A Constructive Argumentation Framework

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Abstract

Dung's argumentation framework is an abstract framework based on a set of arguments and a binary attack relation defined over the set. One instantiation, among many others, of Dung's framework consists in constructing the arguments from a set of propositional logic formulas. Thus an argument is seen as a reason for or against the truth of a particular statement. Despite its advantages, the argumentation approach for inconsistency handling also has important shortcomings. More precisely, in some applications what one is interested in are not so much only the conclusions supported by the arguments but also the precise explications of such conclusions. We show that argumentation framework applied to classical logic formulas is not suitable to deal with this problem. On the other hand, intuitionistic logic appears to be a natural alternative candidate logic (instead of classical logic) to instantiate Dung's framework. We develop *constructive argumentation framework*. We show that intuitionistic logic offers nice and desirable properties of the arguments. We also provide a characterization of the arguments in this setting in terms of minimal inconsistent subsets when intuitionistic logic is embedded in the modal logic S4.

Introduction

Argumentation theory is a reasoning process based on constructing arguments, determining conflicts between arguments and determining acceptable arguments. Dung's argumentation framework is an abstract framework based on a set of arguments and a binary attack relation defined over the set (Dung 1995). In this framework, an argument is an abstract entity whose origin and structure are not known. The role of an argument is only described by its conflicts with other arguments. The abstract nature of Dung's framework accounts for the broad range of its applications.

We distinguish between two main trends for extending/instantiating Dung's framework: those argumentation frameworks which use (as in Dung's framework) abstract arguments, e.g. (Bench-Capon 2003), and those which take into account the internal structure of the arguments, e.g., (Simari and Loui 1992; Besnard and Hunter 2008). The present paper follows the second trend. In particular we focus on the case where the arguments are built from a set of classical propositional logic formulas. Therefore an argument is seen as a reason for or against the truth of a particular statement. In particular, an argument is a pair of (1) a set of classical propositional formulas obeying some conditions, called the support of the argument, and (2) a logical formula inferred from the set, called the conclusion of the argument. Given a set of arguments (constructed from a set of classical propositional formulas), the classical treatment of argumentation framework consists in determining conflicts over the arguments and determining acceptable arguments. Conclusions supported by acceptable arguments are consistent and considered reliable. Although works around argumentation frameworks have shown great promises for reasoning with inconsistent knowledge, the argumentation approach also has important shortcomings in this setting. More precisely, in some applications what one is interested in are not so much only the conclusions supported by the acceptable arguments but also the explications of such conclusions. Unfortunately, in some situations the internal structure of the acceptable arguments is not sufficient to provide such information. Let us consider the following example to illustrate the problem:

- If John has taken his medication (M), then his situation is stable (S).
- If John has not taken his medication, then a doctor administers the medication to him (D).
- If a doctor administers the medication to John, then the situation of John is stable.

From the previous statements, we can deduce that the situation of John is stable (S) using classical reasoning. Indeed, this can be easily obtained using the law of excluded middle ($M \vee \neg M$): on one hand, if M is true then S is true. On the other hand, if $\neg M$ is true then D is true which allows us to conclude that S is true too. It goes without saying that this situation is extremely confusing as, although we know that the situation of John is stable, we are not able to provide the precise reason for such a conclusion if we don't know whether John has taken his medication or not! For instance, if John lost consciousness, we, including the doctor, cannot determine which of M or $\neg M$ is true in $M \vee \neg M$. Clearly we need an argumentation framework over a logic which, contrary to classical logic, goes beyond the classical deduction of the conclusion of an argument from the support of the argument. This consideration calls for more sophisticated logics. In other words, what we need is not only the information from which a certain conclusion is derived but also that information which permit to construct the conclusion at hand. This consideration is formally known in the literature as "constructivism".

In mathematics, *constructivism* is a philosophical approach which consists in requiring that in order to prove that a mathematical object exists, it is necessary to be able to construct it. In this context, intuitionistic propositional logic can be seen as a formalization of constructive mathematics of Brouwer and Heyting (Beeson 1984; Troelstra and van Dalen 1988). It is defined starting from classical logic by excluding certain standard forms of reasoning, in particular, the law of excluded middle and double-negation elimination. A statement in intuitionistic logic is considered as true if we are able to provide a constructive proof, and as false if we are able to prove that it does not have a constructive proof. In a sense, the truth value of a statement is more related to our knowledge than in classical logic. For instance, the formula $M \vee \neg M$ is true in intuitionistic logic only if we provide a constructive proof of M or a constructive proof of $\neg M$, whereas this formula is true in classical logic without such proofs. Thus, if we consider the previous example, it is not possible to derive S in intuitionistic logic from the given statements.

In the setting of argumentation framework for handling inconsistent logical formulas, intuitionistic logic appears to be a suitable tool to address the above considerations. In this paper, we develop a *constructive argumentation framework* which builds on Dung's framework and intuitionistic logic.

Intuitionistic Propositional Logic

We first define the syntax of intuitionistic propositional logic (IPL for short). Let Prop be a denumerable set of propositional variables whose elements are denoted by p, q, r, etc. The symbols $\vee, \wedge, \rightarrow$ denote disjunction, conjunction and implication, respectively. The set of formulæ of IPL, denoted Form, is defined from Prop and the constant $\perp$, denoting absurdity, by using the following grammar:

$$A ::= p \mid \perp \mid A \wedge A \mid A \vee A \mid A \rightarrow A.$$

The logical connective of negation $\neg$ is defined by using $\perp$ and the connective $\rightarrow$: $\neg A \equiv A \rightarrow \perp$. The logical connective of equivalence, denoted $\leftrightarrow$, is defined as usual.

We provide here a possible world semantics, called Kripke semantics, of intuitionistic logic. In this semantics, we use a universe of worlds where each propositional variable has a truth value in each world.

Definition 1 (Kripke Model). *Let W be the universe of worlds. A Kripke model is defined as a triple $(W, \preceq, V)$, where $\preceq$ is a preorder over W and $V : W \rightarrow 2^{\mathsf{Prop}}$ is an interpretation such that, for all w and w' in W with $w \preceq w'$, $V(w) \subseteq V(w')$.*

We associate to each Kripke model $\mathcal{M} = (W, \preceq, V)$ a forcing relation, denoted $\vDash_{\mathcal{M}}$, between W and Form. It is defined by induction on formula structure as follows:

$w \vDash_{\mathcal{M}} p$ iff $p \in V(w)$;

$w \vDash_{\mathcal{M}} \perp$ never holds;

$w \vDash_{\mathcal{M}} A \wedge B$ iff $w \vDash_{\mathcal{M}} A$ and $w \vDash_{\mathcal{M}} B$;

$w \vDash_{\mathcal{M}} A \vee B$ iff $w \vDash_{\mathcal{M}} A$ or $w \vDash_{\mathcal{M}} B$;

$w \vDash_{\mathcal{M}} A \rightarrow B$ iff, for all $w' \in W$ with $w \preceq w'$, if $w' \vDash_{\mathcal{M}} A$ then $w' \vDash_{\mathcal{M}} B$.

Note that $\vDash_{\mathcal{M}}$ satisfies Kripke monotonicity property:

Proposition 1. *Let A be a formula, $\mathcal{M} = (W, \preceq, V)$ and $w, w' \in W$. If $w \vDash_M A$ and $w \preceq w'$, then $w' \vDash A$.*

A formula A is satisfiable in IPL if there exists a Kripke model $\mathcal{M} = (W, \preceq, V)$ and a world w in W such that $w \vDash_M A$. A is valid in IPL if, for all Kripke model $\mathcal{M} = (W, \preceq, V)$ and for all w in W, $w \vDash_M A$. Satisfiability and validity in IPL are Polynomial-Space Complete (Statman 1979).

For logical consequence concept, we use the turnstile symbol "$\vdash_i$" (i for intuitionistic propositional logic), i.e., $\{A_1, \ldots, A_n\} \vdash_i B$ and $(A_1 \wedge \cdots \wedge A_n) \rightarrow B$ are equivalent. Similarly to classical logic, we have the deduction property in intuitionistic logic:

Proposition 2. *Let Γ be a set of formulæ, and A and B two formulæ. Then, $\Gamma \vdash_i A \rightarrow B$ iff $\Gamma \cup \{A\} \vdash_i B$.*

Classical logic is stronger than intuitionistic logic. Therefore a valid formula in the latter is also valid in the former.

Classical vs Constructive Arguments

In this section, we first present the usage of Dung's framework for handling inconsistency in classical propositional logic knowledge bases (Besnard and Hunter 2008). We then discuss how this instantiation can or can not be directly applicable with intuitionistic propositional logic. Essentially, a logic-based argumentation framework operates in the following steps:

1. constructing arguments (in favor of/against a conclusion) from knowledge bases,
2. determining the conflicts, called an attack relation, between the arguments,
3. and determining the acceptable arguments from which justified conclusions are concluded.

Dung's argumentation framework is a pair $\langle \mathbb{A}, Att \rangle$, where $\mathbb{A}$ is the set of arguments and Att is the attack relation over $\mathbb{A} \times \mathbb{A}$. Acceptability semantics define sets of arguments that should satisfy some conditions in order to represent a justifiable point of view on the acceptance of the arguments. Due to the lack of space we do not recall these semantics and refer the reader to (Dung 1995). The notion of argument is defined on the basis of the underlying logical language and its associated logical consequence.

Classical Argumentation Framework

When Dung's argumentation framework is used to deal with inconsistent knowledge encoded in classical propositional logic, an argument is defined in the following way:

Definition 2 (Argument). *Let Γ be a set of classical propositional formulas. An argument over Γ is a pair $\mathcal{A} = \langle \Delta, C \rangle$ such that $\Delta \subseteq \Gamma$, $\Delta \nvdash_c \bot$, $\Delta \vdash_c C$ (c for classical propositional logic) and, for all $\Delta' \subset \Delta$, $\Delta' \nvdash_c C$.*

The set Δ is called the support of the argument and C its conclusion. We say that $\mathcal{A}$ is an argument for C. In this setting, $\langle \Delta, C \rangle$ is called a classical argument. For instance, consider $\Gamma = \{A \to C, \neg A \to B, B \to C\}$. The pair $\langle \Delta, C \rangle$, with $\Delta = \Gamma$, is a classical argument. Given two arguments $\langle \Delta, C \rangle$ and $\langle \Delta', C' \rangle$, we say that $\langle \Delta, C \rangle$ undercuts $\langle \Delta', C' \rangle$ iff for some $\phi \in \Delta'$, $C \equiv \neg\phi$. $\langle \Delta, C \rangle$ rebuts $\langle \Delta', C' \rangle$ iff $C \equiv \neg C'$. Then, $\langle \Delta, C \rangle$ attacks $\langle \Delta', C' \rangle$ iff $\langle \Delta, C \rangle$ rebuts/undercuts $\langle \Delta', C' \rangle$[1].

The practical computation of the arguments is far from being a straightforward problem. Given the setting of classical propositional logic and the properties of an argument, the authors of (Besnard et al. 2010) have proposed an approach based on insights from SAT problem (Grégoire, Mazure, and Piette 2009). More precisely, let $\langle \Delta, C \rangle$ be a classical argument. From Definition 2 we have that Δ is minimal. Moreover we have $\Delta \vdash_c C$ which is equivalent to $\Delta \cup \{\neg C\}$ being inconsistent, i.e., $\Delta \cup \{\neg C\} \vdash_c \bot$. Therefore $\Delta \cup \{\neg C\}$ is minimally inconsistent. The subset is inconsistent but all its proper subsets are consistent. From the previous equivalence, a classical argument is characterized in the following way (Besnard and Hunter 2008):

$\langle \Delta, C \rangle$ is a classical argument if and only if $\Delta \cup \{\neg C\}$ is a minimal inconsistent subset.

In SAT terminology, a minimal inconsistent subset is called MUS (for Minimally Unsatisfiable Subset). An algorithm is provided in (Besnard et al. 2010) to generate the set of arguments in an efficient way.

Constructive Argumentation Framework

Let us now focus our attention on our main problem, namely argumentation for intuitionistic propositional logic. We define a constructive argumentation framework as an instantiation of Dung's framework over intuitionistic propositional logic. It also operates in the three steps previously described. The main difference however consists in the logical consequence. More precisely due to the use of intuitionistic (instead of classical) propositional logic, the logical consequence $\vdash_c$ in Definition 2 is replaced with $\vdash_i$. The argument is then called constructive. The definitions of the attack relation and acceptability semantics remain unchanged.

Having defined what constructive argumentation framework is, we naturally come to the question "How do classical and constructive argumentation frameworks relate to each other?" While the definition of the attack relation and acceptability semantics are identical, we will show that the notion of argument and its characterization with a minimally inconsistent subset fall down in the setting of intuitionistic propositional logic. For this purpose, readers need not to have a strong background on IPL. What they need to know in this section are the following properties[2]: *(i)* $\{\neg\neg A\} \vdash_i A$ does not hold, *(ii)*$A \to \neg\neg A$ is a theorem of IPL and *(iii)* $A \vee \neg A$ is not a theorem of IPL.

With this in mind, we can first state that:

a classical argument is not necessarily a constructive argument.

For example $\langle \{A \to C, \neg A \to B, B \to C\}, C \rangle$ is a classical argument but not a constructive one.

Moreover the characteristic property of a classical argument stating that $\langle \Delta, C \rangle$ is a classical argument if and only if $\Delta \cup \{\neg C\}$ is a minimal inconsistent subset does not hold with constructive arguments. For instance, the set $\{\neg\neg p, \neg p\}$ is a minimal inconsistent subset, but $\langle \{\neg\neg p\}, p \rangle$ is not a constructive argument. This is because of $\{\neg\neg p\} \nvdash_i p$. Indeed, the formula $\neg\neg p \to p$ has as a counter-model $\mathcal{M} = (\{w, w'\}, \preceq, V)$ with $w \preceq w'$, $V(w) = \emptyset$ and $V(w') = \{p\}$. In this Kripke model, we have $w \vDash_{\mathcal{M}} \neg\neg p$, since $w' \vDash_{\mathcal{M}} p$, and $w \nvDash_{\mathcal{M}} p$.

However, we have the following property:

Proposition 3. *If $\langle \Delta, C \rangle$ is a constructive argument, then $\Delta \cup \{\neg C\}$ is an inconsistent subset.*

Proof. We have $\Delta \vdash_i C$, since the pair $\langle \Delta, C \rangle$ is a constructive argument. Moreover, we have $C \vdash_i \neg\neg C$ ($C \to \neg\neg C$ is a valid formula in IPL). Thus, using $\Delta \vdash_i C$ and $C \vdash_i \neg\neg C$, we obtain $\Delta \vdash_i \neg\neg C$. From Prop. 2, we deduce that $\Delta \cup \{\neg C\}$ is an inconsistent set, since $\neg\neg C \equiv \neg C \to \bot$. □

The inconsistent subset in the previous proposition is not necessarily minimal. Indeed, the pair $\langle \{\neg\neg p, \neg\neg p \to p\}, p \rangle$ is a constructive argument, but $\{\neg\neg p, \neg\neg p \to p, \neg p\}$ is not a minimal inconsistent set because $\{\neg\neg p, \neg p\}$ is an inconsistent set.

Nonetheless, constructive arguments can be characterized by means of minimal inconsistent subsets when the conclusion of the argument has the form $\neg C$. Formally, we have:

Proposition 4. *The pair $\langle \Delta, \neg C \rangle$ is a constructive argument iff $\Delta \cup \{C\}$ is a minimal inconsistent subset.*

Proof.

Part $\Rightarrow$. Using Prop. 2, we have if $\Delta \vdash_i \neg C$ then $\Delta \cup \{C\} \vdash_i \bot$. Hence, $\Delta \cup \{C\}$ is an inconsistent subset. If $\Delta \cup \{C\}$ is not a minimal inconsistent subset then there exists $\Delta' \subset \Delta$ such that $\Delta' \vdash_i \neg C$. We get a contradiction since Δ is minimal from Def. 2, i.e., no proper subset of Δ deduces $\neg C$. Therefore, $\Delta \cup \{C\}$ is a minimal inconsistent subset.

Part $\Leftarrow$. If $\Delta \cup \{C\}$ is a minimal inconsistent subset, then we have $\Delta \nvdash_i \bot$, $\Delta \vdash_i \neg C$ (Prop. 2) and, for all $\Delta' \subset \Delta$, $\Delta' \nvdash_i \neg C$. Thus, the pair $\langle \Delta, \neg C \rangle$ is a constructive argument. □

Proposition 4 comes from the fact that constructing the negation of a formula corresponds to having a contradiction ($\bot$) from this formula considered as an hypothesis, since $\neg C$ is seen as the formula $C \to \bot$ in IPL.

Lastly, let us emphasize that a constructive argument can be seen as a classical argument augmented with additional

[1] In some argumentation systems, called preference-based argumentation frameworks, the attack relation is derived from a conflict relation and a preference relation over the arguments. However this is not the main focus in this paper.

[2] A formal exposition of IPL and its consequence in argumentative reasoning will be given in the next section.

information that allow us to comply with the principles of constructivism.

Proposition 5. *If $\langle \Delta, C \rangle$ is a constructive argument, then there exists $\Delta' \subseteq \Delta$ s.t. $\langle \Delta', C \rangle$ is a classical argument.*

The proof of the previous proposition relies on the fact that each valid formula in IPL is also valid in classical logic.

Besides the fact that constructive arguments provide a way to construct a given conclusion, they also prevent some undesirable justifications when classical arguments are dealt with. By undesirability we mean here that the justification is misleading. In order to illustrate this point, we consider a simple medical diagnostic framework. We use rules of the form $S_1 \wedge \cdots \wedge S_n \to D$, where $S_1, \ldots, S_n$ denote symptoms and D a disease. A diagnosis consists in searching for a rule for which the left part matches symptoms of a patient. Consider the following rules and symptoms of a patient: $S_1 \to D$, $(\neg S_1 \wedge S_2) \to D$, S_1, S_2. We fix Γ as the set of the previous formulæ. The pair $\mathcal{A} = \langle \{S_1 \to D, (\neg S_1 \wedge S_2) \to D, S_2\}, D \rangle$ is a classical argument over Γ. Clearly this argument cannot be considered as such. This is because it is based on the law of excluded middle. Indeed, by using the validity of $S_1 \vee \neg S_1$, if S_1 then D is true because of the rule $S_1 \to D$; otherwise, $\neg S_1$ is true and, by using the rule $(\neg S_1 \wedge S_2) \to D$, D is also true, since we have S_2 in the support. Therefore this argument suggests that D is true because S_1 is true or $\neg S_1 \wedge S_2$ is true. However only S_1 is true and $\neg S_1 \wedge S_2$ cannot by no mean considered as a possible justification of D. The argument $\mathcal{A}$ is not constructive because it is based on the law of excluded middle. There exists a single constructive argument over Γ having D as conclusion which is the pair $\langle \{S_1, S_1 \to D\}, D \rangle$. This argument provides the justification explaining why the patient has the disease D, i.e., the patient has the symptom S_1. Indeed we can say that constructive arguments get rid of imprecision (S_1 or $\neg S_1 \wedge S_2$) and provide arguments with a precise justification, namely S_1 in the previous example.

Properties of Intuitionistic Logic: Application to Constructive Arguments

We have shown in the previous section that instantiating Dung's framework with intuitionistic logic defines a new argumentation framework in which the notion of argument and its characterization with a minimal inconsistent set completely differs from classical argumentation framework. Not only does the new framework compute an argument in favor of a statement but it also builds the precise justification for that statement. Given the virtues of IPL, in this section we go into a more detailed exposition of this logic. We illustrate its applicability on argumentation reasoning.

Non-Interdefinability of Connectives

In classical logic, it is possible to define the logical connective $\wedge$ (resp. $\vee$) by using $\vee$ and $\neg$ (resp. $\wedge$ and $\neg$) which is not possible in intuitionistic logic. Indeed in intuitionistic logic, it is not possible to reformulate the logical connectives $\wedge$, $\vee$ and $\to$. The most of de Morgan laws are not valid in this logic. For instance, the formula $A \to B$ is not equivalent to $\neg A \vee B$. Moreover, double-negation elimination is excluded from IPL, i.e., A is not equivalent to $\neg\neg A$.

Intuitively reasoning in intuitionistic logic is not carried out in terms of "true" and "false", but in terms of "proof" and "contradiction". For instance, the formula $\neg(A \wedge B)$ means that we can prove from both A and B that we get a contradiction. However it does not mean that we can prove that (i) we get a contradiction from A or (ii) we get a contradiction from B as it may be suggested by $(\neg A \vee \neg B)$.

For instance, consider the formula $\neg(p \wedge q) \to (\neg p \vee \neg q)$ which is valid in classical logic. This formula is not valid in intuitionistic logic because it admits the countermodel $\mathcal{M} = (\{w, w', w''\}, \preceq, V)$ where $\preceq$ is defined by $w \preceq w'$ and $w \preceq w''$, and $V(w) = \emptyset$, $V(w') = \{p\}$ and $V(w') = \{q\}$. Indeed, we have $w \vDash_{\mathcal{M}} \neg(p \wedge q)$, because of $w' \nvDash_{\mathcal{M}} p \wedge q$ and $w'' \nvDash_{\mathcal{M}} p \wedge q$; and we have $w \nvDash_{\mathcal{M}} \neg p$ because of $w' \vDash_{\mathcal{M}} p$, and $w \nvDash_{\mathcal{M}} \neg q$ because of $w'' \vDash_{\mathcal{M}} q$.

Let us consider the following statements:

1. Peter cannot be an owner and a tenant of an apartment: $\neg(O \wedge T)$.
2. Peter is an owner of an apartment: O.

We put $\Gamma = \{\neg(O \wedge T), O\}$. The pair $\mathcal{A} = \langle \{\neg(O \wedge T)\}, \neg O \vee \neg T \rangle$ is a classical argument over Γ. This means that from $\neg(O \wedge T)$ we obtain that Peter is not an owner of an apartment or he is not a tenant of an apartment. Note that the conclusion of $\mathcal{A}$ is obtained by using one of the following instances of the law of excluded middle: $(O \vee \neg O)$ and $(T \vee \neg T)$. Indeed, if we have O (resp. T) then, by using $\neg(O \wedge T)$, we have $\neg T$ (resp. $\neg O$). Otherwise, we have $\neg O$ (resp. $\neg T$). Thus, the argument $\mathcal{A}$ allows us to know $\neg O \vee \neg T$ without precisely knowing whether Peter is not an owner of an apartment ($\neg O$) or he is not a tenant of an apartment ($\neg T$). The unique constructive argument over Γ having $\neg O \vee \neg T$ as conclusion is $\langle \{O, \neg(O \wedge T)\}, \neg O \vee \neg T \rangle$. This argument is constructive because we know that Peter is not a tenant ($\neg T$) and, *a fortiori*, we have $\neg O \vee \neg T$.

As a second example, consider the formula $(p \to q) \to (\neg p \vee q)$ which is not valid in intuitionistic logic. A countermodel, among others, of this formula is $\mathcal{M} = (\{w, w'\}, \preceq, V)$ where $V(w) = \emptyset$, $V(w') = \{p, q\}$ and $\preceq$ is defined by $w \preceq w'$. This is obtained from $w \vDash_{\mathcal{M}} p \to q$, $w \nvDash_{\mathcal{M}} \neg p$ because of $w' \vDash_{\mathcal{M}} p$, and $w \nvDash_{\mathcal{M}} p$.

In order to illustrate the fact that the formula $(p \to q) \to (\neg p \vee q)$ is not valid in intuitionistic logic, consider the following statements:

- If Peter is a tenant, then he will buy an apartment: $T \to B$.
- Peter is a tenant: T.

We put $\Gamma = \{T \to B, T\}$. In the classical argument $\langle \{T \to B\}, \neg T \vee B \rangle$ we know that Peter is not a tenant or he will buy an apartment without exactly knowing which of these statements is true. In this context, constructivism requires to know this information in order to obtain the conclusion $\neg T \vee B$. For instance, $\langle \{T, T \to B\}, \neg T \vee B \rangle$ is a constructive

$$\frac{}{\Gamma, A \vdash A}\ [Id] \qquad \frac{}{\Gamma, \bot \vdash C}\ [\bot]$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \land B \vdash C}\ [\land_L] \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \land B}\ [\land_R]$$

$$\frac{\Gamma \vdash A_i}{\Gamma \vdash A_1 \lor A_2}\ [\lor_R^{i \in \{1,2\}}] \qquad \frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash A}{\Gamma, A \lor B \vdash C}\ [\lor_L]$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\ [\to_R]$$

$$\frac{\Gamma, A \to B \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}\ [\to_L]$$

Figure 1: Sequent Calculus G_{IPL}

argument because its support allows us to know that Peter will buy an apartment.

As a third example, consider the following statement: it is note true that if the suspect is guilty then he confesses his crime ($\neg(G \to C)$). From this statement, one can deduce that the suspect is guilty in classical reasoning. Indeed, the pair $\langle\{\neg(G \to C)\}, G\rangle$ is a classical argument, since we have $\neg(G \to C) \equiv \neg(\neg G \lor C) \equiv G \land \neg C$. This comes from the interdefinability of connectives in classical logic and the double-negation elimination. However, the previous pair is not a constructive argument, since we are not able to construct G from $\neg(G \to C)$.

Disjunction Property

The disjunction property is one of the most important properties satisfied in intuitionistic logic. It says that if a formula $A \lor B$ is valid, then A is valid or B is valid. This property is not satisfied in classical logic. Indeed, the formula $p \lor \neg p$ is valid without p and $\neg p$ being individually valid in classical logic. From the point of view of constructivism, the disjunction property says that to construct the object $A \lor B$, it is necessary to be able to construct at least one of the objects A and B.

Here we use the fact that intuitionistic logic enjoys the disjunction property to show that if a constructive argument has a support Δ which does not contain the disjunction connective, and a conclusion of the form $A \lor B$, then from Δ we can construct one of the formulas A and B, i.e., $\langle\Delta, A\rangle$ or $\langle\Delta, B\rangle$ is a constructive argument. In order to show this property, we use a proof system for IPL in the sequent calculus formalism-style.

Let us recall that an *inference rule* has the following form:

$$\frac{P_1 \ \cdots \ P_n}{C}\ [R]$$

where $[R]$ is its name, C its *conclusion* and $P_1, \ldots, P_n$ its *premises*. An *axiom* can be seen as a rule without premises. A *proof system* is defined as a set of inference rules. Proof-search in a sequent calculus corresponds to a bottom-up construction of derivations using its inference rules, i.e., a construction from the conclusion to axioms.

We consider here the sequent calculus G_{IPL} for IPL described in Figure 1 (see (Troelstra and Schwichtenberg 1996)). A sequent $\mathcal{S}$ has a proof in G_{IPL} if it has a finite derivation in G_{IPL} where each leaf node is labeled with an axiom. For instance, we provide a proof of $\{p \to r, q \to r\} \vdash p \lor q \to r$ using G_{IPL} in Figure 2.

We now show a property satisfied by constructive arguments that comes from the disjunction property.

Proposition 6. *If $\langle\Delta, A \lor B\rangle$ is a constructive argument and $\lor$ does not appear in Δ, then $\langle\Delta, A\rangle$ or $\langle\Delta, B\rangle$ is a constructive argument.*

Proof. By induction on the proof of $\Delta \vdash A \lor B$ in the sequent calculus G_{IPL}. Note that $\Delta \vdash A \lor B$ can not be an instance of an axiom ($[\bot]$ and $[Id]$), since $\Delta \nvdash \bot$ and $A \lor B$ is not a subformula of Δ. If the last application rule is a right rule, then it is an instance of $[\lor_R]$. Hence, we have a proof of $\Delta \vdash A$ or $\Delta \vdash B$ in G_{IPL}. Consequently, $\langle\Delta, A\rangle$ is a constructive argument or $\langle\Delta, B\rangle$ is a constructive argument, since $\Delta \nvdash \bot$ and Δ is minimal. We now consider the case where the last application rule is an instance of a left rule. In this case, the last application rule is an instance of either $[\land_L]$ or $[\to_L]$.

In the case of $[\land_L]$:

$$\frac{\Delta', C, D \vdash A \lor B}{\Delta', C \land D \vdash A \lor B}\ [\land_L]$$

where $\Delta = \Delta', A \land B$, by applying the induction hypothesis on $\Delta', C, D \vdash A \lor B$, we obtain a proof of $\Delta', C, D \vdash A$ or $\Delta', C, D \vdash B$ in G_{IPL}. Hence, we have a proof of $\Delta', C \land D \vdash A$ or $\Delta', C \land D \vdash B$ in G_{IPL}. Therefore, (Δ, A) is a constructive argument or (Δ, B) is a constructive argument.

In the case of $[\to_L]$:

$$\frac{\Delta', C \to D \vdash C \qquad \Delta', D \vdash A \lor B}{\Delta', C \to D \vdash A \lor B}\ [\to_L]$$

where $\Delta = \Delta', A \to B$, by applying the induction hypothesis on $\Delta', D \vdash A \lor B$, we obtain a proof of $\Delta', D \vdash A$ or $\Delta', D \vdash B$ in G_{IPL}. Hence, using the rule $[\to_L]$, we have a proof of $\Delta', C \to D \vdash A$ or $\Delta', C \to D \vdash B$ in G_{IPL}. Therefore, $\langle\Delta, A\rangle$ or $\langle\Delta, B\rangle$ is a constructive argument. □

Note that Proposition 6 is not satisfied in classical logic. For instance, the pair $\langle\{\neg(p \land q)\}, \neg p \lor \neg q\rangle$ is a classical argument where the support does not contain the disjunction connective. However, neither $\langle\{\neg(p \land q)\}, \neg p\rangle$ nor $\langle\{\neg(p \land q)\}, \neg q\rangle$ are classical arguments.

Computing Constructive Arguments using Modal Logic S4

In this section we borrow from classical argumentation framework the characterization of classical arguments in terms of minimal inconsistent sets. As shown in (Besnard et al. 2010) this characterization offers nice tractability properties for computing classical arguments. We provide such a characterization for constructive arguments using modal logic S4 (Blackburn, de Rijke, and Venema 2001).

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \to r, q \to r, p \vdash p}[Id] \quad \dfrac{}{q \to r, p, r \vdash r}[Id]}{p \to r, q \to r, p \vdash r}[\to_L] \qquad \dfrac{\dfrac{}{p \to r, q \to r, q \vdash q}[Id] \quad \dfrac{}{p \to r, q, r \vdash r}[Id]}{p \to r, q \to r, q \vdash r}[\to_L]}{p \to r, q \to r, p \vee q \vdash r}[\vee_L]}{p \to r, q \to r \vdash p \vee q \to r}[\to_R]$$

Figure 2: A proof in G_{IPL}

Modal Logic S4

The set of S4 formulæ is obtained by extending the propositional language with the modal connectives $\Box$ et $\Diamond$:

$$A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A \mid \Diamond A.$$

Similarly to intuitionistic logic, modal logic S4 has a possible world semantics where we use a universe of worlds with an accessibility relation between worlds which is reflexive and transitive. More precisely, an S4 model is a triple $\mathcal{M} = (W, \preceq, V)$, where $\preceq$ is a preorder over W and $V : W \to 2^{\mathsf{Prop}}$ is an interpretation. Hence, a S4 model can be seen as a Kripke model of intuitionistic logic without Kripke monotonicity property.
The forcing relation, denoted $\vDash^{S4}_{\mathcal{M}}$, is inductively defined on formula structure as follows:

- $w \vDash^{S4}_{\mathcal{M}} p$ iff $p \in V(w)$; $w \vDash^{S4}_{\mathcal{M}} \bot$ never holds;
- $w \vDash^{S4}_{\mathcal{M}} A \wedge B$ iff $w \vDash^{S4}_{\mathcal{M}} A$ and $w \vDash^{S4}_{\mathcal{M}} B$;
- $w \vDash^{S4}_{\mathcal{M}} A \vee B$ iff $w \vDash^{S4}_{\mathcal{M}} A$ or $w \vDash^{S4}_{\mathcal{M}} B$;
- $w \vDash^{S4}_{\mathcal{M}} A \to B$ iff if $w \vDash^{S4}_{\mathcal{M}} A$ then $w \vDash^{S4}_{\mathcal{M}} B$;
- $w \vDash^{S4}_{\mathcal{M}} \Diamond A$ iff $\exists w' \in W$ s.t. $w \preceq w'$ and $w' \vDash^{S4}_{\mathcal{M}} A$;
- $w \vDash^{S4}_{\mathcal{M}} \Box A$ iff $\forall w' \in W$ with $w \preceq w'$, $w' \vDash^{S4}_{\mathcal{M}} A$;

Embedding Intuitionistic Logic into S4

We describe here Gödel's embedding of intuitionistic logic into modal logic S4 (see, e.g., (Troelstra and Schwichtenberg 1996)). Intuitively, this embedding comes from the fact that the Kripke models of intuitionistic logic are S4 models. The definition of the embedding $(\cdot)^{S4}$ is by induction on formula structure as follows:

$(p)^{S4} = \Box p$; $(\bot)^{S4} = \bot$; $(A \wedge B)^{S4} = (A)^{S4} \wedge (B)^{S4}$;
$(A \vee B)^{S4} = (A)^{S4} \vee (B)^{S4}$;
$(A \to B)^{S4} = \Box((A)^{S4} \to (B)^{S4})$.

We have the following property:

Proposition 7. *$\Gamma \vdash_i C$ is valid in intuitionistic logic iff $(\Gamma)^{S4} \vdash (C)^{S4}$ is valid in S4.*

Hence, since the logical consequence concept in S4 is classical, we obtain the following proposition:

Proposition 8. *The pair $\langle \Delta, C \rangle$ is a constructive argument iff $(\Delta)^{S4} \cup \{\neg (C)^{S4}\}$ is a minimal inconsistent subset in S4.*

Prop. 8 states that the use of $S4$ allows to provide a simple characterization of being a constructive argument similar to that of being a classical argument. Such a characterization can be used in a constructive argument generation in the same way as in (Besnard et al. 2010). However, notice that the computation of minimal inconsistent subsets in modal logics is much less studied than in classical propositional logic. An interesting idea would be exploring how the MUS computation methods in classical propositional logic could benefit to modal logics. This is left for future work.

Conclusion and Future works

Constructivism is an approach which requires that to prove the existence of an object, it is necessary to be able to construct it. So far the main application of intuitionistic logic in computer science is using the Curry-Howard correspondence (Howard 1980) which corresponds to a direct relationship between constructive proofs and computer programs (Nordström, Petersson, and Smith 1990; Paulin-Mohring and Werner 1993).

In this paper we show the benefits of using intuitionistic logic to reason about inconsistency in argumentation theory. In particular Dung's framework is instantiated with this logic. In this setting, not only does the support of an argument deduce the conclusion of that argument but also constructs that conclusion. The present paper comes to complete existing works studying the validity of the logic-based instantiations of Dung's framework (Amgoud and Besnard 2013). While the focus of these works has been on the quality of the output of the logic-based argumentation frameworks (in terms of postulates), no attention has been paid on the argument itself, in particular the support of the argument. In our setting, a set of formulas which deduces a conclusion would not be a support of an argument for that conclusion if the reasons for such a deduction are not exactly identified. For example $\langle \{\neg(O \wedge T)\}, \neg O \vee \neg T \rangle$ is not an argument in our setting while $\langle \{O, \neg(O \wedge T)\}, \neg O \vee \neg T \rangle$ is. In addition, we provided a characterization of being a constructive argument in terms of a minimal inconsistent subset using Gödel's embedding of intuitionistic logic into the modal logic S4. Our work should be useful in diagnosis-based applications and law reasoning, to cite few.

As a future work, we intend to investigate the use of the Curry-Howard correspondence in encoding the proofs of the constructive arguments. Indeed, we know that each proof of a constructive argument can be encoded as a λ-term in a typed λ-calculus (Howard 1980). In this context, we plan to consider a constructive argumentation framework where we associate to each constructive argument a λ-term encoding a method used in the argument to construct its conclusion from its support. In this case, one of the perspectives consists in defining a new type of attack relations over the λ-terms.

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