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Geometric Extensions of Cutwidth in any Dimension

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Abstract

We define a multi-dimensional geometric extension of cutwidth. A graph has \( d \)-cutwidth at most \( k \) if it can be embedded in the \( d \)-dimensional euclidean space so that no hyperplane can intersect more than \( k \) of its edges. We prove a series of combinatorial results on \( d \)-cutwidth which imply that for every \( d \) and \( k \), there is a linear time algorithm checking whether the \( d \)-cutwidth of a graph \( G \) is at most \( k \).

1 Introduction

The cutwidth of a (total) vertex ordering of a graph is the maximum number of edges connecting vertices on opposite sides of any of the “gaps” between successive vertices in the linear layout. The cutwidth of a graph \( G \), denoted by \( cw(G) \), is the minimum cutwidth over all its possible vertex orderings. The problem that asks, given a \( n \)-vertex graph \( G \) and an integer \( k \), whether \( cw(G) \leq k \), is an NP-complete problem known in the literature as the Minimum Cut Linear Arrangement problem [4]. From the parameterized complexity point of view, the same problem is fixed parameter tractable, as an algorithm that checks whether \( cw(G) \leq k \) in \( f(k) \cdot n \) steps was given in [10]. Cutwidth has been extensively studied both from its combinatorial (see e.g. [2, 7, 1]) as well as its algorithmic point of view [8, 11, 3, 6].

\( d \)-dimensional cutwidth In this note we introduce a multi-dimensional geometric extension of cutwidth, namely the \( d \)-dimensional cutwidth (or, simply, \( d \)-cutwidth) that, roughly, instead of mono-dimensional linear arrangements of the graph \( G \), we consider embeddings of \( G \) in the \( d \)-dimensional Euclidean space \( \mathbb{R}^d \) and define the \( d \)-cutwidth of such an embedding to be the maximum number of edges a hyperplane of \( \mathbb{R}^d \) can intersect. Then, the \( d \)-cutwidth of \( G \), denoted by \( cw_d(G) \), is the minimum \( d \)-cutwidth over all such embeddings. Our results are summarized in the following.

Theorem 1 The following hold:

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Observe that any hyperplane \( \Pi \) of \( \mathbb{R}^d \) then there are \( a_0, a_1, \ldots, a_d \in \mathbb{R} \) such that \( \Pi = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1 x_1 + \cdots + a_d x_d + a_0 = 0 \} \). We denote by \( H(d) \) the set of all hyperplanes of \( \mathbb{R}^d \). A hypersphere, \( S(c, r) \), with center \( c \) and radius \( r \) in \( \mathbb{R}^d \) is the set \( \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} (x_i - c_i)^2 = r^2 \} \). We denote by \( S(d) \) the set of all hyperspheres of \( \mathbb{R}^d \).

### 2 Preliminaries and definitions

Every \((d - 1)\)-dimensional subspace \( \Pi \) of a \( d \)-dimensional space \( X \) is called a hyperplane of \( X \). Here we are interested in hyperplanes of \( \mathbb{R}^d \) (which are known to be isomorphic to \( \mathbb{R}^{d-1} \)). Let \( \Pi \) be a hyperplane in \( \mathbb{R}^d \), then there are \( a_0, a_1, \ldots, a_d \in \mathbb{R} \) such that \( \Pi = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1 x_1 + \cdots + a_d x_d + a_0 = 0 \} \). We denote by \( H(d) \) the set of all hyperplanes of \( \mathbb{R}^d \). A hypersphere, \( S(c, r) \), with center \( c \) and radius \( r \) in \( \mathbb{R}^d \) is the set \( \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} (x_i - c_i)^2 = r^2 \} \). We denote by \( S(d) \) the set of all hyperspheres of \( \mathbb{R}^d \).

We call a continuous function \( C : [0, 1] \rightarrow \mathbb{R}^d \) a curve of \( \mathbb{R}^d \) with ends \( C(0) \) and \( C(1) \).

Let \( G = (V, E) \) be a graph. An embedding of \( G \), denoted by \( \mathcal{E}_d(G) \), in the euclidean space \( \mathbb{R}^d \) is a tuple \( (f, \mathcal{C}) \), where \( f : V \rightarrow \mathbb{R}^d \) is an injection, mapping the vertices of \( G \) to \( \mathbb{R}^d \) and \( \mathcal{C} = \{C_e \mid e \in E\} \) is a set of curves of \( \mathbb{R}^d \) with the following properties: (a) for every \( e = \{u, v\} \in E \), the ends of \( C_e \) are \( f(u) \) and \( f(v) \), and (b) for all \( x \in (0, 1) \) and \( v \in V \) it holds that \( f_e(x) \neq f(v) \). For simplicity, we may sometimes refer to the elements of \( f(V) \) and \( \mathcal{C} \) as the vertices and edges of \( \mathcal{E}_d(G) \) respectively. We denote by \( \mathcal{E}_d(G) \) the set of all embeddings \( \mathcal{E}_d(G) = (f, \mathcal{C}) \), of \( G \) in \( \mathbb{R}^d \), such that for every positive integer \( i \leq d \), if \( S \) is a subset of \( V \) with \( |S| \geq i \), then the dimension of the subspace defined by \( \{f(u) \mid u \in S\} \) is \( i - 1 \). We call an element of \( \mathcal{E}_d(G) \) essential-embedding of \( G \) in \( \mathbb{R}^d \). Let \( \mathcal{E}_d(G) \) be a essential-embedding of \( G \) in \( \mathbb{R}^d \), then if \( \Pi \) is a hyperplane of \( \mathbb{R}^d \) (resp. \( \Sigma \) is a hypersphere of \( \mathbb{R}^d \) that does not intersect any \( f(v) \), \( v \in V \), we denote by \( \partial_G(\mathcal{E}_d(G), \Pi) \) (resp. \( \partial_G(\mathcal{E}_d(G), \Sigma) \)) the set of curves of \( \mathcal{E}_d(G) \) that are intersected by \( \Pi \) (resp. \( \Sigma \)).

**Definition 1** Let \( G = (V, E) \) be a graph and \( k, d \) be positive integers, where \( d \geq 2 \). Then we define the \( d \)-dimensional cutwidth of \( G \), or simply \( d \)-cutwidth, to be

\[
\text{cw}_d(G) = \min_{\mathcal{E}_d(G) \in \mathcal{E}_d(G)} \max_{\Pi \in H(d)} |\partial_G(\mathcal{E}_d(G), \Pi)|
\]

Observe that any hyperplane \( \Pi \) of \( \mathbb{R}^d \) that meets a curve \( C_e \in \mathcal{C} \) once, also meets the unique straight line segment of \( \mathbb{R}^d \) with parametric equation \( \sigma_e(t) = t \cdot C_e(0) + (1 - t) \cdot C_e(1), t \in \mathbb{R} \), i.e., the straight line segment of \( \mathbb{R}^d \) that is defined by the “images” of the endpoints of edge \( e \). Therefore, without loss of generality, we can consider only straight-line embeddings where \( \mathcal{C} = \{\sigma_e \mid e \in E\} \). Notice that every straight line embedding \( \mathcal{E}_d(G) = (f, \mathcal{C}) \) is fully defined by the function \( f \), therefore, for simplicity, for now on we will omit \( \mathcal{C} \). Observe that the definition of 1-cutwidth, where hyperplanes degenerate to subspaces of \( \mathbb{R} \) of dimension 0 (i.e., points) is equivalent to the usual definition of

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1 A graph \( H \) is an immersion of a graph \( G \) if it can be obtained from \( G \) after a sequence of vertex/edge removals or edge lifts (the operation of lifting two edges \( \{x, y\} \) and \( \{y, z\} \) incident to the same vertex \( y \) is the operation of replacing these edges by the edge \( \{x, z\} \)). A graph invariant is immersion closed if its value on a graph \( G \) is always smaller or equal than its value on its immersions.
cutwidth. Therefore, $d$-cutwidth is the intuitive generalization of the notion of cutwidth in any dimension $d \geq 2$. Also observe that our demand of essential embeddings is expressed here by our demand of injective functions.

3 Properties of $d$-cutwidth

This section is devoted to the last two statements of Theorem 1.

Proof of Theorem 1.iii. Consider the $d$-dimensional moment curve $C$ with parametric equation $C(t) := (t, t^2, t^3, \ldots, t^d)$, $t \in \mathbb{R}$. Consider also an ordering of the nodes of $G$ that realizes the cutwidth of $G$. Embed a node $v_i$ of $G$ to the point $p_i = C(t_i)$, for an appropriate value $t_i$. By appropriate we mean that if a node $v_i$ is after a node $v_j$ in the cutwidth ordering, then the parametric value $t_i$ corresponding to $v_i$ is strictly greater than the parameter value $t_j$ corresponding to node $v_j$. Now embed an edge $e_{ij} = (v_i, v_j)$ of $G$ by connecting the points $p_i$ and $p_j$ on $C$ with the minimum length arc of $C$ connecting these points.

Consider a generic hyperplane $\Pi$ with equation $a_1x_1 + a_2x_2 + \ldots + a_dx_d + a_0 = 0$, where, for all $i$, $a_i \in \mathbb{R}$. $\Pi$ can cut $C$ at at most $d$ points. To see that, solve the system of equations $a_1x_1 + a_2x_2 + \ldots + a_dx_d + a_0 = 0$, and $x_i = t^i$, $i = 1, \ldots, d$ for $t$. This gives the polynomial equation $q(t) := a_0 + a_1t + a_2t^2 + \ldots + a_dt^d = 0$, in $t$ of maximum degree $d$. Since $q(t) = 0$ has at most $d$ real roots, we deduce that $\Pi$ intersects $C$ at at most $d$ points. At each point of intersection at most $\text{cw}(G)$ edges of the embedding of $G$ pass through that point. Hence, $\Pi$ intersects at most $d \cdot \text{cw}(G)$ edges of $G$, i.e., $\text{cw}_d(G) \leq d \cdot \text{cw}(G)$. □

Spherical $d$-cutwith Given a graph $G = (V, E)$ and two positive integers $k$ and $d$, where $d \geq 2$, we define the spherical $d$-dimensional cutwidth of $G$, or simply spherical $d$-cutwidth, to be equal to

$$\text{scw}_d(G) = \min_{E_{d}(G) \in \mathcal{E}_{d}(G)} \max\{|\partial_G(E_{d}(G), \Sigma)| \mid \Sigma \in S(d)\}$$

The proof of Theorem 1.iv is a consequence of Theorem 1.iii and the following two lemmata.

Lemma 1 For every graph $G$ and any $d \geq 2$, $\text{cw}_{d+1}(G) \leq \text{scw}_d(G) \leq (d + 1) \cdot \text{cw}(G)$.

Lemma 2 For every graph $G$ and every $d \geq 1$, $\text{cw}_{d+1}(G) \leq \text{scw}_d(G)$.

The above results clarify the relation between $d$-cutwidth and spherical $d$-cutwidth and we believe that they have independent interest. We omit the proofs as they are too lengthy to fit in this extended abstract.
4 Algorithmic remarks about $d$-cutwidth

As a consequence of the result in [9], for every $k$, the class of immersion minimal graphs with $d$-cutwidth bigger than $k$ contains a finite set of graphs. We call this class immersion obstruction set for cutwidth at most $k$ and we denote it by $O_k$. This fact, combined with Theorem 1.i, implies that $\text{cw}_d(G) \leq k$ if and only if none of the graphs in $O_k$ is contained in $G$ as an immersion. According to the result of Grohe, Kawarabayashi, Marx, and Wollan in [5], checking whether an $n$-vertex graph contains as an immersion some $k$-vertex graph $H$, can be done in $f(k) \cdot n^3$ steps. As a consequence, checking whether $\text{cw}_d(G) \leq k$ can be done in $f(k) \cdot n^3$ steps. This running time can become linear (on $n$) using the first inequality of Theorem 1.iii. Indeed, the algorithm first checks whether $\text{cw}(G) \leq k$. If the answer is negative then we can safely report that $\text{cw}_d(G) > k$. If not, then it is known (see e.g. [10]) that $G$ has a tree decomposition of width $\leq k$ and to check whether some of the graphs in $O_k$ is contained in $G$ as an immersion can be done using dynamic programming in $f(k) \cdot n$ steps.

Unfortunately, the above algorithm is non-constructive as we have no other knowledge about the set $O_k$, except from the fact that it is finite. To obtain a constructive $f(k) \cdot n$ step algorithm for $d$-cutwidth remains an insisting open problem.

References