Geometric Extensions of Cutwidth in any Dimension
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We define a multi-dimensional geometric extension of cutwidth. A graph has $d$-cutwidth at most $k$ if it can be embedded in the $d$-dimensional euclidean space so that no hyperplane can intersect more than $k$ of its edges. We prove a series of combinatorial results on $d$-cutwidth which imply that for every $d$ and $k$, there is a linear time algorithm checking whether the $d$-cutwidth of a graph $G$ is at most $k$.

1 Introduction

The cutwidth of a (total) vertex ordering of a graph is the maximum number of edges connecting vertices on opposite sides of any of the “gaps” between successive vertices in the linear layout. The cutwidth of a graph $G$, denoted by $cw(G)$, is the minimum cutwidth over all its possible vertex orderings. The problem that asks, given a $n$-vertex graph $G$ and an integer $k$, whether $cw(G) \leq k$, is an NP-complete problem known in the literature as the MINIMUM CUT LINEAR ARRANGEMENT problem [4]. From the parameterized complexity point of view, the same problem is fixed parameter tractable, as an algorithm that checks whether $cw(G) \leq k$ in $f(k) \cdot n$ steps was given in [10]. Cutwidth has been extensively studied both from its combinatorial (see e.g. [2, 7, 1]) as well as its algorithmic point of view [8, 11, 3, 6].

$d$-dimensional cutwidth In this note we introduce a multi-dimensional geometric extension of cutwidth, namely the $d$-dimensional cutwidth (or, simply, $d$-cutwidth) that, roughly, instead of mono-dimensional linear arrangements of the graph $G$, we consider embeddings of $G$ in the $d$-dimensional Euclidean space $\mathbb{R}^d$ and define the $d$-cutwidth of such an embedding to be the maximum number of edges a hyperplane of $\mathbb{R}^d$ can intersect. Then, the $d$-cutwidth of $G$, denoted by $cw_d(G)$, is the minimum $d$-cutwidth over all such embeddings. Our results are summarized in the following.

Theorem 1 The following hold:

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Observe that any hyperplane \( \Pi \) of \( R^d \) (resp. \( \Sigma \)).

i. \( d \)-cutwidth is immersion closed\(^1\).

ii. For every graph \( G \) and every \( d \geq 1 \), \( \text{cw}_d(G) \leq \text{cw}_{d+1}(G) \).

iii. For every graph \( G \) and every \( d \geq 1 \), \( \text{cw}_d(G) \leq d \cdot \text{cw}(G) \).

iv. For every graph \( G \), \( \text{cw}_3(G) \leq 2 \cdot \text{cw}_2(G) \).

2 Preliminaries and definitions

Every \((d-1)\)-dimensional subspace \( \Pi \) of a \( d \)-dimensional space \( \mathcal{X} \) is called a hyperplane of \( \mathcal{X} \). Here we are interested in hyperplanes of \( R^d \) (which are known to be isomorphic to \( R^{d-1} \)). Let \( \Pi \) be a hyperplane in \( R^d \), then there are \( a_0, a_1, \ldots, a_d \in \mathbb{R} \) such that \( \Pi = \{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1 x_1 + \cdots + a_d x_d + a_0 = 0 \} \). We denote by \( H(d) \) the set of all hyperplanes of \( \mathbb{R}^d \). A hypersphere, \( S(c, r) \), with center \( c \) and radius \( r \) in \( \mathbb{R}^d \) is the set \( \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} (x_i - c_i)^2 = r^2 \} \). We denote by \( S(d) \) the set of all hyperspheres of \( \mathbb{R}^d \). We call a continuous function \( C : [0, 1] \to \mathbb{R}^d \) a curve of \( \mathbb{R}^d \) with ends \( C(0) \) and \( C(1) \).

Let \( G = (V, E) \) be a graph. An embedding of \( G \), denoted by \( E_d(G) \), in the euclidean space \( \mathbb{R}^d \) is a tuple \((f, \mathcal{C})\), where \( f : V \to \mathbb{R}^d \) is an injection, mapping the vertices of \( G \) to \( \mathbb{R}^d \) and \( \mathcal{C} = \{ C_e \mid e \in E \} \) is a set of curves of \( \mathbb{R}^d \) with the following properties: (a) for every \( e = \{u, v\} \in E \), the ends of \( C_e \) are \( f(u) \) and \( f(v) \), and (b) for all \( x \in (0, 1) \) and \( v \in V \) it holds that \( f_e(x) \neq f(v) \). For simplicity, we may sometimes refer to the elements of \( f(V) \) and \( \mathcal{C} \) as the vertices and edges of \( E_d(G) \) respectively. We denote by \( E_d(G) \) the set of all embeddings \( E_d(G) = (f, \mathcal{C}) \), of \( G \) in \( \mathbb{R}^d \), such that for every positive integer \( i \leq d \), if \( S \) is a subset of \( V \) with \( |S| \geq i \), then the dimension of the subspace defined by \( \{ f(u) \mid u \in S \} \) is \( i - 1 \). We call an element of \( E_d(G) \) essential-embedding of \( G \) in \( \mathbb{R}^d \). Let \( E_d(G) \) be a essential-embedding of \( G \) in \( \mathbb{R}^d \), then if \( \Pi \) is a hyperplane of \( \mathbb{R}^d \) (resp. \( \Sigma \) is a hypersphere of \( \mathbb{R}^d \)) that does not intersect any \( f(v) \), \( v \in V \), we denote by \( \partial_G(E_d(G), \Pi) \) (resp. \( \partial_G(E_d(G), \Sigma) \)) the set of curves of \( E_d(G) \) that are intersected by \( \Pi \) (resp. \( \Sigma \)).

**Definition 1** Let \( G = (V, E) \) be a graph and \( k, d \) be positive integers, where \( d \geq 2 \). Then we define the \( d \)-dimensional cutwidth of \( G \), or simply \( d \)-cutwidth, to be

\[
\text{cw}_d(G) = \min_{E_d(G)\in \mathcal{E}_d(G)} \max \{|\partial_G(E_d(G), \Pi)| \mid \Pi \in H(d)\}
\]

Observe that any hyperplane \( \Pi \) of \( \mathbb{R}^d \) that meets a curve \( C_e \in \mathcal{C} \) once, also meets the unique straight line segment of \( \mathbb{R}^d \) with parametric equation \( \sigma_e(t) = t \cdot C_e(0) + (1 - t) \cdot C_e(1), t \in \mathbb{R} \), i.e., the straight line segment of \( \mathbb{R}^d \) that is defined by the “images” of the endpoints of edge \( e \). Therefore, without loss of generality, we can consider only straight-line embeddings \( \mathcal{C} = \{ \sigma_e \mid e \in E \} \). Notice that every straight line embedding \( E_d(G) = (f, \mathcal{C}) \) is fully defined by the function \( f \), therefore, for simplicity, for now on we will omit \( \mathcal{C} \). Observe that the definition of 1-cutwidth, where hyperplanes degenerate to subspaces of \( \mathbb{R} \) of dimension 0 (i.e., points) is equivalent to the usual definition of

\(^1\)A graph \( H \) is an immersion of a graph \( G \) if it can be obtained from \( G \) after a sequence of vertex/edge removals or edge lifts (the operation of lifting two edges \( \{x, y\} \) and \( \{y, z\} \) incident to the same vertex \( y \) is the operation of replacing these edges by the edge \( \{x, z\} \)). A graph invariant is immersion closed if its value on a graph \( G \) is always smaller or equal than its value on its immersions.
cutwidth. Therefore, $d$-cutwidth is the intuitive generalization of the notion of cutwidth in any dimension $d \geq 2$. Also observe that our demand of essential embeddings is expressed here by our demand of injective functions.

3 Properties of $d$-cutwidth

This section is devoted to the last two statements of Theorem 1.

Proof of Theorem 1.iii. Consider the $d$-dimensional moment curve $C$ with parametric equation $C(t) := (t, t^2, t^3, \ldots, t^d)$, $t \in \mathbb{R}$. Consider also an ordering of the nodes of $G$ that realizes the cutwidth of $G$. Embed a node $v_i$ of $G$ to the point $p_i = C(t_i)$, for an appropriate value $t_i$. By appropriate we mean that if a node $v_i$ is after a node $v_j$ in the cutwidth ordering, then the parametric value $t_i$ corresponding to $v_i$ is strictly greater than the parameter value $t_j$ corresponding to node $v_j$. Now embed an edge $e_{ij} = (v_i, v_j)$ of $G$ by connecting the points $p_i$ and $p_j$ on $C$ with the minimum length arc of $C$ connecting these points.

Consider a generic hyperplane $\Pi$ with equation $a_1x_1 + a_2x_2 + \ldots + a_dx_d + a_0 = 0$, where, for all $i$, $a_i \in \mathbb{R}$. $\Pi$ can cut $C$ at at most $d$ points. To see that, solve the system of equations $a_1x_1 + a_2x_2 + \ldots + a_dx_d + a_0 = 0$, and $x_i = t^i$, $i = 1, \ldots, d$ for $t$. This gives the polynomial equation $q(t) := a_0 + a_1t + a_2t^2 + \ldots + a_dt^d = 0$, in $t$ of maximum degree $d$. Since $q(t) = 0$ has at most $d$ real roots, we deduce that $\Pi$ intersects $C$ at at most $d$ points. At each point of intersection at most $\text{cw}(G)$ edges of the embedding of $G$ pass through that point. Hence, $\Pi$ intersects at most $d \cdot \text{cw}(G)$ edges of $G$, i.e., $\text{cw}_d(G) \leq d \text{cw}(G)$. □

Spherical $d$-cutwith Given a graph $G = (V, E)$ and two positive integers $k$ and $d$, where $d \geq 2$, we define the spherical $d$-dimensional cutwidth of $G$, or simply spherical $d$-cutwidth, to be equal to

$$\text{scw}_d(G) = \min_{E_d(G) \in E_d(G)} \max\{\partial_G(E_d(G), \Sigma) \mid \Sigma \in S(d)\}$$

The proof of Theorem 1.iv is a consequence of Theorem 1.iii and the following two lemmata.

Lemma 1 For every graph $G$ and any $d \geq 2$, $\text{cw}_d(G) \leq \text{scw}_d(G) \leq (d + 1) \text{cw}(G)$.

Lemma 2 For every graph $G$ and every $d \geq 1$, $\text{cw}_{d+1}(G) \leq \text{scw}_d(G)$.

The above results clarify the relation between $d$-cutwidth and spherical $d$-cutwidth and we believe that they have independent interest. We omit the proofs as they are too lengthy to fit in this extended abstract.
Algorithmic remarks about $d$-cutwidth

As a consequence of the result in [9], for every $k$, the class of immersion minimal graphs with $d$-cutwidth bigger than $k$ contains a finite set of graphs. We call this class immersion obstruction set for cutwidth at most $k$ and we denote it by $O_k$. This fact, combined with Theorem 1.i, implies that $cw_d(G) \leq k$ if and only if none of the graphs in $O_k$ is contained in $G$ as an immersion. According to the result of Grohe, Kawarabayashi, Marx, and Wollan in [5], checking whether an $n$-vertex graph contains as an immersion some $k$-vertex graph $H$, can be done in $f(k) \cdot n^3$ steps. As a consequence, checking whether $cw_d(G) \leq k$ can be done in $f(k) \cdot n^3$ steps. This running time can become linear (on $n$) using the first inequality of Theorem 1.iii. Indeed, the algorithm first checks whether $cw(G) \leq k$. If the answer is negative then we can safely report that $cw_d(G) > k$. If not, then it is known (see e.g. [10]) that $G$ has a tree decomposition of width $\leq k$ and to check whether some of the graphs in $O_k$ is contained in $G$ as an immersion can be done using dynamic programming in $f(k) \cdot n$ steps.

Unfortunately, the above algorithm is non-constructive as we have no other knowledge about the set $O_k$, except from the fact that it is finite. To obtain a constructive $f(k) \cdot n$ step algorithm for $d$-cutwidth remains an insisting open problem.

References