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A New Formulation of Degree-Constrained Spanning Problems (Extended Abstract)

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Abstract

Given a graph with edge-costs, searching for a minimum cost structure that connects a subset of vertices is a classic problem. We examine the spanning problems under constraints on the vertex degrees. Spanning tree solutions were generally investigated to solve them. However, for some applications the solution is not necessarily a sub-graph. Assuming that the degree constraint is due to the limited instantaneous capacity of the vertex and that the only other constraint on the spanning structure is its connectivity, we propose a reformulation of some spanning problems. To find the optimal coverage of the concerned vertices, an extension of the tree concept has been proposed. A hierarchy is obtained by a graph homomorphism between a tree and a target graph. Since this spanning structure may refer vertices (and edges) of the target graph several times, it is more flexible to satisfy constraints and nevertheless pertinent for network applications. Hierarchies correspond to the optimal solutions of the new problems. Here we resume our first promising results on the degree-constrained spanning hierarchies. They can solve network related cases where trees meeting the constraints do not exist. In other cases, hierarchies outperform trees. Furthermore, the degree constrained spanning hierarchy problem can be approximated within a constant ratio (while it is not possible with trees).

Keywords: Degree-constrained spanning problems, minimum spanning hierarchies, networks

1. Motivation

They are network applications as broadcasting and multicast routing that several inspired spanning problems in graphs. When a set of more than 2 vertices should be connected and there is no further constraint, the minimum cost spanning structure (the optimal route) is a sub-graph: a tree. In this paper, we focus on degree-constrained minimum spanning problems, where the degree of vertices participating in the span is limited.

The degree-constrained minimum spanning tree (DCMST) problem was introduced in [2]. The vertex set V of a graph $G = (V, E)$ must be covered by a tree in which the maximal degree of any vertices is limited by an integer constant $D > 1$. The problem is NP-hard. (Solving the DCMST problem with the degree bound $D = 2$ is equivalent to find the minimum Hamiltonian path. By reducing the DCMST problem from an equivalent symmetric traveling salesman problem (TSP), Garey and Johnson [3] showed that this problem is NP-hard for any fixed constant $2 \leq D \leq |V - 1|$.) The NP-hard degree-constrained Steiner tree (DCST) problem was first extensively studied by Voß in [11]. In [1] the authors argument the particular interest of

this problem with the limited capacity of switches in high speed networks (where switches can duplicate packets only for a limited outgoing links) to perform multicast routing. To solve the problems, spanning trees or sets of trees are supposed as minimum cost solutions. Unfortunately, considering only trees as potential solutions implies two main drawbacks.

The degree-constrained minimum spanning tree problems (DCMST and DCST) may not always have a feasible solution. For instance, with $D = 2$, the DCMST problem can be solved if and only if the graph G has a Hamiltonian path.

The degree-constrained minimum spanning problems are not approximable. In [9], the authors present the problems in the frame of a generic bi-criteria optimization (A, B, S) where the first objective A corresponds to the respect of a budget (degree) constraint, the second (B) is the objective (minimization of the cost) and S describes the class of sub-graphs candidates to solve the problem. The investigated classes in the paper (as usually) are spanning trees, Steiner trees and generalized Steiner trees. Hence, the solution is always supposed to be a tree. The authors prove the hardness of the approximability in the investigated optimizations. The degree-constrained minimum spanning tree problem is not polynomially approximable within a constant factor ρ: for all $\rho \geq 1$, there is no polynomial time ρ-approximation algorithm regarding cost when respecting the degree constraint even when the same upper bound is given for all the vertices.

However, network related applications (as routing) do not always necessitate a sub-graph as solution. Indeed, the route must always be connected from the source to the destination(s) but it can follow a scheme different from a simple sub-graph. We suppose that the route can return to vertices several times and the degree constraint concerns each passage in the spanning structure. For example, in optical routing, an optical switch can be traversed several times by the same wavelength but for each passage, the splitting of the signal is limited [12]. In order to efficiently solve the routing, we try to find a connected minimum cost spanning structure without the hypothesis that it is a sub-graph. For this, we reformulate the problems and we solve them using a hierarchical structure related to graphs (called *hierarchy*) which can refer vertices and edges in the topology graph several times.

In the following, we propose a fast overview of degree-constrained spanning hierarchies (Section 2). Section 3 resumes our most important results solving these problems. The paper is closed by some perspectives.

2. Hierarchies and degree-constrained spanning structures in graphs

Let us examine a simple analogy. In *elementary walks* vertices (and thus edges) are not repeated but *non-elementary walks* can contain vertices and edges several times. Let $B = (W, F)$ and $G = (V, E)$ be two (undirected) graphs. Remember, that an application $h : W \rightarrow V$ associating a vertex in V to each vertex in W is a homomorphism if the mapping preserves the adjacency: $(u, v) \in F$ implies $(h(u), h(v)) \in E$. A triplet (B, h, G) using a homomorphic mapping h can be applied to define spanning structures in G. In [4], both elementary and non elementary walks have been defined with the help of a homomorphic mapping of elementary paths to graphs. The homomorphic mapping between a tree T and a graph G may be used to span G.

Let $T = (W, F)$ be a tree. Let $h : W \rightarrow V$ be a homomorphism which associates a vertex $v \in V$ to each vertex $w \in W$. The application (T, h, G) defines a *hierarchy* in G.

It corresponds to a "non-elementary tree" in the graph G, which can return to some vertices and edges. Unlike walks, in hierarchies some vertices are eventually branching vertices (there are

the vertex occurrences corresponding to the branching vertices of the tree T). Figure 1 illustrates a hierarchy.

Regarding the mapping of vertices from G to the original tree T, a hierarchy can also be given by two multi-sets: $H = (U, D)$ where U is the multi-set of the concerned vertices and D is the multi-set of edges in H using the labels from G. More details can be found in [7].

In the above mentioned routing problems, the route may be a connected spanning structure different from a tree. Messages must follow the route (which can be different from a sub-graph) from the source to the destinations. For this, the problem can be reformulated. We are looking at connected minimum cost spanning structures covering the desired vertex set $M \subseteq V$ s.t. each vertex occurrence in the structure respects a degree constraint $D > 1$.

3. First results

Analysing and solving the new problem, the following important results are obtained.

A. The minimum cost connected structure spanning M and respecting the constraints is always a hierarchy (cf. the proof in [7]).

B. A first ILP-based formulation of the optimal spanning hierarchy for homogeneous degree bound was proposed in [6]. Since the optimal hierarchy can return to vertices and edges several times, its computation is not trivial. Similarly to the graph $2G = (V, 2E)$ discussed in [10], we solved a first flow problem in the graph $mG = (V, mE)$, where m is the maximum number of the edge repetitions in the optimal hierarchy. Then a second greedy decision permits to obtain the optimal hierarchy from the flow obtained in mG. We compared the spanning tree and spanning hierarchy solutions. In randomly generated sparse graphs with $15 \leq |V| \leq 50$ and with density 2 the following results have been obtained.

C. Spanning hierarchies satisfying the constraints can always successfully solve the routing even if spanning trees do not exist. In the sparse random graphs, the failure rate for spanning trees varies between 45% and 90% with $D = 2$ and between 4% and 46% with $D = 3$.

D. Spanning hierarchies may be cheaper than the existing spanning trees satisfying the constraints. Comparing the costs of the minimum spanning trees and minimum spanning hierarchies in random graphs (if both solutions exist), we found a clear advantage for hierarchies. The average percentage of improvement of the cost using hierarchies varied between 18% and 30% when $D = 2$ and between 8% and 22% when $D = 3$. The improvement increases with the graph size. The same results have been found in a similar spanning problem [5].

E. The DCMSH problem is in APX while the DCMST is not. We proposed an approximation offering a guarantee of $\frac{D}{D-1}$ in [8]. The outline of the approximation is as follows. Since the MST corresponds to a lower bound of the optimal cost, we span an MST with a hierarchy. This hierarchy respects the degree constraints by returning to some branching vertices of the MST if needed. To compute the overhead of returns, the MST is decomposed into a set of stars and cost efficient (cost limited) edge repetitions are decided to cover the stars. An example of this kind of span of a tree (of an MST) is illustrated in Figure 1. Since the the degree constrained minimum cost spanning hierarchy may be significantly different from the MST, the approximation is not the best but spanning an MST, it is the best possible.

4. Conclusions and Perspectives

The newly proposed hierarchy concept is advantageous to solve degree-constrained spanning problems. Trees do not permit the repetition of graph elements but hierarchies, obtained from

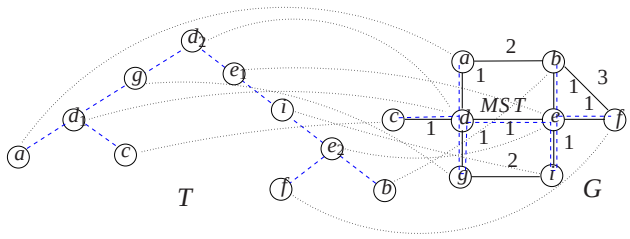

Figure 1: Mapping of vertices for a hierarchy spanning an MST

trees by homomorphism, allow it. In degree bounded spanning problems, we demonstrated that spanning hierarchies exist even if spanning trees can not meet the constraints and hierarchies provide better solutions even when spanning trees exist. The spanning hierarchy problem is in APX (while DCMST and DCST are not) and has a direct application in WDM multicast routing.

The analysis of minimum cost spanning problems under constraints promises further interesting challenges. Generally, these constrained connected spanning problems are NP-hard and the solutions are spanning hierarchies. By reformulating the constrained spanning problems and proposing hierarchies instead of trees new conditions are obtained and new problems can be met. In future works, the different spanning hierarchy problems should be analyzed from the point of view of their computational hardness, complexity, approximability, etc. In some applications, the computation of optimal hierarchies can be expensive and cannot be tolerated. Important research work should investigate the fast computation of advantageous spanning hierarchies for constrained spanning problems and related applications.

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