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To cite this version:
Menelaos Karavelas, Dimitris Zoros, Spyridon Maniatis, Dimitrios M. Thilikos. Geometric Extensions of Cutwidth in any Dimension. ICGT: International Colloquium on Graph Theory and combinatorics, Jun 2014, Grenoble, France. lirmm-01083698

HAL Id: lirmm-01083698
https://hal-lirmm.ccsd.cnrs.fr/lirmm-01083698
Submitted on 17 Nov 2014

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Geometric Extensions of Cutwidth in any Dimension*

Menelaos I. Karavelas†, Spyridon Maniatis‡, Dimitrios M. Thilikos‡§, Dimitris Zoros‡

Abstract

We define a multi-dimensional geometric extension of cutwidth. A graph has \(d\)-cutwidth at most \(k\) if it can be embedded in the \(d\)-dimensional euclidean space so that no hyperplane can intersect more than \(k\) of its edges. We prove a series of combinatorial results on \(d\)-cutwidth which imply that for every \(d\) and \(k\), there is a linear time algorithm checking whether the \(d\)-cutwidth of a graph \(G\) is at most \(k\).

1 Introduction

The cutwidth of a (total) vertex ordering of a graph is the maximum number of edges connecting vertices on opposite sides of any of the “gaps” between successive vertices in the linear layout. The cutwidth of a graph \(G\), denoted by \(cw(G)\), is the minimum cutwidth over all its possible vertex orderings. The problem that asks, given a \(n\)-vertex graph \(G\) and an integer \(k\), whether \(cw(G) \leq k\), is an NP-complete problem known in the literature as the MINIMUM CUT LINEAR ARRANGEMENT problem [4]. From the parameterized complexity point of view, the same problem is fixed parameter tractable, as an algorithm that checks whether \(cw(G) \leq k\) in \(f(k) \cdot n\) steps was given in [10]. Cutwidth has been extensively studied both from its combinatorial (see e.g. [2, 7, 1]) as well as its algorithmic point of view [8, 11, 3, 6].

\(d\)-dimensional cutwidth

In this note we introduce a multi-dimensional geometric extension of cutwidth, namely the \(d\)-dimensional cutwidth (or, simply, \(d\)-cutwidth) that, roughly, instead of mono-dimensional linear arrangements of the graph \(G\), we consider embeddings of \(G\) in the \(d\)-dimensional Euclidean space \(\mathbb{R}^d\) and define the \(d\)-cutwidth of such an embedding to be the maximum number of edges a hyperplane of \(\mathbb{R}^d\) can intersect. Then, the \(d\)-cutwidth of \(G\), denoted by \(cw_d(G)\), is the minimum \(d\)-cutwidth over all such embeddings. Our results are summarized in the following.

**Theorem 1** The following hold:

*The last three authors were co-financed by the E.U. (European Social Fund - ESF) and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF) - Research Funding Program: “Thales. Investing in knowledge society through the European Social Fund”. Emails: mkaravel@iacm.forth.gr, spyridon.maniatis@gmail.com, sedthilk@thilikos.info, dzoros@math.uoa.gr.

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i. $d$-cutwidth is immersion closed\(^1\).

ii. For every graph $G$ and every $d \geq 1$, $\text{cw}_d(G) \leq \text{cw}_{d+1}(G)$.

iii. For every graph $G$ and every $d \geq 1$, $\text{cw}_d(G) \leq d \cdot \text{cw}(G)$.

iv. For every graph $G$, $\text{cw}_3(G) \leq 2 \cdot \text{cw}_2(G)$.

2 Preliminaries and definitions

Every $(d-1)$-dimensional subspace $\Pi$ of a $d$-dimensional space $\mathcal{X}$ is called a hyperplane of $\mathcal{X}$. Here we are interested in hyperplanes of $\mathbb{R}^d$ (which are known to be isomorphic to $\mathbb{R}^{d-1}$). Let $\Pi$ be a hyperplane in $\mathbb{R}^d$, then there are $a_0,a_1,\ldots,a_d \in \mathbb{R}$ such that $\Pi = \{(x_1,\ldots,x_d) \in \mathbb{R}^d \mid a_1x_1 + \cdots + a_dx_d + a_0 = 0\}$. We denote by $H(d)$ the set of all hyperplanes of $\mathbb{R}^d$. A hypersphere, $S(c,r)$, with center $c$ and radius $r$ in $\mathbb{R}^d$ is the set $\{(x_1,\ldots,x_d) \in \mathbb{R}^d \mid \sum_{i=1}^d (x_i-c_i)^2 = r^2\}$. We denote by $S(d)$ the set of all hyperspheres of $\mathbb{R}^d$. We call a continuous function $C: [0,1] \to \mathbb{R}^d$ a curve of $\mathbb{R}^d$ with ends $C(0)$ and $C(1)$.

Let $G = (V,E)$ be a graph. An embedding of $G$, denoted by $E_d(G)$, in the euclidean space $\mathbb{R}^d$ is a tuple $(f,\mathcal{C})$, where $f: V \to \mathbb{R}^d$ is an injection, mapping the vertices of $G$ to $\mathbb{R}^d$ and $\mathcal{C} = \{C_e \mid e \in E\}$ is a set of curves of $\mathbb{R}^d$ with the following properties: (a) for every $e = \{u,v\} \in E$, the ends of $C_e$ are $f(u)$ and $f(v)$, and (b) for all $x \in (0,1)$ and $v \in V$ it holds that $f_x(x) \neq f(v)$. For simplicity, we may sometimes refer to the elements of $f(V)$ and $\mathcal{C}$ as the vertices and edges of $E_d(G)$ respectively. We denote by $E_d(G)$ the set of all embeddings $E_d(G) = (f,\mathcal{C})$, of $G$ in $\mathbb{R}^d$, such that for every positive integer $i \leq d$, if $S$ is a subset of $V$ with $|S| \geq i$, then the dimension of the subspace defined by $\{f(u) \mid u \in S\}$ is $i-1$. We call an element of $E_d(G)$ essential-embedding of $G$ in $\mathbb{R}^d$. Let $E_d(G)$ be a essential-embedding of $G$ in $\mathbb{R}^d$, then if $\Pi$ is a hyperplane of $\mathbb{R}^d$ (resp. $\Sigma$ is a hypersphere of $\mathbb{R}^d$) that does not intersect any $f(v)$, $v \in V$, we denote by $\partial_G(E_d(G),\Pi)$ (resp. $\partial_G(E_d(G),\Sigma)$) the set of curves of $E_d(G)$ that are intersected by $\Pi$ (resp. $\Sigma$).

**Definition 1** Let $G = (V,E)$ be a graph and $k,d$ be positive integers, where $d \geq 2$. Then we define the $d$-dimensional cutwidth of $G$, or simply $d$-cutwidth, to be

$$\text{cw}_d(G) = \min_{E_d(G) \in E_d(G)} \max \{ |\partial_G(E_d(G),\Pi)| \mid \Pi \in H(d) \}$$

Observe that any hyperplane $\Pi$ of $\mathbb{R}_d$ that meets a curve $C_e \in \mathcal{C}$ once, also meets the unique straight line segment of $\mathbb{R}^d$ with parametric equation $\sigma_e(t) = t \cdot C_e(0) + (1-t) \cdot C_e(1)$, $t \in \mathbb{R}$, i.e., the straight line segment of $\mathbb{R}^d$ that is defined by the “images” of the endpoints of edge $e$. Therefore, without loss of generality, we can consider only straight-line embeddings where $\mathcal{C} = \{C_e \mid e \in E\}$. Notice that every straight line embedding $E_d(G) = (f,\mathcal{C})$ is fully defined by the function $f$, therefore, for simplicity, for now on we will omit $\mathcal{C}$. Observe that the definition of 1-cutwidth, where hyperplanes degenerate to subspaces of $\mathbb{R}$ of dimension 0 (i.e., points) is equivalent to the usual definition of

\(^1\)A graph $H$ is an immersion of a graph $G$ if it can be obtained from $G$ after a sequence of vertex/edge removals or edge lifts (the operation of lifting two edges $\{x,y\}$ and $\{y,z\}$ incident to the same vertex $y$ is the operation of replacing these edges by the edge $\{x,z\}$). A graph invariant is immersion closed if its value on a graph $G$ is always smaller or equal than its value on its immersions.
cutwidth. Therefore, $d$-cutwidth is the intuitive generalization of the notion of cutwidth in any dimension $d \geq 2$. Also observe that our demand of essential embeddings is expressed here by our demand of injective functions.

3 Properties of $d$-cutwidth

This section is devoted to the last two statements of Theorem 1.

Proof of Theorem 1.iii. Consider the $d$-dimensional moment curve $C$ with parametric equation $C(t) := (t, t^2, t^3, \ldots, t^d)$, $t \in \mathbb{R}$. Consider also an ordering of the nodes of $G$ that realizes the cutwidth of $G$. Embed a node $v_i$ of $G$ to the point $p_i = C(t_i)$, for an appropriate value $t_i$. By appropriate we mean that if a node $v_i$ is after a node $v_j$ in the cutwidth ordering, then the parametric value $t_i$ corresponding to $v_i$ is strictly greater than the parameter value $t_j$ corresponding to node $v_j$. Now embed an edge $e_{ij} = (v_i, v_j)$ of $G$ by connecting the points $p_i$ and $p_j$ on $C$ with the minimum length arc of $C$ connecting these points.

Consider a generic hyperplane $\Pi$ with equation $a_1x_1 + a_2x_2 + \ldots + a_dx_d + a_0 = 0$, where, for all $i$, $a_i \in \mathbb{R}$. $\Pi$ can cut $C$ at at most $d$ points. To see that, solve the system of equations $a_1x_1 + a_2x_2 + \ldots + a_dx_d + a_0 = 0$, and $x_i = t^i$, $i = 1, \ldots, d$ for $t$. This gives the polynomial equation $q(t) := a_0 + a_1t + a_2t^2 + \ldots + a_dt^d = 0$, in $t$ of maximum degree $d$. Since $q(t) = 0$ has at most $d$ real roots, we deduce that $\Pi$ intersects $C$ at at most $d$ points. At each point of intersection at most $\text{cw}(G)$ edges of the embedding of $G$ pass through that point. Hence, $\Pi$ intersects at most $d \cdot \text{cw}(G)$ edges of $G$, i.e., $\text{cw}_d(G) \leq d \cdot \text{cw}(G)$. □

Spherical $d$-cutwidth. Given a graph $G = (V, E)$ and two positive integers $k$ and $d$, where $d \geq 2$, we define the spherical $d$-dimensional cutwidth of $G$, or simply spherical $d$-cutwidth, to be equal to

$$\text{scw}_d(G) = \min_{E_d(G) \in \mathcal{E}_d(G)} \max\{|\partial_G(E_d(G), \Sigma)| | \Sigma \in S(d)\}$$

The proof of Theorem 1.iv is a consequence of Theorem 1.iii and the following two lemmata.

Lemma 1 For every graph $G$ and any $d \geq 2$, $\text{cw}_d(G) \leq \text{scw}_d(G) \leq (d + 1) \cdot \text{cw}(G)$.

Lemma 2 For every graph $G$ and every $d \geq 1$, $\text{cw}_{d+1}(G) \leq \text{scw}_d(G)$.

The above results clarify the relation between $d$-cutwidth and spherical $d$-cutwidth and we believe that they have independent interest. We omit the proofs as they are too lengthy to fit in this extended abstract.
4 Algorithmic remarks about $d$-cutwidth

As a consequence of the result in [9], for every $k$, the class of immersion minimal graphs with $d$-cutwidth bigger than $k$ contains a finite set of graphs. We call this class immersion obstruction set for cutwidth at most $k$ and we denote it by $O_k$. This fact, combined with Theorem 1.i, implies that $\text{cw}_{d}(G) \leq k$ if and only if none of the graphs in $O_k$ is contained in $G$ as an immersion. According to the result of Grohe, Kawarabayashi, Marx, and Wollan in [5], checking whether an $n$-vertex graph contains as an immersion some $k$-vertex graph $H$, can be done in $f(k) \cdot n^3$ steps. As a consequence, checking whether $\text{cw}_{d}(G) \leq k$ can be done in $f(k) \cdot n^3$ steps. This running time can become linear (on $n$) using the first inequality of Theorem 1.iii. Indeed, the algorithm first checks whether $\text{cw}(G) \leq k$. If the answer is negative then we can safely report that $\text{cw}_{d}(G) > k$. If not, then it is known (see e.g. [10]) that $G$ has a tree decomposition of width $\leq k$ and to check whether some of the graphs in $O_k$ is contained in $G$ as an immersion can be done using dynamic programming in $f(k) \cdot n$ steps.

Unfortunately, the above algorithm is non-constructive as we have no other knowledge about the set $O_k$, except from the fact that it is finite. To obtain a constructive $f(k) \cdot n$ step algorithm for $d$-cutwidth remains an insisting open problem.

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