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# Geometric Extensions of Cutwidth in any Dimension\*

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## Abstract

We define a multi-dimensional geometric extension of cutwidth. A graph has  $d$ -cutwidth at most  $k$  if it can be embedded in the  $d$ -dimensional euclidean space so that no hyperplane can intersect more than  $k$  of its edges. We prove a series of combinatorial results on  $d$ -cutwidth which imply that for every  $d$  and  $k$ , there is a linear time algorithm checking whether the  $d$ -cutwidth of a graph  $G$  is at most  $k$ .

## 1 Introduction

The cutwidth of a (total) vertex ordering of a graph is the maximum number of edges connecting vertices on opposite sides of any of the “gaps” between successive vertices in the linear layout. The cutwidth of a graph  $G$ , denoted by  $\text{CW}(G)$ , is the minimum cutwidth over all its possible vertex orderings. The problem that asks, given a  $n$ -vertex graph  $G$  and an integer  $k$ , whether  $\text{CW}(G) \leq k$ , is an NP-complete problem known in the literature as the MINIMUM CUT LINEAR ARRANGEMENT problem [4]. From the parameterized complexity point of view, the same problem is fixed parameter tractable, as an algorithm that checks whether  $\text{cutwidth}(G) \leq k$  in  $f(k) \cdot n$  steps was given in [10]. Cutwidth has been extensively studied both from its combinatorial (see e.g. [2, 7, 1]) as well as its algorithmic point of view [8, 11, 3, 6].

**$d$ -dimensional cutwidth** In this note we introduce a multi-dimensional geometric extension of cutwidth, namely the  $d$ -dimensional cutwidth (or, simply,  $d$ -cutwidth) that, roughly, instead of mono-dimensional linear arrangements of the graph  $G$ , we consider embeddings of  $G$  in the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$  and define the  $d$ -cutwidth of such an embedding to be the maximum number of edges a hyperplane of  $\mathbb{R}^d$  can intersect. Then, the  $d$ -cutwidth of  $G$ , denoted by  $\text{CW}_d(G)$ , is the minimum  $d$ -cutwidth over all such embeddings. Our results are summarized in the following.

**Theorem 1** *The following hold:*

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- i.  $d$ -cutwidth is immersion closed<sup>1</sup>.
- ii. For every graph  $G$  and every  $d \geq 1$ ,  $\text{CW}_d(G) \leq \text{CW}_{d+1}(G)$ .
- iii. For every graph  $G$  and every  $d \geq 1$ ,  $\text{CW}_d(G) \leq d \cdot \text{CW}(G)$ .
- iv. For every graph  $G$ ,  $\text{CW}_3(G) \leq 2 \cdot \text{CW}_2(G)$ .

## 2 Preliminaries and definitions

Every  $(d-1)$ -dimensional subspace  $\Pi$  of a  $d$ -dimensional space  $\mathcal{X}$  is called a *hyperplane* of  $\mathcal{X}$ . Here we are interested in hyperplanes of  $\mathbb{R}^d$  (which are known to be isomorphic to  $\mathbb{R}^{d-1}$ ). Let  $\Pi$  be a hyperplane in  $\mathbb{R}^d$ , then there are  $a_0, a_1, \dots, a_d \in \mathbb{R}$  such that  $\Pi = \{(x_1, \dots, x_d) \in \mathbb{R}^d \mid a_1x_1 + \dots + a_dx_d + a_0 = 0\}$ . We denote by  $H(d)$  the set of all hyperplanes of  $\mathbb{R}^d$ . A *hypersphere*,  $S(c, r)$ , with *center*  $c$  and *radius*  $r$  in  $\mathbb{R}^d$  is the set  $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^d (x_i - c_i)^2 = r^2\}$ . We denote by  $S(d)$  the set of all hyperspheres of  $\mathbb{R}^d$ . We call a continuous function  $C : [0, 1] \rightarrow \mathbb{R}^d$  a *curve* of  $\mathbb{R}^d$  with ends  $C(0)$  and  $C(1)$ .

Let  $G = (V, E)$  be a graph. An embedding of  $G$ , denoted by  $\mathcal{E}_d(G)$ , in the euclidean space  $\mathbb{R}^d$  is a tuple  $(f, \mathcal{C})$ , where  $f : V \rightarrow \mathbb{R}^d$  is an injection, mapping the vertices of  $G$  to  $\mathbb{R}^d$  and  $\mathcal{C} = \{C_e \mid e \in E\}$  is a set of curves of  $\mathbb{R}^d$  with the following properties: **(a)** for every  $e = \{u, v\} \in E$ , the ends of  $C_e$  are  $f(u)$  and  $f(v)$ , and **(b)** for all  $x \in (0, 1)$  and  $v \in V$  it holds that  $f_e(x) \neq f(v)$ . For simplicity, we may sometimes refer to the elements of  $f(V)$  and  $\mathcal{C}$  as the vertices and edges of  $\mathcal{E}_d(G)$  respectively. We denote by  $\mathbf{E}_d(G)$  the set of all embeddings  $\mathcal{E}_d(G) = (f, \mathcal{C})$ , of  $G$  in  $\mathbb{R}^d$ , such that for every positive integer  $i \leq d$ , if  $S$  is a subset of  $V$  with  $|S| \geq i$ , then the dimension of the subspace defined by  $\{f(u) \mid u \in S\}$  is  $i - 1$ . We call an element of  $\mathbf{E}_d(G)$  *essential-embedding of  $G$  in  $\mathbb{R}^d$* . Let  $\mathcal{E}_d(G)$  be a essential-embedding of  $G$  in  $\mathbb{R}^d$ , then if  $\Pi$  is a hyperplane of  $\mathbb{R}^d$  (resp.  $\Sigma$  is a hypersphere of  $\mathbb{R}^d$ ) that does not intersect any  $f(v)$ ,  $v \in V$ , we denote by  $\partial_G(\mathcal{E}_d(G), \Pi)$  (resp.  $\partial_G(\mathcal{E}_d(G), \Sigma)$ ) the set of curves of  $\mathcal{E}_d(G)$  that are intersected by  $\Pi$  (resp.  $\Sigma$ ).

**Definition 1** Let  $G = (V, E)$  be a graph and  $k, d$  be positive integers, where  $d \geq 2$ . Then we define the  $d$ -dimensional cutwidth of  $G$ , or simply  $d$ -cutwidth, to be

$$\text{CW}_d(G) = \min_{\mathcal{E}_d(G) \in \mathbf{E}_d(G)} \max\{|\partial_G(\mathcal{E}_d(G), \Pi)| \mid \Pi \in H(d)\}$$

Observe that any hyperplane  $\Pi$  of  $\mathbb{R}^d$  that meets a curve  $C_e \in \mathcal{C}$  once, also meets the unique straight line segment of  $\mathbb{R}^d$  with parametric equation  $\sigma_e(t) = t \cdot C_e(0) + (1 - t) \cdot C_e(1)$ ,  $t \in \mathbb{R}$ , i.e., the straight line segment of  $\mathbb{R}^d$  that is defined by the “images” of the endpoints of edge  $e$ . Therefore, without loss of generality, we can consider only *straight-line embeddings* where  $\mathcal{C} = \{\sigma_e \mid e \in E\}$ . Notice that every straight line embedding  $\mathcal{E}_d(G) = (f, \mathcal{C})$  is fully defined by the function  $f$ , therefore, for simplicity, for now on we will omit  $\mathcal{C}$ . Observe that the definition of 1-cutwidth, where hyperplanes degenerate to subspaces of  $\mathbb{R}$  of dimension 0 (i.e., points) is equivalent to the usual definition of

<sup>1</sup>A graph  $H$  is an *immersion* of a graph  $G$  if it can be obtained from  $G$  after a sequence of vertex/edge removals or edge lifts (the operation of *lifting* two edges  $\{x, y\}$  and  $\{y, z\}$  incident to the same vertex  $y$  is the operation of replacing these edges by the edge  $\{x, z\}$ ). A graph invariant is *immersion closed* if its value on a graph  $G$  is always smaller or equal than its value on its immersions.

cutwidth. Therefore,  $d$ -cutwidth is the intuitive generalization of the notion of cutwidth in any dimension  $d \geq 2$ . Also observe that our demand of essential embeddings is expressed here by our demand of injective functions.

### 3 Properties of $d$ -cutwidth

This section is devoted to the last two statements of Theorem 1.

*Proof of Theorem 1.iii.* Consider the  $d$ -dimensional *moment curve*  $C$  with parametric equation  $C(t) := (t, t^2, t^3, \dots, t^d)$ ,  $t \in \mathbb{R}$ . Consider also an ordering of the nodes of  $G$  that realizes the cutwidth of  $G$ . Embed a node  $v_i$  of  $G$  to the point  $p_i = C(t_i)$ , for an appropriate value  $t_i$ . By appropriate we mean that if a node  $v_i$  is after a node  $v_j$  in the cutwidth ordering, then the parametric value  $t_i$  corresponding to  $v_i$  is strictly greater than the parameter value  $t_j$  corresponding to node  $v_j$ . Now embed an edge  $e_{ij} = (v_i, v_j)$  of  $G$  by connecting the points  $p_i$  and  $p_j$  on  $C$  with the minimum length arc of  $C$  connecting these points.

Consider a generic hyperplane  $\Pi$  with equation  $a_1x_1 + a_2x_2 + \dots + a_dx_d + a_0 = 0$ , where, for all  $i$ ,  $a_i \in \mathbb{R}$ .  $\Pi$  can cut  $C$  at at most  $d$  points. To see that, solve the system of equations

$$a_1x_1 + a_2x_2 + \dots + a_dx_d + a_0 = 0, \quad \text{and} \quad x_i = t^i, \quad i = 1, \dots, d$$

for  $t$ . This gives the polynomial equation  $q(t) := a_0 + a_1t + a_2t^2 + \dots + a_dt^d = 0$ , in  $t$  of maximum degree  $d$ . Since  $q(t) = 0$  has at most  $d$  real roots, we deduce that  $\Pi$  intersects  $C$  at at most  $d$  points. At each point of intersection at most  $\text{cw}(G)$  edges of the embedding of  $G$  pass through that point. Hence,  $\Pi$  intersects at most  $d \cdot \text{cw}(G)$  edges of  $G$ , i.e.,  $\text{cw}_d(G) \leq d \text{cw}(G)$ .  $\square$

**Spherical  $d$ -cutwith** Given a graph  $G = (V, E)$  and two positive integers  $k$  and  $d$ , where  $d \geq 2$ , we define the *spherical  $d$ -dimensional cutwidth* of  $G$ , or simply spherical  $d$ -cutwidth, to be equal to

$$\text{scw}_d(G) = \min_{\mathcal{E}_d(G) \in \mathbf{E}_d(G)} \max\{|\partial_G(\mathcal{E}_d(G), \Sigma)| \mid \Sigma \in S(d)\}$$

The proof of Theorem 1.iv is a consequence of Theorem 1.iii and the following two lemmata.

**Lemma 1** *For every graph  $G$  and any  $d \geq 2$ ,  $\text{cw}_d(G) \leq \text{scw}_d(G) \leq (d+1) \text{cw}(G)$ .*

**Lemma 2** *For every graph  $G$  and every  $d \geq 1$ ,  $\text{cw}_{d+1}(G) \leq \text{scw}_d(G)$ .*

The above results clarify the relation between  $d$ -cutwidth and spherical  $d$ -cutwidth and we believe that they have independent interest. We omit the proofs as they are too lengthy to fit in this extended abstract.

## 4 Algorithmic remarks about $d$ -cutwidth

As a consequence of the result in [9], for every  $k$ , the class of immersion minimal graphs with  $d$ -cutwidth bigger than  $k$  contains a finite set of graphs. We call this class *immersion obstruction set* for cutwidth at most  $k$  and we denote it by  $\mathcal{O}_k$ . This fact, combined with Theorem 1.i, implies that  $\text{cw}_d(G) \leq k$  if and only if none of the graphs in  $\mathcal{O}_k$  is contained in  $G$  as an immersion. According to the result of Grohe, Kawarabayashi, Marx, and Wollan in [5], checking whether an  $n$ -vertex graph contains as an immersion some  $k$ -vertex graph  $H$ , can be done in  $f(k) \cdot n^3$  steps. As a consequence, checking whether  $\text{cw}_d(G) \leq k$  can be done in  $f(k) \cdot n^3$  steps. This running time can become linear (on  $n$ ) using the first inequality of Theorem 1.iii. Indeed, the algorithm first checks whether  $\text{CW}(G) \leq k$ . If the answer is negative then we can safely report that  $\text{cw}_d(G) > k$ . If not, then it is known (see e.g. [10]) that  $G$  has a tree decomposition of width  $\leq k$  and to check whether some of the graphs in  $\mathcal{O}_k$  is contained in  $G$  as an immersion can be done using dynamic programming in  $f(k) \cdot n$  steps.

Unfortunately, the above algorithm is non-constructive as we have no other knowledge about the set  $\mathcal{O}_k$ , except from the fact that it is finite. To obtain a constructive  $f(k) \cdot n$  step algorithm for  $d$ -cutwidth remains an insisting open problem.

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