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# Geometric Extensions of Cutwidth in any Dimension* 

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#### Abstract

We define a multi-dimensional geometric extension of cutwidth. A graph has $d$ cutwidth at most $k$ if it can be embedded in the $d$-dimensional euclidean space so that no hyperplane can intersect more than $k$ of its edges. We prove a series of combinatorial results on $d$-cutwidth which imply that for every $d$ and $k$, there is a linear time algorithm checking whether the $d$-cutwidth of a graph $G$ is at most $k$.


## 1 Introduction

The cutwidth of a (total) vertex ordering of a graph is the maximum number of edges connecting vertices on opposite sides of any of the "gaps" between successive vertices in the linear layout. The cutwidth of a graph $G$, denoted by $\mathrm{Cw}(G)$, is the minimum cutwidth over all its possible vertex orderings. The problem that asks, given a $n$-vertex graph $G$ and an integer $k$, whether $\mathrm{Cw}(G) \leq k$, is an NP-complete problem known in the literature as the Minimum Cut Linear Arrangement problem [4]. From the parameterized complexity point of view, the same problem is fixed parameter tractable, as an algorithm that checks whether cutwidth $(G) \leq k$ in $f(k) \cdot n$ steps was given in [10]. Cutwidth has been extensively studied both from its combinatorial (see e.g. [2, 7, 1]) as well as its algorithmic point of view $[8,11,3,6]$.
$d$-dimensional cutwidth In this note we introduce a multi-dimensional geometric extension of cutwidth, namely the $d$-dimensional cutwidth (or, simply, $d$-cutwidth) that, roughly, instead of mono-dimensional linear arrangements of the graph $G$, we consider embeddings of $G$ in the $d$-dimensional Euclidean space $\mathbb{R}^{d}$ and define the $d$-cutwidth of such an embedding to be the maximum number of edges a hyperplane of $\mathbb{R}^{d}$ can intersect. Then, the $d$-cutwidth of $G$, denoted by $\mathrm{CW}_{d}(G)$, is the minimun $d$-cutwidth over all such embeddings. Our results are summarized in the following.

## Theorem 1 The following hold:

[^0]i. $d$-cutwidth is immersion closed ${ }^{1}$.
ii. For every graph $G$ and every $d \geq 1, \mathrm{CW}_{d}(G) \leq \mathrm{CW}_{d+1}(G)$.
iii. For every graph $G$ and every $d \geq 1, \mathrm{CW}_{d}(G) \leq d \cdot \mathrm{CW}(G)$.
iv. For every graph $G, \mathrm{CW}_{3}(G) \leq 2 \cdot \mathrm{CW}_{2}(G)$.

## 2 Preliminaries and definitions

Every $(d-1)$-dimensional subspace $\Pi$ of a $d$-dimensional space $\mathcal{X}$ is called a hyperplane of $\mathcal{X}$. Here we are interested in hyperplanes of $\mathbb{R}^{d}$ (which are known to be isomorphic to $\left.\mathbb{R}^{d-1}\right)$. Let $\Pi$ be a hyperplane in $\mathbb{R}^{d}$, then there are $a_{0}, a_{1}, \ldots, a_{d} \in \mathbb{R}$ such that $\Pi=\left\{\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d} \mid a_{1} x_{1}+\cdots+a_{d} x_{d}+a_{0}=0\right\}$. We denote by $H(d)$ the set of all hyperplanes of $\mathbb{R}^{d}$. A hypersphere, $S(c, r)$, with center $c$ and radius $r$ in $\mathbb{R}^{d}$ is the set $\left\{\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d} \mid \sum_{i=1}^{d}\left(x_{i}-c_{i}\right)^{2}=r^{2}\right\}$. We denote by $S(d)$ the set of all hyperspheres of $\mathbb{R}^{d}$. We call a continuous function $C:[0,1] \rightarrow \mathbb{R}^{d}$ a curve of $\mathbb{R}^{d}$ with ends $C(0)$ and $C(1)$.

Let $G=(V, E)$ be a graph. An embedding of $G$, denoted by $\mathcal{E}_{d}(G)$, in the euclidean space $\mathbb{R}^{d}$ is a tuple $(f, \mathcal{C})$, where $f: V \rightarrow \mathbb{R}^{d}$ is an injection, mapping the vertices of $G$ to $\mathbb{R}^{d}$ and $\mathcal{C}=\left\{C_{e} \mid e \in E\right\}$ is a set of curves of $\mathbb{R}^{d}$ with the following properties: (a) for every $e=\{u, v\} \in E$, the ends of $C_{e}$ are $f(u)$ and $f(v)$, and (b) for all $x \in(0,1)$ and $v \in V$ it holds that $f_{e}(x) \neq f(v)$. For simplicity, we may sometimes refer to the elements of $f(V)$ and $\mathcal{C}$ as the vertices and edges of $\mathcal{E}_{d}(G)$ respectively. We denote by $\mathbf{E}_{d}(G)$ the set of all embeddings $\mathcal{E}_{d}(G)=(f, \mathcal{C})$, of $G$ in $\mathbb{R}^{d}$, such that for every positive integer $i \leq d$, if $S$ is a subset of $V$ with $|S| \geq i$, then the dimension of the subspace defined by $\{f(u) \mid u \in S\}$ is $i-1$. We call an element of $\mathbf{E}_{d}(G)$ essential-embedding of $G$ in $\mathbb{R}^{d}$. Let $\mathcal{E}_{d}(G)$ be a essential-embedding of $G$ in $\mathbb{R}^{d}$, then if $\Pi$ is a hyperplane of $\mathbb{R}^{d}$ (resp. $\Sigma$ is a hypersphere of $\mathbb{R}^{d}$ ) that does not intersect any $f(v), v \in V$, we denote by $\partial_{G}\left(\mathcal{E}_{d}(G), \Pi\right)\left(\operatorname{resp} . \partial_{G}\left(\mathcal{E}_{d}(G), \Sigma\right)\right)$ the set of curves of $\mathcal{E}_{d}(G)$ that are intersected by $\Pi$ (resp. $\Sigma$ ).

Definition 1 Let $G=(V, E)$ be a graph and $k, d$ be positive integers, where $d \geq 2$. Then we define the d-dimensional cutwidth of $G$, or simply d-cutwidth, to be

$$
\mathrm{CW}_{d}(G)=\min _{\mathcal{E}_{d}(G) \in \mathbf{E}_{d}(G)} \max \left\{\left|\partial_{G}\left(\mathcal{E}_{d}(G), \Pi\right)\right| \mid \Pi \in H(d)\right\}
$$

Observe that any hyperplane $\Pi$ of $\mathbb{R}_{d}$ that meets a curve $C_{e} \in \mathcal{C}$ once, also meets the unique straight line segment of $\mathbb{R}^{d}$ with parametric equation $\sigma_{e}(t)=t \cdot C_{e}(0)+(1-t)$. $C_{e}(1), t \in \mathbb{R}$, i.e., the straight line segment of $\mathbb{R}^{d}$ that is defined by the "images" of the endpoints of edge $e$. Therefore, without loss of generality, we can consider only straightline embeddings where $\mathcal{C}=\left\{\sigma_{e} \mid e \in E\right\}$. Notice that every straight line embeding $\mathcal{E}_{d}(G)=(f, \mathcal{C})$ is fully defined by the function $f$, therefore, for simplicity, for now on we will omit $\mathcal{C}$. Observe that the definition of 1-cutwidth, where hyperplanes degenerate to subspaces of $\mathbb{R}$ of dimension 0 (i.e., points) is equivalent to the usual definition of

[^1]cutwidth. Therefore, $d$-cutwidth is the intuitive generalization of the notion of cutwidth in any dimension $d \geq 2$. Also observe that our demand of essential embeddings is expressed here by our demand of injective functions.

## 3 Properties of $d$-cutwidth

This section is devoted to the last two statements of Theorem 1.
Proof of Theorem 1.iii. Consider the $d$-dimensional moment curve $C$ with parametric equation $C(t):=\left(t, t^{2}, t^{3}, \ldots, t^{d}\right), \quad t \in \mathbb{R}$. Consider also an ordering of the nodes of $G$ that realizes the cutwidth of $G$. Embed a node $v_{i}$ of $G$ to the point $p_{i}=C\left(t_{i}\right)$, for an appropriate value $t_{i}$. By appropriate we mean that if a node $v_{i}$ is after a node $v_{j}$ in the cutwidth ordering, then the parametric value $t_{i}$ corresponding to $v_{i}$ is strictly greater than the parameter value $t_{j}$ corresponding to node $v_{j}$. Now embed an edge $e_{i j}=\left(v_{i}, v_{j}\right)$ of $G$ by connecting the points $p_{i}$ and $p_{j}$ on $C$ with the minimum length $\operatorname{arc}$ of $C$ connecting these points.

Consider a generic hyperplane $\Pi$ with equation $a_{1} x_{1}+a_{2} x_{2}+\ldots+a_{d} x_{d}+a_{0}=0$, where, for all $i, a_{i} \in \mathbb{R} . \Pi$ can cut $C$ at at most $d$ points. To see that, solve the system of equations

$$
a_{1} x_{1}+a_{2} x_{2}+\ldots+a_{d} x_{d}+a_{0}=0, \quad \text { and } \quad x_{i}=t^{i}, \quad i=1, \ldots, d
$$

for $t$. This gives the polynomial equation $q(t):=a_{0}+a_{1} t+a_{2} t^{2}+\ldots+a_{d} t^{d}=0$, in $t$ of maximum degree $d$. Since $q(t)=0$ has at most $d$ real roots, we deduce that $\Pi$ intersects $C$ at at most $d$ points. At each point of intersection at $\operatorname{most} \operatorname{Cw}(G)$ edges of the embedding of $G$ pass through that point. Hence, $\Pi$ intersects at most $d \cdot \mathrm{Cw}(G)$ edges of $G$, i.e., $\mathrm{CW}_{d}(G) \leq d \mathrm{CW}(G)$.

Spherical $d$-cutwith Given a graph $G=(V, E)$ and two positive integers $k$ and $d$, where $d \geq 2$, we define the spherical $d$-dimensional cutwidth of $G$, or simply spherical $d$-cutwidth, to be equal to

$$
\operatorname{SCW}_{d}(G)=\min _{\mathcal{E}_{d}(G) \in \mathbf{E}_{d}(G)} \max \left\{\left|\partial_{G}\left(\mathcal{E}_{d}(G), \Sigma\right)\right| \mid \Sigma \in S(d)\right\}
$$

The proof of Theorem 1.iv is a consequence of Theorem 1.iii and the following two lemmata.

Lemma 1 For every graph $G$ and any $d \geq 2, \mathrm{CW}_{d}(G) \leq \operatorname{SCW}_{d}(G) \leq(d+1) \mathrm{Cw}(G)$.
Lemma 2 For every graph $G$ and every $d \geq 1, \mathrm{CW}_{d+1}(G) \leq \operatorname{sCW}_{d}(G)$.
The above results clarify the relation between $d$-cutwidth and spherical $d$-cutwidth and we believe that they have independent interest. We omit the proofs as they are too lengthy to fit in this extended abstract.

## 4 Algorithmic remarks about $d$-cutwidth

As a consequence of the result in [9], for every $k$, the class of immersion minimal graphs with $d$-cutwidth bigger than $k$ contains a finite set of graphs. We call this class immersion obstruction set for cutwidth at most $k$ and we denote it by $\mathcal{O}_{k}$. This fact, combined with Theorem $1 . i$, implies that $\mathrm{CW}_{d}(G) \leq k$ if and only if none of the graphs in $\mathcal{O}_{k}$ is contained in $G$ as an immersion. According to the result of Grohe, Kawarabayashi, Marx, and Wollan in [5], checking whether an $n$-vertex graph contains as an immersion some $k$-vertex graph $H$, can be done in $f(k) \cdot n^{3}$ steps. As a consequence, checking whether $\mathrm{Cw}_{d}(G) \leq k$ can be done in $f(k) \cdot n^{3}$ steps. This running time can become linear (on $n$ ) using the first inequality of Theorem 1.iii. Indeed, the algorithm first checks whether $\operatorname{cw}(G) \leq k$. If the answer is negative then we can safely report that $\mathrm{CW}_{d}(G)>k$. If not, then it is known (see e.g. [10]) that $G$ has a tree decomposition of width $\leq k$ and to check whether some of the graphs in $\mathcal{O}_{k}$ is contained in $G$ as an immersion can be done using dynamic programming in $f(k) \cdot n$ steps.

Unfortunately, the above algorithm is non-constructive as we have no other knowledge about the set $\mathcal{O}_{k}$, except from the fact that it is finite. To obtain a constructive $f(k) \cdot n$ step algorithm for $d$-cutwidth remains an insisting open problem.

## References

[1] F. R. K. Chung and Paul D. Seymour. Graphs with small bandwidth and cutwidth. Discrete Mathematics, 75(1-3):113-119, 1989.
[2] Fan R. K. Chung. On the cutwidth and the topological bandwidth of a tree. SIAM Journal on Algebraic and Discrete Methods, 6(2):268-277, 1985.
[3] Moon Jung Chung, Fillia Makedon, Ivan Hal Sudborough, and Jonathan Turner. Polynomial time algorithms for the MIN CUT problem on degree restricted trees. SIAM Journal on Computing, 14(1):158-177, 1985.
[4] Michael R. Garey and David S. Johnson. Computers and intractability. A guide to the theory of NP-completeness. W. H. Freeman and Co., San Francisco, Calif., 1979.
[5] Martin Grohe, Ken-ichi Kawarabayashi, Dániel Marx, and Paul Wollan. Finding topological subgraphs is fixed-parameter tractable. In Proceedings of the $43 r d$ ACM Symposium on Theory of Computing, (STOC 2011), pages 479-488, 2011.
[6] Dimitrios M. Thilikos Hans L. Bodlaender, Michael R. Fellows. Derivation of algorithms for cutwidth and related graph layout parameters. Journal of Computer and System Sciences, 75(4):231-244, 2009.
[7] Ephraim Korach and Nir Solel. Tree-width, path-width, and cutwidth. Discrete Applied Mathematics, 43(1):97-101, 1993.
[8] B. Monien and I. H. Sudborough. Min cut is NP-complete for edge weighted trees. Theoretical Computer Science, 58(1-3):209-229, 1988.
[9] Neil Robertson and Paul D. Seymour. Graph minors XXIII. Nash-Williams' immersion conjecture. J. Combin. Theory Ser. B, 100(2):181-205, 2010.
[10] Dimitrios M. Thilikos, Maria J. Serna, and Hans L. Bodlaender. Cutwidth I: A linear time fixed parameter algorithm. Journal of Algorithms, To appear.
[11] Mihalis Yannakakis. A polynomial algorithm for the min-cut linear arrangement of trees. Journal of the ACM, 32(4):950-988, 1985.


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[^1]:    ${ }^{1}$ A graph $H$ is an immersion of a graph $G$ if it can be obtained from $G$ after a sequence of vertex/edge removals or edge lifts (the operation of lifting two edges $\{x, y\}$ and $\{y, z\}$ incident to the same vertex $y$ is the operation of replacing these edges by the edge $\{x, z\})$. A graph invariant is immersion closed if its value on a graph $G$ is always smaller or equal than its value on its immersions.

