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Geometric Extensions of Cutwidth in any Dimension

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Abstract

We define a multi-dimensional geometric extension of cutwidth. A graph has $d$-cutwidth at most $k$ if it can be embedded in the $d$-dimensional euclidean space so that no hyperplane can intersect more than $k$ of its edges. We prove a series of combinatorial results on $d$-cutwidth which imply that for every $d$ and $k$, there is a linear time algorithm checking whether the $d$-cutwidth of a graph $G$ is at most $k$.

1 Introduction

The cutwidth of a (total) vertex ordering of a graph is the maximum number of edges connecting vertices on opposite sides of any of the “gaps” between successive vertices in the linear layout. The cutwidth of a graph $G$, denoted by $cw(G)$, is the minimum cutwidth over all its possible vertex orderings. The problem that asks, given a $n$-vertex graph $G$ and an integer $k$, whether $cw(G) \leq k$, is an NP-complete problem known in the literature as the MINIMUM CUT LINEAR ARRANGEMENT problem [4]. From the parameterized complexity point of view, the same problem is fixed parameter tractable, as an algorithm that checks whether $cw(G) \leq k$ in $f(k) \cdot n$ steps was given in [10]. Cutwidth has been extensively studied both from its combinatorial (see e.g. [2, 7, 1]) as well as its algorithmic point of view [8, 11, 3, 6].

$d$-dimensional cutwidth In this note we introduce a multi-dimensional geometric extension of cutwidth, namely the $d$-dimensional cutwidth (or, simply, $d$-cutwidth) that, roughly, instead of mono-dimensional linear arrangements of the graph $G$, we consider embeddings of $G$ in the $d$-dimensional Euclidean space $\mathbb{R}^d$ and define the $d$-cutwidth of such an embedding to be the maximum number of edges a hyperplane of $\mathbb{R}^d$ can intersect. Then, the $d$-cutwidth of $G$, denoted by $cw_d(G)$, is the minimum $d$-cutwidth over all such embeddings. Our results are summarized in the following.

Theorem 1 The following hold:

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i. \(d\)-cutwidth is immersion closed\(^1\).

ii. For every graph \(G\) and every \(d \geq 1\), \(\text{cw}_d(G) \leq \text{cw}_{d+1}(G)\).

iii. For every graph \(G\) and every \(d \geq 1\), \(\text{cw}_d(G) \leq d \cdot \text{cw}(G)\).

iv. For every graph \(G\), \(\text{cw}_3(G) \leq 2 \cdot \text{cw}_2(G)\).

## 2 Preliminaries and definitions

Every \((d-1)\)-dimensional subspace \(\Pi\) of a \(d\)-dimensional space \(\mathcal{X}\) is called a hyperplane of \(\mathcal{X}\). Here we are interested in hyperplanes of \(\mathbb{R}^d\) (which are known to be isomorphic to \(\mathbb{R}^{d-1}\)). Let \(\Pi\) be a hyperplane in \(\mathbb{R}^d\), then there are \(a_0, a_1, \ldots, a_d \in \mathbb{R}\) such that \(\Pi = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1 x_1 + \cdots + a_d x_d + a_0 = 0\}\). We denote by \(H(d)\) the set of all hyperplanes of \(\mathbb{R}^d\). A hypersphere, \(S(c, r)\), with center \(c\) and radius \(r\) in \(\mathbb{R}^d\) is the set \(\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^d (x_i - c_i)^2 = r^2\}\). We denote by \(S(d)\) the set of all hyperspheres of \(\mathbb{R}^d\). We call a continuous function \(C : [0, 1] \rightarrow \mathbb{R}^d\) a curve of \(\mathbb{R}^d\) with ends \(C(0)\) and \(C(1)\).

Let \(G = (V, E)\) be a graph. An embedding of \(G\), denoted by \(\mathcal{E}_d(G)\), in the euclidean space \(\mathbb{R}^d\) is a tuple \((f, \mathcal{C})\), where \(f : V \rightarrow \mathbb{R}^d\) is an injection, mapping the vertices of \(G\) to \(\mathbb{R}^d\) and \(\mathcal{C} = \{C_e \mid e \in E\}\) is a set of curves of \(\mathbb{R}^d\) with the following properties: (a) for every \(e = \{u, v\} \in E\), the ends of \(C_e\) are \(f(u)\) and \(f(v)\), and (b) for all \(x \in (0, 1)\) and \(v \in V\) it holds that \(f_e(x) \neq f(v)\). For simplicity, we may sometimes refer to the elements of \(f(V)\) and \(\mathcal{C}\) as the vertices and edges of \(\mathcal{E}_d(G)\) respectively. We denote by \(\mathbf{E}_d(G)\) the set of all embeddings \(\mathcal{E}_d(G) = (f, \mathcal{C})\), of \(G\) in \(\mathbb{R}^d\), such that for every positive integer \(i \leq d\), if \(S\) is a subset of \(V\) with \(|S| \geq i\), then the dimension of the subspace defined by \(\{f(u) \mid u \in S\}\) is \(i - 1\). We call an element of \(\mathbf{E}_d(G)\) essential-embedding of \(G\) in \(\mathbb{R}^d\). Let \(\mathcal{E}_d(G)\) be a essential-embedding of \(G\) in \(\mathbb{R}^d\), then if \(\Pi\) is a hyperplane of \(\mathbb{R}^d\) (resp. \(\Sigma\) is a hypersphere of \(\mathbb{R}^d\)) that does not intersect any \(f(v), v \in V\), we denote by \(\partial_G(\mathcal{E}_d(G), \Pi)\) (resp. \(\partial_G(\mathcal{E}_d(G), \Sigma)\)) the set of curves of \(\mathcal{E}_d(G)\) that are intersected by \(\Pi\) (resp. \(\Sigma\)).

**Definition 1** Let \(G = (V, E)\) be a graph and \(k, d\) be positive integers, where \(d \geq 2\). Then we define the \(d\)-dimensional cutwidth of \(G\), or simply \(d\)-cutwidth, to be

\[
\text{cw}_d(G) = \min_{\mathcal{E}_d(G) \in \mathbf{E}_d(G)} \max\{|\partial_G(\mathcal{E}_d(G), \Pi)| \mid \Pi \in H(d)\}
\]

Observe that any hyperplane \(\Pi\) of \(\mathbb{R}^d\) that meets a curve \(C_e \in \mathcal{C}\) once, also meets the unique straight line segment of \(\mathbb{R}^d\) with parametric equation \(\sigma_e(t) = t \cdot C_e(0) + (1 - t) \cdot C_e(1), t \in \mathbb{R}\), i.e., the straight line segment of \(\mathbb{R}^d\) that is defined by the “images” of the endpoints of edge \(e\). Therefore, without loss of generality, we can consider only straight-line embeddings where \(\mathcal{C} = \{\sigma_e \mid e \in E\}\). Notice that every straight line embedding \(\mathcal{E}_d(G) = (f, \mathcal{C})\) is fully defined by the function \(f\), therefore, for simplicity, for now on we will omit \(\mathcal{C}\). Observe that the definition of 1-cutwidth, where hyperplanes degenerate to subspaces of \(\mathbb{R}\) of dimension 0 (i.e., points) is equivalent to the usual definition of

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\(^1\)A graph \(H\) is an immersion of a graph \(G\) if it can be obtained from \(G\) after a sequence of vertex/edge removals or edge lifts (the operation of lifting two edges \(\{x, y\}\) and \(\{y, z\}\) incident to the same vertex \(y\) is the operation of replacing these edges by the edge \(\{x, z\}\)). A graph invariant is immersion closed if its value on a graph \(G\) is always smaller or equal than its value on its immersions.
cutwidth. Therefore, $d$-cutwidth is the intuitive generalization of the notion of cutwidth in any dimension $d \geq 2$. Also observe that our demand of essential embeddings is expressed here by our demand of injective functions.

3 Properties of $d$-cutwidth

This section is devoted to the last two statements of Theorem 1.

Proof of Theorem 1.iii. Consider the $d$-dimensional moment curve $C$ with parametric equation $C(t) := (t, t^2, t^3, \ldots, t^d)$, $t \in \mathbb{R}$. Consider also an ordering of the nodes of $G$ that realizes the cutwidth of $G$. Embed a node $v_i$ of $G$ to the point $p_i = C(t_i)$, for an appropriate value $t_i$. By appropriate we mean that if a node $v_i$ is after a node $v_j$ in the cutwidth ordering, then the parametric value $t_i$ corresponding to $v_i$ is strictly greater than the parameter value $t_j$ corresponding to node $v_j$. Now embed an edge $e_{ij} = (v_i, v_j)$ of $G$ by connecting the points $p_i$ and $p_j$ on $C$ with the minimum length arc of $C$ connecting these points.

Consider a generic hyperplane $\Pi$ with equation $a_1 x_1 + a_2 x_2 + \ldots + a_d x_d + a_0 = 0$, where, for all $i$, $a_i \in \mathbb{R}$. $\Pi$ can cut $C$ at at most $d$ points. To see that, solve the system of equations $a_1 x_1 + a_2 x_2 + \ldots + a_d x_d + a_0 = 0$, $x_i = t^i$, $i = 1, \ldots, d$. This gives the polynomial equation $q(t) := a_0 + a_1 t + a_2 t^2 + \ldots + a_d t^d = 0$, in $t$ of maximum degree $d$. Since $q(t) = 0$ has at most $d$ real roots, we deduce that $\Pi$ intersects $C$ at at most $d$ points. At each point of intersection at most $\text{cw}(G)$ edges of the embedding of $G$ pass through that point. Hence, $\Pi$ intersects at most $d \cdot \text{cw}(G)$ edges of $G$, i.e., $\text{cw}_d(G) \leq d \cdot \text{cw}(G)$. □

Spherical $d$-cutwith\thinspace\thinspace Given a graph $G = (V, E)$ and two positive integers $k$ and $d$, where $d \geq 2$, we define the spherical $d$-dimensional cutwidth of $G$, or simply spherical $d$-cutwidth, to be equal to

$$\text{scw}_d(G) = \min_{E_d(G) \in \mathbb{E}_d(G)} \max\{|\partial_G(E_d(G), \Sigma)| \mid \Sigma \in S(d)\}$$

The proof of Theorem 1.iv is a consequence of Theorem 1.iii and the following two lemmata.

Lemma 1 For every graph $G$ and any $d \geq 2$, $\text{cw}_d(G) \leq \text{scw}_d(G) \leq (d+1) \cdot \text{cw}(G)$.

Lemma 2 For every graph $G$ and every $d \geq 1$, $\text{cw}_{d+1}(G) \leq \text{scw}_d(G)$.

The above results clarify the relation between $d$-cutwidth and spherical $d$-cutwidth and we believe that they have independent interest. We omit the proofs as they are too lengthy to fit in this extended abstract.
4 Algorithmic remarks about $d$-cutwidth

As a consequence of the result in [9], for every $k$, the class of immersion minimal graphs with $d$-cutwidth bigger than $k$ contains a finite set of graphs. We call this class immersion obstruction set for cutwidth at most $k$ and we denote it by $O_k$. This fact, combined with Theorem 1.i, implies that $cw_d(G) \leq k$ if and only if none of the graphs in $O_k$ is contained in $G$ as an immersion. According to the result of Grohe, Kawarabayashi, Marx, and Wollan in [5], checking whether an $n$-vertex graph contains as an immersion some $k$-vertex graph $H$, can be done in $f(k) \cdot n^3$ steps. As a consequence, checking whether $cw_d(G) \leq k$ can be done in $f(k) \cdot n^3$ steps. This running time can become linear (on $n$) using the first inequality of Theorem 1.iii. Indeed, the algorithm first checks whether $cw(G) \leq k$. If the answer is negative then we can safely report that $cw_d(G) > k$. If not, then it is known (see e.g. [10]) that $G$ has a tree decomposition of width $\leq k$ and to check whether some of the graphs in $O_k$ is contained in $G$ as an immersion can be done using dynamic programming in $f(k) \cdot n$ steps.

Unfortunately, the above algorithm is non-constructive as we have no other knowledge about the set $O_k$, except from the fact that it is finite. To obtain a constructive $f(k) \cdot n$ step algorithm for $d$-cutwidth remains an insisting open problem.

References


