' \to V, h : W \to W'$$

and

$$\begin{aligned} (f_1 + f_2)^\dagger &= f_1^\dagger + f_2^\dagger \\ q_1 \circ q_1^\dagger + q_2 \circ q_2^\dagger &= 1_{V \oplus W} \,. \end{aligned} \tag{3}$$

It follows that any biproduct category has a *matrix calculus*, i.e. the following equalities hold

$$M_{f+g} = M_f + M_g \text{ and } M_{g \circ f} = M_g M_f \,, \tag{4}$$

where the right-hand matrices are defined the usual way, entry by entry. For example, if f_{kj} are the entries of M_f and g_{ik} those of M_g the entry h_{ij} of $M_g M_f$ satisfies $h_i = g_{i1} \circ f_{1j} + g_{i2} \circ f_{2j}$.

Define the n-ary biproduct with canonical injections $q_{i\,n} : V_i \to V_1 \oplus \ldots \oplus V_n$ by induction thus

$$\begin{aligned} & V_1 \oplus \ldots \oplus V_0 := 0 \qquad V_1 \oplus \ldots \oplus V_1 := V_1 \\ & V_1 \oplus \ldots \oplus V_n := (V_1 \oplus \ldots \oplus V_{n-1}) \oplus V_n \text{ for } n \geq 2 \,, \end{aligned}$$

$$\begin{aligned} & q_{0\,0} = 1_0 \quad q_{1\,1} = 1_{V_1} \qquad q_{i\,2} = q_i : V_i \to V_1 \oplus V_2 \text{ for } i = 1, 2 \\ & q_{i\,n} = q_1 \circ q_{i\,(n-1)}, \text{ for } i = 1, \ldots, n-1 \text{ and } q_{n\,n} = q_2, \\ & \text{where } q_1 : (V_1 \oplus \ldots \oplus V_{n-1}) \to (V_1 \oplus \ldots \oplus V_{n-1}) \oplus V_n, \\ & \qquad q_2 : V_n \to (V_1 \oplus \ldots \oplus V_{n-1}) \oplus V_n \,. \end{aligned}$$

If context permits, write q_i instead of $q_{i\,n}$.

In the case where $V_i = V$ for all $i = 1, \ldots, n$, write $n \cdot V = V_1 \oplus \ldots \oplus V_n$[1]. Adopt a similar convention for $n \cdot f$, where $f : V \to W$.

Equalities (1) - (3) generalise to n-ary biproducts. Together, they constitute the generalised *Dagger Biproduct Calculus*. For example, the generalised version of (1) is

$$q_i^\dagger \circ q_i = 1_{V_i}, \ q_i^\dagger \circ q_j = 0_{V_j V_i}, \text{ for } i, j = 1, \ldots, n, i \neq j \,.$$

Any morphism $f : V_1 \oplus \ldots \oplus V_m \to W_1 \oplus \ldots \oplus W_n$ is completely determined by the nm morphisms $q_i^\dagger \circ f \circ q_j$, for $j = 1, \ldots, m$, $i = 1, \ldots, n$. Hence, $f = g$ if and only if the following *Matrix Equalities* hold

$$q_i^\dagger \circ g \circ q_j = q_i^\dagger \circ f \circ q_j, \text{ for } j = 1, \ldots, m, \ i = 1, \ldots, n \,. \tag{5}$$

The nm morphisms $q_i^\dagger \circ f \circ q_j$ may be displayed in the form of an nm-matrix

$$M_f = \begin{pmatrix} q_1^\dagger \circ f \circ q_1 & \ldots & q_1^\dagger \circ f \circ q_m \\ \\ q_n^\dagger \circ f \circ q_1 & \ldots & q_n^\dagger \circ f \circ q_m \end{pmatrix} .$$

The equalities (2) - (4) then generalise to arbitrary biproducts.

[1] The notation V^n is reserved for the n-ary tensor product introduced in Section 3.3.

The most interesting case is when $V_j = W_i = I$ for all indices j, i, where I is any object non-isomorphic to 0. Then the entries of the matrix are endomorphisms of I, i.e. $q_i^\dagger \circ f \circ q_j : I \to I$. If every object is isomorphic to a finite coproduct of some distinguished object, say I, then the entries of a matrix refer to the coproducts $V = m \cdot I$ and $W = n \cdot I$.

2.2 Examples of biproduct dagger categories

The categories $\mathcal{H}_I$ of linear maps and finite dimensional Hilbert spaces over the fields $I = \mathbb{R}$ or $I = \mathbb{C}$ are biproduct dagger categories. Every object is a finite biproduct of I, identified with a one-dimensional space. The adjoint of $\alpha \in I$ is the conjugate of α. If $I = \mathbb{R}$ then $\alpha^\dagger = \alpha$. The matrix of the adjoint $f^\dagger$ of a linear map f in $\mathcal{H}_I$ is the transpose of the conjugate matrix of f.

The category $\mathit{2}\mathcal{SF}$ of two-sorted functions

Two-sorted first order logic has two sorts of variables, one for elements x, and one for sets X. Besides an equality symbol for each sort, there is a binary symbol $\in$ requiring elements on the left and sets on the right, $x \in X$. There are two sorts of quantifiers, $\forall_x$, $\forall_X$ etc. Functional symbols accept both sorts as arguments.

Models interpret every function symbol by a *two-sorted function* $f : A \to B$ satisfying

$$\begin{aligned} f(\{x\}) &= f(x) \text{ for } x \in A \\ f(\emptyset) &= \emptyset \\ f(X \cup Y) &= f(X) \cup f(Y) \text{ for } X, Y \subseteq A\,. \end{aligned}$$

The category $\mathit{2}\mathcal{SF}$ of two-sorted functions as morphisms and finite sets as objects is a biproduct dagger category. Any two-sorted function is determined by its values on elements, because all sets are finite. The adjoint $f^\dagger : B \to A$ of $f : A \to B$ is given by

$$f^\dagger(b) = \{a \in A \,:\, f(a) = b \text{ or } b \in f(a)\}\ .$$

The biproduct is the disjoint union of sets, with $\emptyset$ as the zero object. With these definitions, any object of $\mathit{2}\mathcal{SF}$ is isomorphic to a finite biproduct of any arbitrarily fixed singleton set I. Note that it has exactly two endomorphisms, namely the identity map and the zero map, which sends the unique element of I to the empty set. Therefore, $\alpha^\dagger = \alpha$ for all endomorphisms $\alpha : I \to I$.

The sum of $f, g : A \to B$ is the set-theoretical union $(f+g)(x) = f(x) \cup g(x)$. The right-hand side of the last equality involves an abuse of notation: if $f(x)$ or $g(x)$ is an element, we should have used the corresponding singleton set. English makes the same abuse. Compare '*apples and pears*' with '*the teacher and the student*'.

The category $\mathcal{RI}$ of semimodules over $[0, 1]$

Recall that the linear order on the real numbers in $[0, 1]$ induces a distributive and implication-complemented lattice structure on $[0,1]$, namely

$$\alpha \vee \beta = \max\{\alpha, \beta\} \qquad \alpha \wedge \beta = \min\{\alpha, \beta\}$$
$$\alpha \to \beta = \max\{\gamma \in I : \ \alpha \wedge \gamma \leq \beta\}$$
$$\neg\alpha = \alpha \to 0 \,.$$

This lattice is not Boolean, because $\neg\neg\alpha = 1 \neq \alpha$ for $0 < \alpha < 1$.

The lattice operations define a semiring structure on $I = [0,1]$ with neutral element 0 and unit 1 by

$$\alpha + \beta = \alpha \vee \beta \qquad \alpha \cdot \beta = \alpha \wedge \beta \,.$$

The category $\mathcal{RI}$ of free semi-modules over the real interval $I = [0,1]$, generated by a finite set is a biproduct dagger category. The zero-object is the semi-module reduced to a single element, $\{0\}$. The interval $I = [0,1]$ is a semi-module, generated by the singleton set $\{1\}$. The endomorphisms of I identify with the elements of I. The biproduct of two semi-modules is generated by the disjoint union of the two semi-modules. Hence every object of $\mathcal{RI}$ is a finite biproduct of I. Every endomorphism/element of I is by definition, its own adjoint

$$\alpha = \alpha^{\dagger} \,.$$

The adjoint $f^{\dagger} : n \cdot I \to m \cdot I$ of a linear map $f : m \cdot I \to n \cdot I$ is the linear map defined by the transpose of the matrix of f.

2.3 Generalising geometrical notions

Several geometrical notions familiar from real or complex vector spaces can be generalised to arbitrary dagger biproduct categories. The letter $\mathcal{C}$ denotes an arbitrary biproduct dagger category in the following.

Definition 2. *Morphisms $f : U \to W$ and $g : V \to W$ are said to be* orthogonal *in W if $f^{\dagger} \circ g = 0$. A* projector *is an idempotent and self-adjoint morphism $p : V \to V$, i.e. $p \circ p = p$ and $p^{\dagger} = p$.*

Orthogonality is a symmetric relation. Every morphism is orthogonal to 0. In general, a morphism can have several distinct orthogonal morphisms. Projectors, as we shall see later, determine a maximal morphism that is orthogonal to them. This maximal orthogonal morphism is itself a projector, the *negation* of the original projector.

The rest of this section is an argument that iterated biproducts of any object $V \not\simeq 0$ internalise propositions and finite subsets. Projectors will play the role of propositions, the canonical injections $q_i : V \to n{\cdot}V$ the role of individuals. Note that the canonical injections q_i and q_j are distinct for $i \neq j$, because $1_V \neq 0_{VV}$. Subsets of individuals are internalised as sums of distinct canonical injections.

A morphism $f : V \to W$ is *unitary* if $f^{\dagger} \circ f = 1_V$. A unitary f is necessarily monic and its adjoint is epic. In the case where f is an isomorphism, f is unitary if and only if $f \circ f^{\dagger} = 1_W$ if and only if $f^{\dagger} = f^{-1}$. For example, a morphism of $\mathit{2}\mathcal{SF}$ is unitary if and only if it maps different elements to nonempty disjoint subsets.

Proposition 2. *Let V be any object of $\mathcal{C}$. Assume $K = \{i_1, \ldots, i_k\}$ and $M = \{l_1, \ldots, l_m\}$ are disjoint subsets of $\{1, \ldots, n\}$.*

Then $q_K = [q_{i_1}, \ldots q_{i_k}] : k \cdot V \to n \cdot V$ is unitary and orthogonal to $q_M = [q_{l_1}, \ldots q_{l_m}] : m \cdot V \to n \cdot V$.

The endomorphism $p_K = q_K \circ q_K^\dagger : n \cdot V \to n \cdot V$ is a projector and

$$p_K + p_M = p_{K \cup M} \,. \tag{6}$$

Proof. First, recall that $[q_{i_1}, \ldots q_{i_k}]^\dagger = \langle q_{i_1}^\dagger, \ldots q_{i_k}^\dagger \rangle$. Hence, the Matrix Equalities (5) characterise p_K as the unique morphism satisfying

$$q_i^\dagger \circ p_K \circ q_j = \begin{cases} 1_V & \text{if } i = j \text{ and } j \in K \\ 0_{VV} & \text{else} \end{cases}, \text{ for } i, j = 1, \ldots, n \,. \tag{7}$$

Next, use the Dagger Biproduct Calculus and the Matrix Equalities to show that

$$\begin{aligned} \langle q_{i_1}^\dagger, \ldots q_{i_k}^\dagger \rangle \circ [q_{i_1}, \ldots q_{i_k}] &= 1 \\ \langle q_{i_1}^\dagger, \ldots q_{i_k}^\dagger \rangle \circ [q_{l_1}, \ldots q_{l_m}] &= 0 \,. \end{aligned} \tag{8}$$

This proves that q_K is unitary and orthogonal to q_M.

Finally, check that p_K is self-adjoint, via the equality recalled initially, and that it is idempotent, via the first equality of (8). Equality (6) follows from the Matrix Equalities and bilinearity of addition. □

Corollary 1. *If $V \not\simeq 0$, the map $K \mapsto p_K$ is a one-to-one correspondence between subsets $K \subseteq \{1, \ldots, n\}$ and the projectors p_K.*

Proof. Use the characterising equalities (7) and the fact that $1_V \neq 0_{vv}$. □

Corollary 2. *If $K \cap M = \emptyset$ and $K \cup M = \{1 \ldots, n\}$ then*

$$p_K + p_M = 1_{n \cdot V} = q_1 \circ q_1^\dagger + \cdots + q_n \circ q_n^\dagger \,.$$

Proof. The equality $p_{K \cup M} = 1_{n \cdot V}$ is a special case of (8). Hence, $p_K + p_M = 1_{n \cdot V}$ follows by (6). □

Make $V = I$ and think of the canonical injections $q_i : I \to n \cdot I = N$ as 'individuals'. Then every p_K assimilates to a 'property of individuals', namely the property that is true for q_i if $p_K \circ q_i = q_i$ and false for q_i if $p_K \circ q_i = 0$.

Recall that a morphism $g : U \to V$ is a *kernel* of $f : V \to W$ if it satisfies $f \circ g = 0$ and is universal for this property. Universality means that for any $h : X \to V$ with $f \circ h = 0$ there is a unique $h' : X \to U$ with $h = g \circ h'$.

Proposition 3. *Let $K = \{i_1, \ldots, i_k\}$ be a subset of $\{1, \ldots, n\} = N$. Then $q_{N \setminus K}$ is a unitary kernel of p_K and $q_K^\dagger$. Moreover, q_K is the image of p_K.*

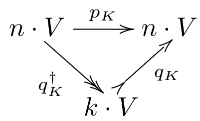
Proof. The equality $p_K \circ q_{N\setminus K} = 0$ is a particular case of (8). To prove the universality of $q_{N\setminus K}$, assume that $g : U \to n \cdot V$ satisfies $p_K \circ g = 0$. Let $h := q^\dagger_{N\setminus K} \circ g : U \to (n-k) \cdot V$. Then

$$g = (p_K + p_{N\setminus K}) \circ g = p_K \circ g + q_{N\setminus K} \circ q^\dagger_{N\setminus K} \circ g = q_{N\setminus K} \circ q^\dagger_{N\setminus K} \circ g = q_{N\setminus K} \circ h\,.$$

This proves that $ker(p_K) = q_{N\setminus K}$. The equality $ker(q^\dagger_K) = q_{N\setminus K}$ follows, because $q^\dagger_K \circ g = 0$ implies $p_K \circ g = 0$.

Finally, using the definition $im(p_K) := ker((ker(p^\dagger_K))^\dagger)$ of [Heunen and Jacobs, 2010], compute

$$im(p_K) = ker((ker(p_K))^\dagger) = ker(q^\dagger_{N\setminus K}) = q_K\,.$$



□

Note that $v : W \to n \cdot V$ is invariant under composition with p_K exactly when v factorises through q_K. Indeed, $v = p_K \circ v$ implies $v = q_K \circ (q^\dagger_K \circ v)$. Conversely, $v = q_K \circ g$ implies $v = q_K \circ (q^\dagger_K \circ q_K) \circ g = p_K \circ v$.

Proposition 3 also implies that the *orthogonal complement* $p_K{}^\perp := ker(p_K) \circ (ker(p_K))^\dagger$ of the projector p_K satisfies

$$p_K{}^\perp = p_{N\setminus K}\,.$$

Proposition 4. *The projectors p_K, p_L satisfy for any $K, L \subseteq \{1, \dots, n\}$*

$$p_K \circ p_L = p_L \circ p_K = p_{K\cap L}\,.$$

The relation given by

$$p_K \leq p_L \Leftrightarrow p_K \circ p_L = p_K \tag{9}$$

is a partial order on the projectors p_K inducing a Boolean Algebra structure such that

$$p_K \wedge p_L = p_K \circ p_L \qquad p_K \vee p_L = p_{K\cup L} \qquad \neg p_K = p_{N\setminus K} = p_K{}^\perp$$

Under the assumption that $V \not\cong 0$, the map $K \mapsto p_K$ is an isomorphism from the Boolean algebra of subsets of N into the projectors of $n \cdot V$. In particular, $\neg p_K$ is the largest projector orthogonal to p_K, i.e. $p_L \leq \neg p_K$ if and only if $p_K \circ p_L = 0$.

Proof. Partition $K \cup L$ into the three disjoint subsets $M = K \cap L$, $M' = K \setminus (K \cap L)$, and $M'' = L \setminus (K \cap L)$. By Proposition (2), $p_K = p_M + p_{M'}$ and

$p_L = p_M + p_{M''}$ and the mixed terms $p_M \circ p_{M''}$, $p_{M'} \circ p_M$, $p_{M'} \circ p_{M''}$ are equal to 0. Therefore

$$p_K \circ p_L = (p_M + p_{M'}) \circ (p_M + p_{M''}) = p_M \circ p_M = p_M \,.$$

Similarly, $p_L \circ p_K = p_M$. This proves the first assertion. The rest is now straight forward. □

Make $V = I$ and think of the canonical injections $q_i : I \to n \cdot I = N$ as individuals. Recall that p_K is true for the individual q_i if $p_K \circ q_i = q_i$ and false for q_i if $p_K \circ q_i = 0$. Then $p_K \circ p_L$ is true for q_i if and only if both p_K and p_L are true for q_i.

Proposition 5. *The partial order of the projectors p_K is isomorphic to the partial order of their canonical images $im(p_K) = q_K$.*

More precisely, for arbitrary subsets $K = \{i_1, \ldots, i_k\}$ and $M = \{j_1, \ldots, j_m\}$ of $\{1, \ldots, n\}$ the following equivalence holds

$$p_K \leq p_M \text{ if and only if } q_K \leq q_M \text{ as subobjects.}$$

Proof. Recall that $p_K \leq p_M$ is equivalent to $K \subseteq M$ by (9). Assume that $q_K \leq q_M$ as subobjects and let $g : k \cdot V \to m \cdot M$ be the morphism such that $q_K = q_M \circ g$. Then $q_K = q_M \circ g$ implies $q_{i_l} = q_K \circ q_l = q_M \circ g \circ q_l$ and therefore $i_l \in M$, by Proposition 3, and this for $l = 1, \ldots, k$. Hence, $K \subseteq M$. Conversely, the inclusion $K \subseteq M$ provides an obvious factorisation $q_K = q_M \circ g$. □

3 Semantic categories

This section mentions well known properties of biproduct dagger categories with a chosen generating object. The most important consequence for natural language semantics is the Property of Explicit Definitions, Proposition 8.

Definition 3. *Let $\mathcal{C}$ be a biproduct dagger category. An object I of $\mathcal{C}$ is a* generating object *if the following holds*

- $\alpha \circ \beta = \beta \circ \alpha$ *for all* $\alpha, \beta : I \to I$
- *for every object V there is an integer $n \geq 0$ and a unitary isomorphism $b_V : n \cdot I \to V$.*

A semantic category *is a biproduct dagger category that has a distinguished generating object $I \not\simeq 0$.*

In the category $\mathcal{2SF}$ of two-sorted functions, I is a distinguished singleton set. In the category $\mathcal{RI}$ of semi-modules over the real interval $[0, 1]$, the distinguished object is this interval, $I = [0, 1]$. For real Hilbert spaces, $I = \mathbb{R}$, for complex Hilbert spaces, $I = \mathbb{C}$.

3.1 Spaces, vectors, scalars

This subsections transfers the terminology familiar from Hilbert spaces to an arbitrary semantic category. Hopefully, this will not confuse the reader.

A *space* is an object V of $\mathcal{C}$ together with a unitary isomorphism $b_V : n \cdot I \to V$, called the *base* of the space. The integer n is the *dimension* of the space. A *vector* of V is a morphism from I to V. A *scalar* is an endomorphism of I. The scalars form a commutative semiring where multiplication is composition and addition is defined by Proposition 1. A scalar β is *positive*, if it has the form $\beta = \alpha^\dagger \circ \alpha$. For example, 1_I and 0_I are positive. The positive scalars are closed under multiplication and addition, e.g.[Selinger, 2007].

These notions have the usual meaning in Hilbert spaces. A space of $\mathit{2}\mathcal{SF}$ is a set B and an enumeration of its elements $B = \{b_1, \ldots, b_n\}$, where $b_i = b_V(q_i)$, for $i = 1, \ldots, n$. The dimension is the cardinal of the set B. The vectors of B are the subsets of B. There are two scalars in $\mathit{2}\mathcal{SF}$, namely the map 0_I that maps the unique element of I to the empty set and the map 1_I that maps the unique element to itself. Both are positive.

Scalar multiplication is defined for any scalar $\alpha : I \to I$ and vector $v : I \to V$ by

$$\alpha v = v \circ \alpha\,.$$

Scalar multiplication is associative and commutes with addition

$$(\alpha\beta)v = \alpha(\beta v) \text{ and } \alpha(v + w) = \alpha v + \alpha w\,.$$

For any morphism $f : V \to W$ and vector $v : I \to V$, the *value* $f(v)$ of f at v is

$$f(v) = f \circ v\,.$$

All morphisms of $\mathcal{C}$ are *linear*, that is to say for $f : V \to W$, $v, w : I \to V$ and $\alpha, \beta : I \to I$

$$f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)\,.$$

Assume that $b_V : m \cdot I \to V$ is the base of V. The vectors $a_j = b_V \circ q_j : I \to V$, $j = 1, \ldots, m$ are *basis vectors* of V and $A = \{a_1, \ldots, a_m\}$ is *the chosen basis* of V. The basis vectors satisfy

$$a_i^\dagger \circ a_j = \delta_{ij}, \text{ for } i, j = 1, \ldots, m, \tag{10}$$

where $\delta_{ii} = 1_I$ and $\delta_{ij} = 0_{II}$ for $i \neq j$.

There are exactly m distinct basis vectors, because otherwise we would have $1_I = 0_{II}$, which contradicts $I \not\simeq 0$. The equalities (10) mean that the basis vectors are unitary and pairwise orthogonal.

Proposition 6. *Every vector of V can be written uniquely as a linear combination of the chosen basis vectors.*

Proof. Let $\{a_1, \ldots, a_m\}$ be the basis of V and $v : I \to V$ and

$$\alpha_i = q_i^\dagger \circ b_V^\dagger \circ v, \text{ for } i = 1, \ldots, m\,.$$

Recall that $q_1 \circ q_1^\dagger + \cdots + q_m \circ q_m^\dagger = 1_{m\cdot I}$, by (6). Hence

$$\begin{aligned} v &= b_V \circ (q_1 \circ q_1^\dagger + \cdots + q_m \circ q_m^\dagger) \circ b_V^\dagger \circ v \\ &= b_V \circ q_1 \circ q_1^\dagger \circ b_V^\dagger \circ v + \cdots + b_V \circ q_m \circ q_m^\dagger \circ b_V^\dagger \circ v \\ &= a_1 \circ \alpha_1 + \cdots + a_m \circ \alpha_m = \alpha_1 a_1 + \cdots + \alpha_m a_m \,. \end{aligned}$$

This proves the existence.

To see the unicity, assume $v = a_1 \circ \beta_1 + \cdots + a_m \circ \beta_m$. Multiplying both sides of the equality on the left by $q_i^\dagger \circ b_V$, we get $q_i^\dagger \circ b_V \circ v = \beta_i$, for $i = 1, \ldots, m$. □

Corollary 3. *Let $K = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, $v : I \to k \cdot I$ and $j \in \{1, \ldots, n\}$. Then $q_K \circ v = q_j$ implies $j \in K$.*

Proof. Recall that $q_K = [q_{i_1}, \ldots q_{i_k}] : k \cdot I \to n \cdot I$ and therefore $q_K \circ q_l = q_{i_l}$ for $l = 1, \ldots, k$. Assume $v : I \to k \cdot I$ and $q_K \circ v = q_j$. Write $v = \sum_{l=1}^k \alpha_l q_l$, where $\alpha_l : I \to I$. Then $q_j = q_K \circ (\sum_{l=1}^k \alpha_l q_l) = \sum_{l=1}^k \alpha_l (q_K \circ q_l) = \sum_{l=1}^k \alpha_l q_{i_l}$. Coordinates are unique, thus $j = i_l$ and $\alpha_l = 1$ for some $l \leq k$ and $\alpha_{l'} = 0$ for $l' \neq l$. Finally, $q_j = q_{i_l}$ implies $j = i_l$, which terminates the proof. □

Refer to the (unique) scalars α_i, $i = 1, \ldots, m$, such that $v = \alpha_1 a_1 + \cdots + \alpha_m a_m$ as the *components* of v . The notation $V = V_A$ expresses that A is the basis of the space V . If the last element in a composition is a vector we may switch to set-theoretical notation to highlight the analogy between categorical and set-theoretical properties, e.g. $p(v) = w$ instead of $p \circ v = w$ etc.

3.2 Matrix calculus

The matrix calculus familiar from Hilbert spaces generalises to arbitrary biproduct dagger categories with a generating object.

Proposition 7. *Every morphism is uniquely determined by its values on the basis vectors.*

Proof. Let $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$ and suppose that $f, g : V_A \to W_B$ coincide on the basis vectors $a_j = b_V \circ q_j$ for $j = 1, \ldots, m$. Then

$$q_i^\dagger \circ b_W^\dagger \circ f \circ b_V \circ q_j = q_i^\dagger \circ b_W^\dagger \circ g \circ b_V \circ q_j \text{ for } i = 1, \ldots, n, j = 1, \ldots, m \,.$$

Hence, $b_W^\dagger \circ f \circ b_V = b_W^\dagger \circ g \circ b_V$, which implies $f = g$. □

Proposition 7 has a converse

Proposition 8 (Explicit Definitions). *Given vectors $w_1, \ldots, w_m$ in W_B, there is a unique morphism $f : V_A \to W_B$ satisfying*

$$f \circ a_j = w_j, \text{ for } j = 1, \ldots, m \,. \tag{11}$$

Proof. The components of $w_j = \phi_{1j} b_1 + \cdots + \phi_{nj} b_n$, for $j = 1, \ldots, m$, define a unique morphism $g : m \cdot I \to n \cdot I$ such that $q_i^\dagger \circ g \circ q_j = \phi_{ij}$, for $i = 1, \ldots, n$, $j = 1, \ldots, m$. Then $f = b_W \circ g \circ b_V^\dagger$ satisfies (11). □

Proposition 8 can be rephrased by saying that semantic categories admit *Explicit Definitions*. The morphism f is explicitly defined by the values $w_j \in W_B$ for the basis vectors $a_j \in V_A$ in (11).

By Proposition 8, every morphism $f : V_A \to W_B$ determines and is determined by the scalars

$$\phi_{ij} = q_i^\dagger \circ b_W^\dagger \circ f \circ b_V \circ q_j, \text{ for } j = 1, \dots, m, \ i = 1, \dots, n .$$

The scalars $\phi_{ij}^\dagger$ then determine $f^\dagger$ with respect to A and B. The corresponding matrices are

$$M_f = \begin{pmatrix} \phi_{11} & \dots & \phi_{1m} \\ \vdots & & \vdots \\ \phi_{n1} & \dots & \phi_{nm} \end{pmatrix} \qquad M_{f^\dagger} = \begin{pmatrix} \phi_{11}^\dagger & \dots & \phi_{n1}^\dagger \\ \vdots & & \vdots \\ \phi_{1m}^\dagger & \dots & \phi_{nm}^\dagger \end{pmatrix} .$$

The Dirac notation can be introduced with its usual properties: Assign to any vector $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$ of $V = V_B$ a row matrix and a column matrix

$$\langle v| = M_{v^\dagger} = \begin{pmatrix} \alpha_1^\dagger & \dots & \alpha_n^\dagger \end{pmatrix}, \quad |v\rangle = M_v = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} .$$

The *inner product* of v and $w = \beta_1 b_1 + \cdots + \beta_n b_n : I \to V$ is

$$\langle v|w\rangle := M_{v^\dagger} M_w = \alpha_1^\dagger \beta_1 + \cdots + \alpha_n^\dagger \beta_n ,$$

and the *outer product* of any vector $u = \gamma_1 a_1 + \cdots + \gamma_m a_m$ of $U = U_A$ and w

$$|w\rangle\langle u| := M_w M_{u^\dagger} = \begin{pmatrix} \beta_1\gamma_1^\dagger & \dots & \beta_1\gamma_m^\dagger \\ \vdots & & \vdots \\ \beta_n\gamma_1^\dagger & \dots & \beta_n\gamma_m^\dagger \end{pmatrix} .$$

Otherwise said, $\langle v|w\rangle$ is the matrix of $v^\dagger \circ w$ and $|w\rangle\langle u|$ is the matrix of $b_V \circ w \circ u^\dagger \circ b_U^\dagger : U \to V$.

The outer product of a basis vector $b_i = \sum_{k=1}^n \delta_{ki} b_k$ of V_B and a basis vector $a_j = \sum_{l=1}^m \delta_{jl} a_l$ of U_A is

$$|b_i\rangle\langle a_j| = (\delta_{kl}^{ij}) ,$$

where $\delta_{ij}^{ij} = 1$ and $\delta_{kl}^{ij} = 0$ for $k \neq i$ or $l \neq j$, $k = 1, \dots, n$, $l = 1, \dots, m$. Indeed, $\delta_{kl}^{ij} = \delta_{ki}\delta_{jl}^\dagger = \delta_{ki}\delta_{jl}$. In particular, the outer product $|b_i\rangle\langle b_i|$ is the matrix of the projector $p_{\{i\}}$, for $i = 1, \dots, n$.

Definition 3 can now be reformulated for vectors in terms of the inner product. Vectors are orthogonal if and only if their inner product equals 0. A vector is unitary if the inner product of the vector with itself equals 1.

Examples of explicitly defined morphisms are the *diagonal* $d_V : V \to V \otimes V$ and the *symmetry* $\sigma_{VW} : V \otimes W \to W \otimes V$ defined for the basis vectors $\{a_1, \ldots, a_m\}$ of V and $\{b_1, \ldots, b_n\}$ of W thus

$$\begin{array}{rl} d_V(a_j) & = a_j \otimes a_j \\ \sigma_{VW}(a_j \otimes b_i) & = b_i \otimes a_j \end{array} \text{, for } j = 1, \ldots, m,\ i = 1, \ldots, n\,.$$

Note that the morphism $d_V^\dagger : V \otimes V \to V$ satisfies

$$d_V^\dagger(a_j \otimes a_i) = \begin{cases} a_i & \text{if } i = j \\ 0 & \text{if } i \neq j\,. \end{cases}$$

Let K, L be subsets of the basis A of V, v_K be the sum of basis vectors in K and similarly for v_L. Then

$$d_V^\dagger \circ (v_K \otimes v_L) = v_{K \cap L} = v_k \wedge v_L\,. \tag{12}$$

The morphisms $d_V^\dagger$ and d_V play the role of multiplication and comultiplication of the Frobenius algebra over the vector space V of $\mathbf{FdVect}_{\mathbb{R}}$. This means that the compositional semantics of [Kartsaklis et al., 2013] can be generalised to an arbitrary semantic category.

3.3 Compact closed categories

Recall that a *monoidal category* consists of a category $\mathcal{C}$, a bifunctor $\otimes$, a distinguished object I and natural isomorphisms $\alpha_{VWU} : (V \otimes W) \otimes U \to V \otimes (W \otimes U)$, $\lambda_V : V \to I \otimes V$ and $\rho_V : V \to V \otimes I$ subject to the coherence conditions of [Mac Lane, 1971]. It is a *compact closed category* if every object has a right and a left adjoint. A compact closed category is *symmetric* if there is a natural isomorphism $\sigma_{VW} : V \otimes W \to W \otimes V$ such that $\sigma_{VW}^{-1} = \sigma_{WV}$ subject to the coherence conditions of [Mac Lane, 1971].

For notational convenience, the associativity isomorphisms α_{VWU} and the unit isomorphisms λ_V and ρ_V are replaced by identities, e.g. $(V \otimes W) \otimes V = V \otimes (W \otimes U)$, $V = I \otimes V$ and $V \otimes I$.

The tensor product is definable in semantic categories. It plays the role of a bookkeeping device and 'internalises' matrices as vectors of a tensor product space.

Let $b_V : m \cdot I \to V$ and $b_W : n \cdot I \to W$ be spaces with chosen basis vectors $a_j = b_V \circ q_j$, $b_i = b_W \circ q_i$, where $q_j : I \to m \cdot I$, $q_i : I \to n \cdot I$, $j = 1, \ldots, m$, $i = 1, \ldots, n$, are the canonical injections.

The *tensor product of* V and W and the dagger isomorphism $b_{V \otimes W} : n \cdot (m \cdot I) \to V \otimes W$ are

$$V \otimes W := n \cdot V, \quad b_{V \otimes W} := n \cdot b_V\,.$$

Let $q_i' : m \cdot I \to n \cdot (m \cdot I)$, $i = 1, \ldots, n$, be the canonical injections. The *tensor product of* a_j *and* b_i is the vector

$$a_j \otimes b_i := b_{V \otimes W} \circ q_i' \circ q_j : I \to V \otimes W\,.$$

Under these definitions, the tensor product distributes over the dagger and the biproduct

$$\begin{array}{ll} (f \otimes g)^\dagger & = f^\dagger \otimes g^\dagger : W \otimes D \to V \otimes U \\ V \otimes (W \oplus U) & \simeq (V \otimes W) \oplus (V \otimes U)\,. \end{array}$$

A *compact closed category* is a symmetric monoidal category $\mathcal{C}$ together with a contra-variant functor * and maps $\eta_V : I \to V^* \otimes V$ and $\epsilon_V : V \otimes V^* \to I$, called *unit* and *counit* respectively, such that

$$\begin{array}{ll} (\epsilon_V \otimes 1_V) \circ (1_V \otimes \eta_V) & = 1_V \\ (\epsilon_{V^*} \otimes 1_{V^*}) \circ (1_{V^*} \otimes \eta_{V^*}) & = 1_{V^*}\,. \end{array}$$

Proposition 9. *Semantic categories are compact closed.*

Proof. (Outline) Follow the construction of the dual space in [Abramsky and Coecke, 2004] for the category of complex Hilbert spaces. First, introduce the *dual scalar multiplication* for $v : I \to V_A$ and $\alpha : I \to I$

$$\alpha * v := v \circ \alpha^\dagger = \alpha^\dagger v\,.$$

This definition creates a dual version of Proposition 6: Every vector can be written uniquely as the sum of dual scalar multiples of basis vectors. Indeed, let $\beta_i = {\alpha_i}^\dagger$ for $j = 1, \ldots, m$. Then

$$\sum_{i=1}^{m} \alpha_i a_i = \sum_{i=1}^{m} {\alpha_i}^{\dagger\dagger} a_i = \sum_{i=1}^{m} \beta_i * a_i\,.$$

The *dual space* ${V_A}^*$ is the space V_A where vectors are given as sums of dual scalar multiples of basis vectors. In the case of $\mathit{2}\mathcal{SF}$, $\mathcal{RI}$ and real Hilbert spaces, we have ${V_A}^* = V_A$, because $\alpha^\dagger = \alpha$ for all $\alpha : I \to I$.

Given $f : V_A \to W_B$, use the principle of Explicit Definitions to introduce the morphisms $f_* : {V_A}^* \to {W_B}^*$ and the *dual* $f^* : {W_B}^* \to {V_A}^*$ such that

$$\begin{array}{lll} f_*(a_j) & = \sum_{i=1}^n \phi_{ij} * b_i & = \sum_{i=1}^n \phi_{ij}^\dagger b_i, \text{ for } j = 1, \ldots, m \\ f^*(b_i) & = \sum_{j=1}^m {\phi_{ij}}^\dagger * a_j & = \sum_{j=1}^m \phi_{ij} a_j, \text{ for } i = 1, \ldots, n\,. \end{array}$$

Then

$$f^* = {f_*}^\dagger = {f^\dagger}_* : {W_B}^* \to {V_A}^*\,.$$

Note that the dual coincides with the dagger in the categories $\mathit{2}\mathcal{SF}$, $\mathcal{RI}$ and $\mathbf{FdVect}_{\mathbb{R}}$, which are the most frequently considered categories in natural language semantics.

The unit $\eta_V : I \to V^* \otimes V$ and counit $\epsilon_V : V \otimes V^* \to I$ are the morphisms defined explicitly thus

$$\begin{array}{r} \eta_V(1_I) = \sum_{i=1}^n a_i \otimes a_i \\ \epsilon_V(a_i \otimes a_j) = \delta_{ij}, \text{ for } i, j = 1, \ldots, n\,, \end{array}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. □

4 The internal logic

An internal logic of a category consists of a class of morphisms, the propositions, and a set of equalities expressing the truth of propositions.

The internal logic of semantic categories follows quantum logic in letting projectors stand for propositions. Logical connectives are defined in such a way that the projectors form an ortho-complemented lattice with the identity as largest element. The approach to logic via predicates also is possible in semantic categories. The equivalence between the notions will be the subject of this section.

In both cases, the basis vectors of the domain play the role of individuals. Basis vectors are generalised to 'Boolean vectors' to capture the plurals of natural language. A vector $v = \alpha_1 b_1 + \cdots + \alpha_n b_n$ is said to be *Boolean* if $\alpha_i = 0$ or $\alpha_i = 1$, for all $i = 1, \ldots, n$.

Every Boolean vector $v : I \to V_B$ determines a unique subset $K = \{i_1, \ldots, i_k\}$ of $N = \{1, \ldots, n\}$ such that

$$v = \sum_{i \in K} b_i = v_K .$$

The propositional connectives are lifted from subsets $K \subseteq N$ to Boolean vectors such that

$$v_K \wedge v_L = v_{K \cap L},\ v_K \vee v_L = v_{K \cup L}\ ,\ \text{etc.}$$

hold. Hence, the map $K \mapsto v_K$ is a Boolean isomorphism. The Boolean vectors form a Boolean algebra with largest element $\overrightarrow{1} = \sum_{i=1}^{n} b_i = v_N$ and smallest element $\overrightarrow{0} = v_\emptyset = 0_{V_B}$.

Sometimes it is convenient to identify a subset $A = \{b_{i_1}, \ldots, b_{i_k}\} \subseteq B$ of basis vectors with the corresponding Boolean vector $A = b_{i_1} + \cdots + b_{i_k} = v_K$.

4.1 The logic of intrinsic projectors

Projectors will stand for propositions in this subsection. The truth of a proposition p is expressed by the equality $p = 1_V$. A semantic category may have projectors that do not intervene in natural language semantics. For example, if all entries of a square matrix are equal to 1, it defines a projector in $\mathcal{2SF}$ and in $\mathcal{RI}$, but not in a Hilbert space. We avoid the unwanted morphisms by limiting interpretations to intrinsic morphisms.

Definition 4 (**Intrinsic morphism**). *A morphism $f : V_A \to W_B$ is* intrinsic *with respect to A and B if it sends every basis vector in A to a basis vector in B or to the zero vector $\overrightarrow{0}$.*

The identity 1_V and the diagonal $d_V : V \to V \otimes V$, which maps any basis vector b of V to $b \otimes b$, are intrinsic. Intrinsic morphisms are closed under composition and tensor products. They are ubiquitous in natural language. Determiners, relative pronouns and verbs, to mention but a few, are interpreted by intrinsic morphisms.

Observe the following properties, which are are straightforward except possibly (?), which is proved in [Preller, 2012].

Proposition 10. *Let $B = \{b_1, \ldots, b_n\}$ be the basis of space V_B in $\mathcal{C}$. Then*
- *a projector $p : V_B \to V_B$ is intrinsic if and only if*

$$p(b_i) = b_i \text{ or } p(b_i) = \overrightarrow{0}, \text{ for } i = 1, \ldots, n$$

- *the entries of the matrix $(\pi_{ij})_{ij}$ of an intrinsic projector p satisfy*

$$\begin{matrix} \pi_{ij} = 1, \text{ if } i = j \text{ and } p(b_i) = b_i \\ \pi_{ij} = 0, \text{ else} \end{matrix} \quad , \quad i, j = 1, \ldots, n$$

- *intrinsic projectors map Boolean vectors to Boolean vectors*
- *every intrinsic projector p has the form $b_V \circ p_K \circ b_V^\dagger$, where*

$$K = \{i : p(b_i) = b_i, \ 1 \leq i \leq n\}$$

- *the morphism $b_V \circ p_K \circ b_V^\dagger$ is an intrinsic projector of V_B for every $K \subseteq \{1, \ldots, n\}$.*

The unitary isomorphism b_V lifts the lattice operators defined on the canonical projectors p_K of $n \cdot I$ to the intrinsic projectors of V_B. Context permitting, we write $p = p_K$ instead of $p = b_V \circ p_K \circ b_V^\dagger$.

Proposition 11. *Let $\mathcal{P}_B$ the set of projectors of V_B that are intrinsic w.r.t. B. The map $K \mapsto p_K$ is a negation preserving lattice isomorphism from the Boolean algebra of subsets of $N = \{1, \ldots, n\}$ onto the lattice $\mathcal{P}_B$ of intrinsic projectors of V_B.*

Moreover, the following properties hold for any $K, L \subseteq N$
- $\neg p_K \vee p_L = 1_{V_B}$ *if and only if* $p_K \circ p_L = p_K$
- $p_K \circ p_L = p_K$ *if and only if* $p_L \circ v_K = v_K$
- $p_L \circ v_K = v_{L \cap K}$. *In particular,* $p_L(\overrightarrow{1}) = v_L$ *and* $p_N \circ v_K = v_K$
- $p_L \circ v_K = v_K$ *if and only if* $p_L(b_i) = b_i$, *for all* $i \in K$
- *intrinsic projectors are monotone increasing on Boolean vectors.*

The restriction to $\mathcal{P}_B$ makes the logic classical. If V_B is a Hilbert space then the elements of $\mathcal{P}_B$ are exactly the projectors for which B is a common basis of eigenvectors. The internal logic proposed here uses only the classical part of quantum logic.

4.2 The two-sorted logic of predicates

Predicates and two-sorted logic are definable in an arbitrary semantic category. Let $S \simeq I \oplus I$ be a fixed two-dimensional space with basis vectors $\top = b_S \circ q_1$ and $\bot = b_S \circ q_2$.

The two-sorted connectives introduced below are morphisms and as such they are determined by their values on the basis vectors.

The *two-sorted negation* $\mathtt{not}_S : S \to S$ is defined explicitly by

$$\mathtt{not}_S(\top) = \bot,\ \mathtt{not}_S(\bot) = \top\,.$$

Recall that the full vector $\overrightarrow{1}$ of S satisfies $\overrightarrow{1} = \top + \bot$. Then

$$\mathtt{not}_S(\overrightarrow{1}) = \overrightarrow{1} \text{ and } \mathtt{not}_S(\overrightarrow{0}) = \overrightarrow{0}\,.$$

More generally,

$$\mathtt{not}_S(\alpha \cdot \top + \beta \cdot \bot) = \beta \cdot \top + \alpha \cdot \bot\,.$$

The *two-sorted conjunction* $\mathtt{and}_S : S \otimes S \to S$ and *two-sorted disjunction* $\mathtt{or}_S : S \otimes S \to S$ are defined explicitly on the four basis vectors of $S \otimes S$

$$\mathtt{and}_S(\top \otimes \top) = \top,\ \mathtt{and}_S(\bot \otimes \top) = \mathtt{and}_S(\top \otimes \bot) = \mathtt{and}_S(\bot \otimes \bot) = \bot$$
$$\mathtt{or}_S(\bot \otimes \bot) = \bot,\ \mathtt{or}_S(\bot \otimes \top) = \mathtt{or}_S(\top \otimes \bot) = \mathtt{or}_S(\top \otimes \top) = \top\,.$$

Note that the two-sorted connectives are different form the set-theoretical connectives introduced for Boolean vectors at the beginning of this section.

Proposition 12. *The two-sorted connectives define a Boolean structure on the vectors of S. In particular, for arbitrary vectors $v : I \to S$ and $w : I \to S$ the following holds*

$$\mathtt{not}_S \circ \mathtt{not}_S \circ v = v,\ \mathtt{not}_S \circ \mathtt{and}_S \circ (v \otimes w) = \mathtt{or}_S \circ (\mathtt{not}_S \circ v \otimes \mathtt{not}_S \circ w)\,.$$

Proof. Show the property for basis vectors and use the fact that morphisms commute with addition and scalar multiplication. □

A morphism $p : V \to S$ is a predicate if it is intrinsic. It is a *predicate on* V if it maps basis vectors of V to basis vectors of S, i.e.

$$p(x) = \top \text{ or } p(x) = \bot, \text{ for every basis vector } x \text{ of } V\,.$$

By an *n-ary predicate on E* we mean a predicate on $V = E \otimes \ldots \otimes E$.

We may think of basis vectors as individuals of the universe of discourse (sort 'one') and of Boolean vectors as subsets of individuals (sort 'two'). A predicate accepts individuals and sets of individuals as arguments. For an individual there are only two possible truth values, namely $\top$ and $\bot$. For sets of individuals there are at least two more.

Proposition 13. *Denote $m = 1 + \cdots + 1$ the m-fold sum of the unit $1 \in I$. Let $p : V_B \to S$ be a predicate on V_B and $A = \{b_{i_1}, \ldots, b_{i_k}\} \subseteq B$ a subset of k distinct basis vectors. Then there are non-negative integers $k_1 \leq k$ and $k_2 \leq k$ such that*

$$p(\sum_{x \in A} x) = k_1 \cdot \top + k_2 \cdot \bot\,. \tag{13}$$

Proof. The predicate p partitions A into two disjoint subsets A_1 and A_2 such that $p(x) = \top$ for all $x \in A_1$ and $p(x) = \bot$ for all $x \in A_2$. Let k_i be the number of elements in A_i. □

COUNTING PROPERTY
In Hilbert spaces, a predicate 'counts' the number of elements of a set for which it is true and also the number of elements for which it is false. The integers k_1 and k_2 are unique and satisfy $k_1 + k_2 = k$.

FUNDAMENTAL PROPERTY in $2\mathcal{SF}$
A two-sorted predicate informs us whether it is always true, always false or partly true and partly false on a set A identified with the Boolean vector $\sum_{x\in A} x$.

$$\begin{array}{lll} p(A) = \overrightarrow{0} & \Leftrightarrow p(x) = \overrightarrow{0} \textit{ for all } x \in A & \Leftrightarrow \ A = \emptyset \\ p(A) = \top & \Leftrightarrow p(x) = \top \textit{ for all } x \in A \textit{ and } A \neq \emptyset & \\ p(A) = \bot & \Leftrightarrow p(x) = \bot \textit{ for all } x \in A \textit{ and } A \neq \emptyset & \\ p(A) = \top + \bot & \Leftrightarrow p(x) = \top \textit{ and } p(y) = \bot \textit{ for some } x, y \in A\,. & \end{array} \tag{14}$$

Indeed, the k-fold sum of the unit of $1 \in I$ is equal to 1 whenever $k > 0$. Hence, $k \cdot \top = \top$.

The counting property and the fundamental property give us a clue to defining truth in semantic catgories.

TRUTH-VALUES
Let p be a linear predicate on V_A and X any vector of V_A. We say that

- $p(X)$ is *true* if there is $\alpha \neq 0$ such that $p(X) = \alpha\top$
- $p(X)$ is *false* if there is $\alpha \neq 0$ such that $p(X) = \alpha\bot$
- $p(X)$ is *mixed* if $p(X) = \alpha\top + \beta\bot$ for some $\alpha \neq 0, \beta \neq 0$
- $p(X)$ is *mute* if $p(X) = 0$.

Proposition 14. *Predicates are closed under composition with the two-sorted connectives.*

More precisely, assume that $p : V_B \to S$ and $r : V_B \to S$ are predicates on $V = V_B$. Then the morphisms

$$\mathtt{not}_S \circ p,\ \mathtt{and}_S \circ (p \otimes r),\ \mathtt{or}_S \circ (p \otimes r)$$

are again predicates on V_B respectively on $V_B \otimes V_B$ and satisfy

$$\begin{array}{rl} \mathtt{not}_S \circ \mathtt{not}_S \circ p = p & \\ \mathtt{not}_S \circ \mathtt{and}_S \circ (p \otimes r) = & \mathtt{or}_S \circ ((\mathtt{not}_S \circ p) \otimes (\mathtt{not}_S \circ r))\,. \end{array} \tag{15}$$

For any $x \in B$, $A \subseteq B$

$$\begin{array}{lll} p(x) & = \bot & \Leftrightarrow \mathtt{not}_S(p(x)) = \top \\ p(\sum_{x\in A} x) & = k_1 \cdot \top + k_2 \cdot \bot & \Leftrightarrow \mathtt{not}_S(p(\sum_{x\in A} x)) = k_2 \cdot \top + k_1 \cdot \bot \end{array}$$

Whereas $\mathtt{not}_S(p(x)) = \top$ is equivalent to $p(x) \neq \top$ for anyy basis vector x, this no longer holds for arbitrary vectors. For the counter example, let a and b be two distinct basis vectors such that $p(a) = \top$ and $p(b) = \bot$.

The predicates $\mathtt{and}_S \circ (p \otimes r)$ and $\mathtt{or}_S \circ (p \otimes r)$ are predicates on $V \otimes V$. Composing them with the diagonal $d_V : V \to V \otimes V$, we obtain the predicates $\mathtt{and}_S \circ (p \otimes r) \circ d_V$ and $\mathtt{or}_S \circ (p \otimes r) \circ d_V$ on V such that the equalities (15) still hold. Hence, the predicates on a given space form a Boolean algebra.

4.3 Intrinsic projectors and predicates

Let $\mathcal{C}$ be an arbitrary semantic category and $V = V_B$ be any space of $\mathcal{C}$ with basis B. For every intrinsic projector $p : V \to V$, define a predicate $\mathcal{I}(p) : V \to S$ by the condition

$$\mathcal{I}(p)(x) = \begin{cases} \top & \text{if } p(x) = x \\ \bot & \text{else} \end{cases}, \text{ for all } x \in B\,.$$

Conversely, given a predicate $p : V \to S$ on V, define an intrinsic projector $\mathcal{J}(p) : V \to V$ by

$$\mathcal{J}(p)(x) = \begin{cases} x & \text{if } p(x) = \top \\ \vec{0} & \text{else} \end{cases}, \text{ for all } x \in B\,. \tag{16}$$

Proposition 15. *The maps $\mathcal{I}$ and $\mathcal{J}$ are inverse of each other and the following holds for any intrinsic projectors $p, r : V \to V$*

$$\begin{aligned} \mathcal{I}(\neg p) &= \mathtt{not}_S \circ \mathcal{I}(p) \\ \mathcal{I}(p \wedge r) &= \mathtt{and}_S \circ (\mathcal{I}(p) \otimes \mathcal{I}(r)) \circ d_V \\ \mathcal{I}(p \vee r) &= \mathtt{or}_S \circ (\mathcal{I}(p) \otimes \mathcal{I}(r)) \circ d_V \\ \mathcal{I}(p \otimes r) &= \mathtt{and}_S \circ (\mathcal{I}(p) \otimes \mathcal{I}(r))\,. \end{aligned} \tag{17}$$

Moreover, for any Boolean vector $v_L = \sum_{i \in L} b_i$ and any intrinsic projector $p : V \to V$

$$\begin{aligned} p(v_L) = v_L &\iff \forall_{i \in L}(\mathcal{I}(p)(b_i) = \top \\ p(v_L) = \vec{0} &\iff \forall_{i \in L}(\mathcal{I}(p)(b_i) = \bot\,. \end{aligned} \tag{18}$$

In particular, if k is the number of basis vectors left invariant by p then

$$\mathcal{I}(p)(\sum_{i=1}^{n} b_i) = k \cdot \top + (n - k) \cdot \bot\,.$$

Proof. It is sufficient to verify (17) for basis vectors, an easy exercise. The equalities (18) follow from (13). □

The switch between predicates and intrinsic projectors is common in natural language. Typically, an adjective in attributive position is interpreted as an intrinsic projector $\mathtt{big} : V_B \to V_B$. The same adjective, when in predicative position, is interpreted by a predicate $\mathtt{big} : V_B \to S$ such that

$$\mathtt{big}(x) = x \iff \mathtt{big}(x) = \top, \text{ for all } x \in B\,.$$

The isomorphism (16) transforming a predicate into a projector is related to the relative pronoun, see Section 5.3.

5 Compositional semantics

5.1 The lexical category

The description of the lexical category given below is a notational variant of the description in [Preller and Lambek, 2007].

Call *lexical category* any free compact bicategory $\mathcal{C}(\mathcal{D})$ with a single 0-cell[2], generated by some category $\mathcal{D}$. Think of the objects of $\mathcal{D}$ as basic types and of the morphisms of $\mathcal{D}$ as basic morphisms. For simplicity, the canonical associativity and unit isomorphisms of the tensor product (1-cell composition) are replaced by identities, for example $A \otimes (B \otimes C) = A \otimes B \otimes C = (A \otimes B) \otimes C$, $A \otimes I = A = I \otimes A$. The iterated tensor products are assimilated to strings of objects. That explains why the symbol for the tensor product may be omitted in lexical categories

Saying that $\mathcal{C}(\mathcal{D})$ is compact means that every 1-cell (object) Γ has a left adjoint Γ^ℓ and a right adjoint Γ^r. Then Γ is a right adjoint to its left adjoint Γ^ℓ, thus $\Gamma^{\ell r} \simeq \Gamma$. Hence the objects (1-cells) of $\mathcal{C}(\mathcal{D})$ are the unit I, the objects of $\mathcal{D}$, their iterated right of left adjoints and the strings built from these. An iterated adjoint $A^{(z)}$ is *even* if $z = (2n)\ell$ or $z = (2n)r$. It is *odd* if $z = (2n+1)\ell$ or $z = (2n+1)r$. By convention, $A^{(0)} = A$. A similar convention applies to the morphisms of $\mathcal{D}$. Capital latin letters designate objects of $\mathcal{D}$, capital greek letters objects of $\mathcal{C}(\mathcal{D})$.

The *morphisms*, i.e. the 2-cells, of $\mathcal{C}(\mathcal{D})$, are represented by graphs where the vertices are labelled by iterated adjoints of objects of $\mathcal{D}$ and the oriented links are labelled by morphisms of $\mathcal{D}$.

The first four rules constitute a cut-free axiomatisation of Compact Bilinear Logic. They imply the fifth, the Cut rule. In the presentation below, each rule is followed by the corresponding morphism (on the left) and its proof-graph (on the right). Graphs display the domain of the morphism above, the codomain below. The links are directed, because they are morphisms of $\mathcal{D}$. The basic type at the tail of the link is the domain of the link and the basic type at the head its codomain in $\mathcal{D}$. In the case where the label is an identity, it is generally omitted. A double line stands for the collection of links of some previously constructed graph.

Axioms

$$\frac{}{\vdash} \qquad\qquad 1_I \ = \ \begin{array}{c} I \\ \\ I \end{array}$$

[2] definitially equivalent to compact closed category

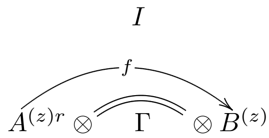
z is even $\dfrac{f:A\to B\in\mathcal{D}}{A^{(z)}\vdash B^{(z)}}$ if z is odd $\dfrac{f:A\to B\in\mathcal{D}}{B^{(z)}\vdash A^{(z)}}$

$$f^{(z)} = \begin{array}{c} A^{(z)} \\ \downarrow f \\ B^{(z)} \end{array} \qquad f^{(z)} = \begin{array}{c} B^{(z)} \\ \uparrow f \\ A^{(z)} \end{array}$$

Units $g : I \to \Gamma,\ f : A \to B \in \mathcal{D}$

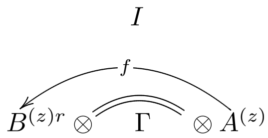
z is even $$\frac{\vdash \Gamma \quad A \vdash B}{\vdash A^{(z)r} \otimes \Gamma \otimes B^{(z)}}$$

$$(1_{A^{(z)r}} \otimes g \otimes 1_{B^{(z)}}) \circ \eta_{f^{(z)}} =$$



z is odd $$\frac{\vdash \Gamma \quad A \vdash B}{\vdash B^{(z)r} \otimes \Gamma \otimes A^{(z)}}$$

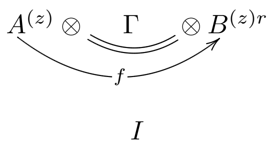
$$(1_{B^{(z)r}} \otimes g \otimes 1_{A^{(z)}}) \circ \eta_{f^{(z)}} =$$



Counits $g : \Gamma \to I,\ f : A \to B \in \mathcal{D}$

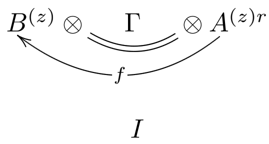
z is even $$\frac{\Gamma \vdash \quad A \vdash B}{A^{(z)} \otimes \Gamma \otimes B^{(z)r} \vdash I}$$

$$\epsilon_{f^{(z)}} \circ (1_{A^{(z)}} \otimes g \otimes 1_{B^{(z)r}}) =$$



z is odd $$\frac{\Gamma \vdash \quad A \vdash B}{B^{(z)} \otimes \Gamma \otimes A^{(z)r} \vdash I}$$

$$\epsilon_{f^{(z)}} \circ (1_{B^{(z)}} \otimes g \otimes 1_{A^{(z)r}}) =$$



1-Cell Composition $g : \Gamma \to \Delta,\ h : \Theta \to \Lambda$

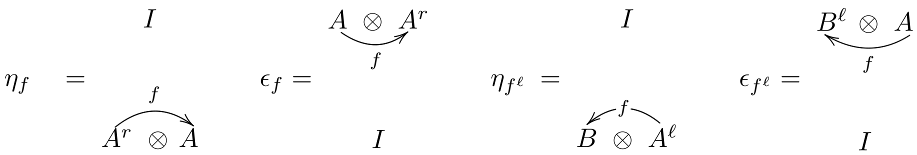
$$\frac{\Gamma \vdash \Delta \quad \Theta \vdash \Lambda}{\Gamma \otimes \Theta\ \vdash \Delta \otimes \Lambda} \qquad\qquad g \otimes h = \begin{array}{ccc} \Gamma & \otimes & \Theta \\ \| & & \| \\ \Delta & \otimes & \Lambda \end{array}$$

Cut $g : \Gamma \to \Delta,\ h : \Delta \to \Theta$

$$\frac{\Gamma \vdash \Delta \quad \Delta \vdash \Theta}{\Gamma\ \vdash\ \Theta} \qquad\qquad h \circ g\ = \begin{array}{c} \Gamma \\ \| \\ \Theta \end{array}$$

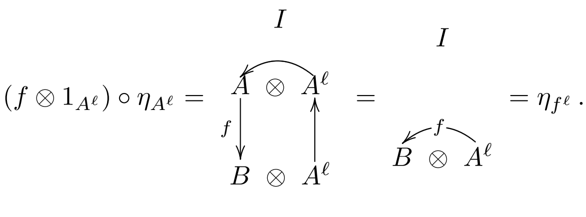
Any morphism of $\mathcal{C}(\mathcal{D})$ can be obtained from the morphisms of $\mathcal{D}$ without Cut by the first four rules only, see [Preller and Lambek, 2007]. All paths in graphs obtained without Cut have length one. We compute the cut-free graph of $g \circ f$ by connecting the graph of $g : \Gamma \to \Delta$ to the graph of $h : \Delta \to \Theta$ at their joint interface Θ and replace any path with endpoints in Γ or Δ by a link with the same endpoints and direction, see the examples below.

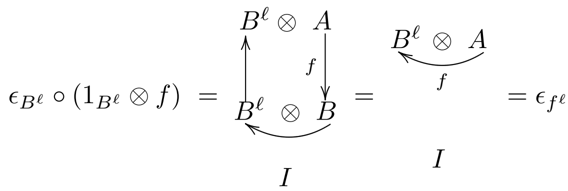
Choosing $z = 0$, $g = 1_I$, but $f : A \to B \in \mathcal{D}$ arbitrary in the Unit and Counit rules we obtain



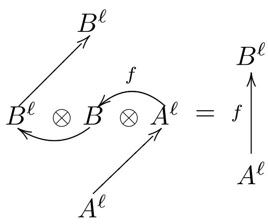
In the particular case where $f = 1_{A^{(z)}}$, the results are the unit $\eta_A : I \to A^r \otimes A$ and the counit $\epsilon_A : A \otimes A^r \to I$ for the right adjoint and, as $A = A^{\ell r}$, the unit η_{A^ℓ} and counit ϵ_{A^ℓ} for the left adjoint.

The following four examples of composition by 'walking paths' illustrate how cut-elimination can be proved. The last of two of them are the axioms concerning units and counits.



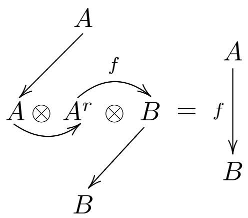


$$(\epsilon_{B^\ell} \otimes 1_{A^\ell}) \circ (1_{B^\ell} \otimes \eta_{f^\ell}) =$$



$$= f^\ell$$

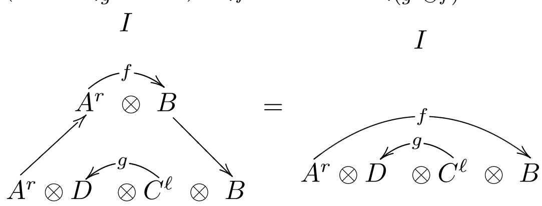
$$(\epsilon_A \otimes 1_B) \circ (1_A \otimes \eta_f) =$$



$$= f$$

Units of adjunction give rise to 'nested' graphs. The same holds for counits. For example, let $f : A \to B$, $g : C \to D$ be morphisms in $\mathcal{D}$

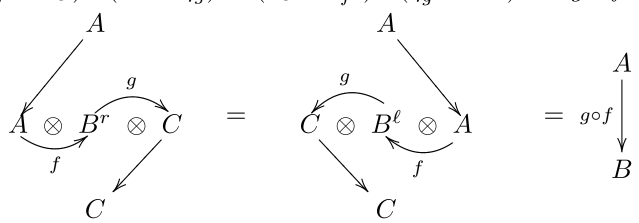
$$(1_{A^r} \otimes \eta_{g^\ell} \otimes 1_B) \circ \eta_f \;=\; \eta_{(g^\ell \otimes f)}$$



etc.

Assume $f : A \to B$, $g : B \to C$. Then

$$(\epsilon_f \otimes 1_C) \circ (1_A \otimes \eta_g) = (1_C \otimes \epsilon_{f^\ell}) \circ (\eta_{g^\ell} \otimes 1_A) \quad = g \circ f$$



The benefit of orienting and labelling links becomes evident when computing the meaning of strings of words where the graphs are given by the grammar in Section 5.2.

5.2 Pregroup grammars

According to [Lambek, 1999], a pregroup grammar is determined by a partially ordered set $\mathcal{B}$ and a dictionary. The elements of $\mathcal{B}$ stand for grammatical notions and are called basic types, e.g. $\boldsymbol{c}_2$, $\boldsymbol{n}_2$ and $\boldsymbol{s}$ for 'plural count noun', 'plural noun phrase' and 'sentence'. The *dictionary* is a finite list of pairs *word* : T where the *type* T is an object of $\mathcal{C}(\mathcal{B})$.

Every functor from $\mathcal{B}$ into a semantic category $\mathcal{C}$ extends to a functor $\mathcal{F} : \mathcal{C}(\mathcal{B}) \to \mathcal{C}$ that preserves right and left adjoints

$$\mathcal{F}(T^\ell) = \mathcal{F}(T)^* = \mathcal{F}(T^r) \qquad \mathcal{F}(f^\ell) = \mathcal{F}(f)^* = \mathcal{F}(f^r)$$

and every derivation of compact bilinear logic to a morphism of $\mathcal{C}$.

A *lexicon* is a finite list of triples *word* : $T :: \overline{\texttt{word}}$, where T a type and $\overline{\texttt{word}}$ a *meaning* expression in the language of compact closed categories with two distinguished objects E and S.

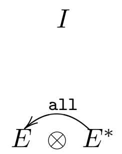
Thus, a pregroup lexicon is a pregroup dictionary, enriched by formal meaning expressions in the language or compact closed categories. Dictionaries are sufficient for recognising languages, but do not assign meanings. The added meaning expressions are the compact bilinear version of the lambda-terms of higher order logic, see [Moot and Retoré, 2011].

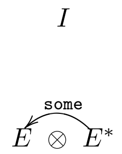
$$\begin{array}{llll} \textit{all} & : \boldsymbol{n}_2\, {\boldsymbol{c}_2}^\ell :: I \xrightarrow{\overline{\texttt{all}}} E \otimes E^* & \textit{birds}: \boldsymbol{c}_2 & :: I \xrightarrow{\overline{\texttt{bird}}} E \\ \textit{some} & : \boldsymbol{n}_2\, {\boldsymbol{c}_2}^\ell :: I \xrightarrow{\overline{\texttt{some}}} E \otimes E^* & \textit{fly} \;\; : {\boldsymbol{n}_2}^r\, \boldsymbol{s} & :: I \xrightarrow{\overline{\texttt{fly}}} E^* \otimes S \end{array}$$

$$\begin{array}{lll} \textit{who} & : {\boldsymbol{c}_2}^r \boldsymbol{c}_2\, \boldsymbol{s}^\ell \boldsymbol{n}_2 & :: I \xrightarrow{\overline{\texttt{who}}} E^* \otimes E \otimes S^* \otimes E \\ \textit{do} & : \boldsymbol{n}^r \boldsymbol{s}\, \boldsymbol{i}^\ell \boldsymbol{d} & :: I \xrightarrow{\overline{\texttt{do}}} E^* \otimes S \otimes S^* \otimes E \\ \textit{not} & : \boldsymbol{d}^r \boldsymbol{i}\boldsymbol{i}^\ell \boldsymbol{d} & :: I \xrightarrow{\overline{\texttt{not}}} E^* \otimes S \otimes S^* \otimes E \end{array}$$

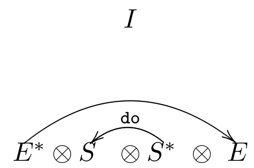
The basic types corresponding to the mini-lexicon above are $\boldsymbol{c}_2, \boldsymbol{n}_2, \boldsymbol{d}, \boldsymbol{s}, \boldsymbol{i}$, partially ordered by the equalities and $\boldsymbol{c}_2 < \boldsymbol{n}_2$. The basic types $\boldsymbol{d}$ and $\boldsymbol{i}$ stand for 'dummy noun phrase' and 'infinitive'. A functor interpreting the basic types $\boldsymbol{c}_2, \boldsymbol{n}_2, \boldsymbol{d}$ by the distinguished space E and the basic types $\boldsymbol{i}$ and $\boldsymbol{s}$ a by the distinguished space S also maps the type T in the entry *word* : $T :: \overline{\texttt{word}}$ to the codomain of $\overline{\texttt{word}}$. Thus, the lexicon defines an obvious functor from $\mathcal{B}$ to the semantic category $\mathcal{C}$, which maps the inequality $\boldsymbol{c}_2 < \boldsymbol{n}_2$ to 1_E .

The meaning morphisms in the lexicon above a formal expressions in the language of compact closed categories, represented by their graphs. For example, let $E = V_B$

$\overline{\texttt{all}} = \eta_{\texttt{all}^*} =$  $\texttt{all} : E \to E$

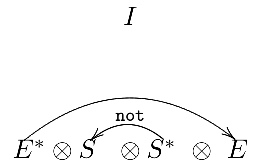
$\overline{\texttt{some}} = \eta_{\texttt{some}^*} =$  $\texttt{some} : E \to E$

$\overline{\texttt{do}} = \eta_{(1_E \otimes \texttt{do}^*)} =$



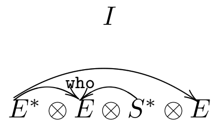
$\texttt{do} : S \to S$

$\overline{\texttt{not}} = \eta_{(1_E \otimes \texttt{not}^*)} =$



$\texttt{not} : S \to S$

$\overline{\texttt{who}} =$



$\texttt{who} : E \otimes S \to E$

The algebraic expression corresponding to the graph above is

$$\overline{\texttt{who}} = c \circ \ulcorner(1_E \otimes \texttt{who}) \circ (d_E \otimes 1_S)\urcorner : I \to E^* \otimes E \otimes S^* \otimes E$$

where the morphism $c : E^* \otimes S^* \otimes E \otimes E \ \to\ E^* \otimes E \otimes S^* \otimes E$ is the permutation that first switches the last two factors of $E^* \otimes S^* \otimes E \otimes E$ and then the third and the second factor.

The meaning of grammatical strings involves besides the meanings of the words a syntactical analysis of the string.

A string of words $word_1 \ldots word_n$ is *grammatical* if there are entries $word_1 : T_1 :: \overline{\texttt{word}}_1, \ldots,\ word_n : T_n :: \overline{\texttt{word}}_n$, and a basic type $\boldsymbol{b}$ such that

$$T_1 \ldots T_n \vdash \boldsymbol{b}$$

is provable in compact bilinear logic.

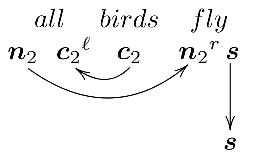
Due to a theorem in [Lambek, 1999] the graph of the proof involves only underlinks. Such a graph is called a *reduction.* It designates a uniquely determined morphism of $\mathcal{C}(\mathcal{B})$.

Any functor $\mathcal{F} : \mathcal{C}(\mathcal{B}) \to \mathcal{C}$ maps the reduction to a morphism of the semantic category $\mathcal{C}$ and thus assigns the following *meaning* to the grammatical string

$$m(word_1 \ldots word_n) = \mathcal{F}(r) \circ (\overline{\texttt{word}}_1 \otimes \ldots \otimes \overline{\texttt{word}}_n)\,.$$

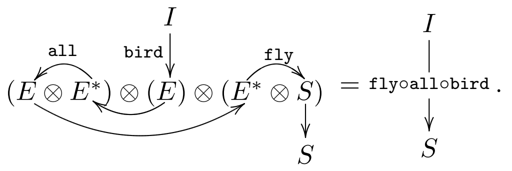
The string *all birds fly* has the following reduction to the sentence type

$r_0 =$



and therefore its meaning in $\mathcal{C}$ is the morphism

$$\mathcal{F}(r_0) \circ (\overline{\mathtt{all}} \otimes \overline{\mathtt{bird}} \otimes \overline{\mathtt{fly}}) =$$



Note that that any functor on $\mathcal{B}$ to a compact closed category assigns a meaning-morphism to grammatical strings via these definitions. An arbitrary functor, however, does not guarantee that the resulting meaning reflects the logical content of natural language.

Therefore we add postulates that make the morphisms reflect the logical content of the words. We require that S is the two-dimensional space of 'truth-values' with basis vectors $\top, \bot$ and that the basis vectors of $E = V_B$ stand for 'individuals', 'pairs of individuals' and so on. The vectors $\mathtt{noun} : I \to E$ are Boolean, morphisms $\mathtt{word} : E \to S$ are predicates on E, the morphisms $\mathtt{word} : E \to E$ are intrinsic projectors. Some words are completely determined by their logical content, namely

$$\mathtt{do} = 1_S, \ \mathtt{not} = \mathtt{not}_S \, .$$

Moreover, $\mathtt{who} : E \otimes S \to E$ is the explicitly defined morphism satisfying

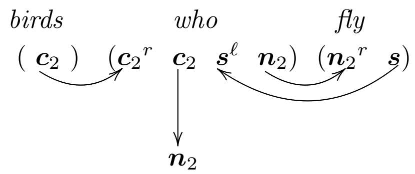
$$\begin{array}{ll} \mathtt{who}(b \otimes \top) & = b \\ \mathtt{who}(b \otimes \bot) & = \overrightarrow{0} \end{array} \, , \text{ for } b \in B \, .$$

Taking the postulates into account, the meaning of the sentence *all birds fly* is

$$m(\textit{all birds fly}) = \mathtt{fly} \circ \mathtt{bird} : I \to S$$

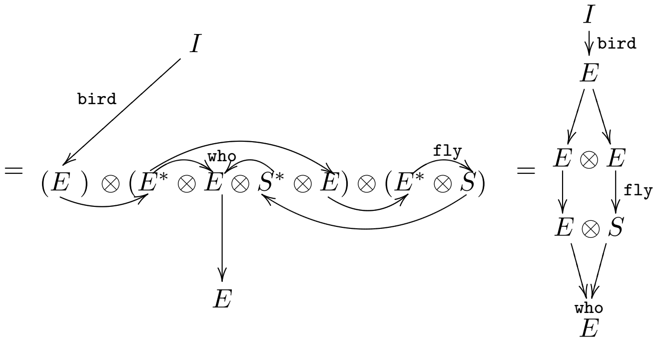
The string *birds who fly* has a reduction to the plural noun phrase type, namely

$$r_1 =$$



and therefore its meaning is

$$\mathcal{F}(r_1) \circ (\overline{\texttt{bird}} \otimes \overline{\texttt{who}} \otimes \overline{\texttt{fly}})$$

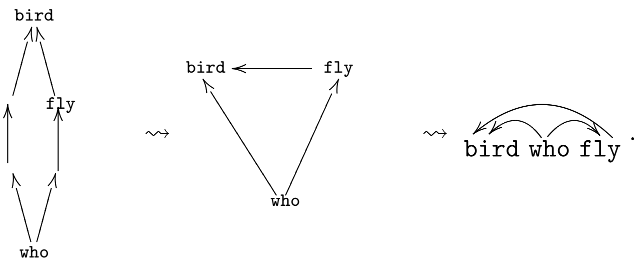


(19)

$$= \texttt{who} \circ (1_E \otimes \texttt{fly}) \circ d_E \circ \texttt{bird} = \texttt{who} \circ \langle \texttt{bird}, \texttt{fly} \circ \texttt{bird} \rangle \,.$$

The right-hand graph above captures the dependency relation in *birds who fly*. Indeed, *who* depends on *birds* and *fly* whereas *fly* depends on *birds*

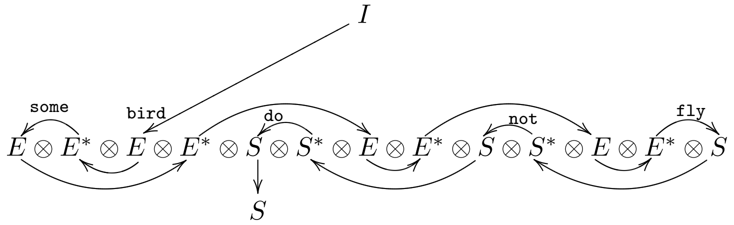
Th algorithm that computes the dependencies implements the following instructions
1) omit domains and codomains of the links and orient them in the opposite direction
2) move the label of a link to the head of the reversed link
3) replace maximal paths by single links
4) write the labels on the same line



(20)

Finally, the last example computed using pregroup syntax and semantics

$$m(\textit{some birds do not fly}) = r \circ (\overline{\texttt{some}} \otimes \overline{\texttt{bird}} \otimes \overline{\texttt{do}} \otimes \overline{\texttt{not}} \otimes \overline{\texttt{fly}}) =$$



$$= \texttt{do} \circ \texttt{not} \circ \texttt{fly} \circ \texttt{some} \circ \texttt{bird} = \texttt{not}_S(\texttt{fly}(\texttt{some}(\texttt{bird})))\,.$$

To sum up: All grammatical strings are interpreted by variable free expressions formed by morphisms and vectors.

The computation of the expression involves a syntactical analysis of the string via a pregroup grammar. There are cubic polynomial algorithms that decide whether the string is grammatical and, if it is grammatical, construct a reduction. The reduction includes a choice of type for each word. Forming the tensor product of the corresponding meanings is proportional to the length of the string. Walking the graph is linear in the number of links, which is proportional to the number of words.

5.3 Internal logic in action

The meanings of the preceding examples can now be compared and translated from predicate logic to quantum logic via the isomorphism $\mathcal{J}$ of Section 4.3.

We start with the the relative pronoun and show that it assimilates to an operator that accepts a predicate and a Boolean vector as an input and returns a Boolean vector.

Proposition 16. *Let p be a predicate on $E = V_B$ and w be a Boolean vector. Then*

$$\texttt{who} \circ (1_E \otimes p) \circ d_E \circ w = \mathcal{J}(p) \circ w\,. \tag{21}$$

Proof. Recall the explicit definition of the morphism $\texttt{who} : E \otimes S \to E$ in Subsection 5.2, namely

$$\begin{matrix} \texttt{who} \circ (x \otimes \top) & = x \\ \texttt{who} \circ (x \otimes \bot) & = \overrightarrow{0} \end{matrix}\;, \text{ for all } x \in B\,.$$

Hence

$$\texttt{who} \circ (1_E \otimes p) \circ d_E \circ x = \texttt{who} \circ (x \otimes p(x)) = \begin{cases} x & \text{if } p(x) = \top \\ \overrightarrow{0} & \text{else} \end{cases}\;, \text{ for all } x \in B\,.$$

The explicit definition of $\mathcal{J}(p)$ implies the equality

$$\texttt{who} \circ (1_E \otimes p) \circ d_E = \mathcal{J}(p)\,,$$

because the subset K of basis vectors left invariant by $\mathcal{J}(p)$ is exactly the set of basis vectors for which p takes value $\top$. The equality (21) follows. □

One can reformulate the preceding proposition by saying that semantic categories satisfy comprehension. It suffices to identify a subset K of distinct basis vectors with their sum v_K. For example, the meaning vector of the noun phrase *birds who fly* satisfies

$$\mathtt{who} \circ (1_E \otimes \mathtt{fly}) \circ d_E \circ \mathtt{bird} = \sum_{\{x \in \mathtt{bird} : \mathtt{fly}(x) = \top\}} x \,.$$

Composition with the morphism $d_V^\dagger$ simulates comprehension, but not every logical connective can be simulated by composition with a morphism in quantum logic. For example, the negation $\mathtt{not}_S$ is replaced by the orthogonal complement in quantum logic, by (17). The orthogonal complement of the projector $0_{EE} : E \to E$ is the identity $1_E : E \to E$ where as $f \circ 0_{EE} = 0_{EE}$ for any endomorphism $f : E \to E$.

Switching from predicate logic to quantum logic via the isomorphism $\mathcal{J}$ of Section 4.3 the comprehension operator becomes the morphism $d_V^\dagger$. Indeed, let v_K be the sum of all basis vectors left invariant by the projector $\mathcal{J}(p)$. Then from Proposition 11 and equality (12) follows

$$\mathcal{J}(p) \circ w = v_K \wedge w = d_V^\dagger \circ (v_K \otimes w) \,.$$

If predicates are replaced the corresponding projectors one can no longer distinguish between sentences and noun-phrases. For example, the sentence *all birds fly* and the noun-phrase *birds who fly* have the same meaning

$$\mathtt{fly} \circ \mathtt{all} \circ \mathtt{bird} = \mathtt{fly} \circ \mathtt{bird} \mapsto \mathcal{J}(\mathtt{fly}) \circ \mathtt{bird} = \mathtt{who} \circ (1_E \otimes \mathtt{fly}) \circ d_E \circ \mathtt{bird}$$

By Proposition 15, is an isomorphism. Moreover, it preserves truth if $\mathcal{C} = \mathcal{2SF}$ or $\mathcal{C} = \mathbf{Fd}\mathrm{vect}_{\mathbb{R}}$

$$\begin{aligned} \mathtt{fly} \circ \mathtt{all} \circ \mathtt{bird} \textit{ is true} &\Leftrightarrow \mathtt{bird} \leq \mathcal{J}(\mathtt{fly}) \circ \mathtt{bird} \\ \mathtt{not}_S \circ \mathtt{fly} \circ \mathtt{bird} \textit{ is true} &\Leftrightarrow \mathtt{bird} \leq \mathcal{J}(\mathtt{fly})^\perp \circ \mathtt{bird} \,. \end{aligned}$$

The situation for *some* is quite different. Note that

$$p(\mathtt{some}(A)) = \top \Rightarrow \exists_X (X \neq \emptyset \,\&\, X \subseteq A \,\&\, \mathtt{fly}(X)) \,,$$

but the determiner *some* does more than guarantee existence in natural language. It introduces a witness and only as a consequence it acts like an existential quantifier. The fact that $\mathtt{some}$ creates a witness is built into categorical semantics. The discourse *Some birds fly. They have wings* is represented by the three expressions $\mathtt{fly}(\mathtt{some}(\mathtt{bird}))$, $\mathtt{have}(\mathtt{they}, \mathtt{wing})$, $\mathtt{they} = \mathtt{some}(\mathtt{bird})$.

On the other hand, the interpretation of $\mathtt{some}$ may change from one occurrence to the next, for instance *some birds fly and some birds do not fly.* The solution to the latter problem is to index the occurrences of *some*. For example, the meaning of the sentence above is

$$\texttt{and}_S \circ ((\texttt{fly} \circ \texttt{some}_1 \circ \texttt{bird}) \otimes (\texttt{not}_S \circ \texttt{fly} \circ \texttt{some}_2 \circ \texttt{bird}))\,.$$

The interpretation of *some bird* as a generalised quantifier by [Barwise and Cooper, 2002] takes into account the change of meaning when occurring in different senteces, but it does not construct the set to which the noun phrase refers.

The preceding suggests the canonical vector space model $\mathcal{V}$ associated to $\mathcal{F}$. For any vector v, projector p_K and predicate p let

$$\mathcal{V}(v) = v \qquad \mathcal{V}(p_K) = v_K \qquad \mathcal{V}(p) = \mathcal{V}(\mathcal{J}(p))$$

Then the map $\mathcal{V} \circ \mathcal{F}$ extends to a partial functor from the lexical category to the Frobenius structure over V_B. For example,

$$\mathcal{V}(\texttt{fly} \circ \texttt{bird}) = d_V^\dagger \circ (\mathcal{V}(\texttt{fly}) \otimes \mathcal{V}(\texttt{bird}))\,.$$

6 Conclusion

If the generating object I includes the semiring of positive rational numbers the 'counting property' of the predicates makes it possible to replace the canonical vector space model by a probabilistic vector space model where the basis vectors stand for arbitrarily chosen 'basic concepts', see [Preller, 2013]. A promising line of future investigations is the step from 'counting' predicates to measuring predicates in Hilbert spaces to capture the notion of 'truth in probability' and its relation to natural language.

Going in the opposite direction, predicates of $\mathcal{2SF}$ and $\mathcal{RI}$ do not 'count' at all, because addition is idempotent. A language that only has a word for 'one' and a word for 'more than one' can only count '0, 1, $^>$1'. The appropriate generating object I of its semantic category is the linearly ordered set consisting of the three elements.

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