Multigraphs without large bonds are wqo by contraction

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Multigraphs without large bonds are wqo by contraction*

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Abstract

We show that the class of multigraphs with at most p connected components and bonds of size at most k is well-quasi-ordered by edge contraction for all positive integers p, k. (A *bond* is a minimal non-empty edge cut.) We also characterize canonical antichains for this relation.

1 Introduction

A *well-quasi-order* (*wqo* for short) is a quasi-order which contains neither infinite decreasing sequences, nor infinite collections of pairwise incomparable elements. The beginnings of the theory of well-quasi-orders go back to the 1950s and some early results on wqos include that of Higman on sequences from a wqo [6], Kruskal's Tree Theorem [8], as well as other (now standard) techniques, for example the *minimal bad sequence* argument of Nash-Williams [9].

A recent result on wqos and arguably one of the most significant results in this field is the theorem by Robertson and Seymour which states that graphs are well-quasi-ordered by the minor relation [11]. Later, the same authors also proved that graphs are well-quasi-ordered by the immersion relation [10].

Nonetheless, most of containment relations do not well-quasi-order the class of all graphs. For example, graphs are not well-quasi-ordered by subgraphs, induced subgraphs, or topological minors. Therefore, attention was naturally brought to classes of graphs where well-quasi-ordering for such relations exists. Damaschke proved that cographs are well-quasi-ordered by induced subgraphs [1] and Ding characterized subgraph ideals that are well-quasi-ordered by the subgraph relation [2]. Finally, Liu and Thomas recently announced that graphs excluding any graph of a class called

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"Robertson chain" as topological minor are well-quasi-ordered by the topological minor relation [7].

An *antichain* is a collection of non-comparable elements. Another line of research is to classify non-wqo containment relations depending on the antichains they contain. Ding introduced the concepts of *canonical antichain* and *fundamental antichain* aimed at extending the study of the existence of obstructions of being well-quasi-ordered in a quasi-order [4]. In particular, he proved that finite graphs do not admit a canonical antichain under the induced subgraph relation but they do under the subgraph relation.

In this paper, we consider finite graphs where parallel edges are allowed, but not loops. Graphs where no edges are parallel are referred to as *simple graphs*. An *edge contraction* is the operation that identifies two adjacent vertices and deletes the loops that were possibly created (but keeps multiple edges). A graph H is said to be a *contraction* of a graph G, denoted $H \trianglelefteq G$, if H can be obtained from G by a sequence of edge contractions. A *bond* is a minimal non-empty edge cut, i.e. a minimal set of edges whose removal increases the number of connected components (cf. Figure 1).

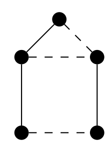

Figure 1: A bond of size 3 (dashed edges) in the house graph.

The contraction relation defines a quasi-order on finite graphs. This is not a wqo, as witnessed by the following infinite antichains, that are also depicted in Figure 2: $\mathcal{A}_\theta$ is the class of connected graphs with two vertices and $\mathcal{A}_{\overline{K}}$ is the class of edgeless graphs with at least one vertex. For the contraction ordering, infinite antichains are the only obstructions to well-quasi-ordering as decreasing sequences are always finite.

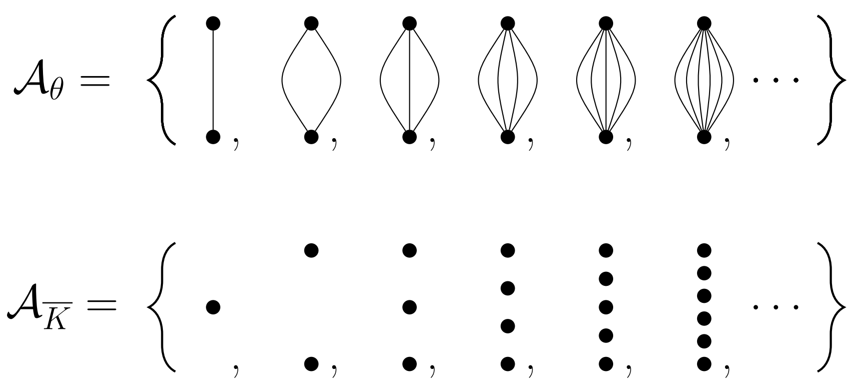

Figure 2: Two infinite antichains for contractions: multiedges and cocliques.

For every $p, k \in \mathbb{N}$, let $\mathcal{G}_{p,k}$ be the class of graphs having at most p connected components and not containing a bond of order more than k. Our main result is the following.

Theorem 1. *For every $p, k \in \mathbb{N}$, the class $\mathcal{G}_{p,k}$ is well-quasi-ordered by $\trianglelefteq$.*

The complement of a simple graph G, denoted $\overline{G}$ is the graph obtained by replacing every edge by a non-edge and vice-versa in G. Remark that a graph has a bond of order k iff it contains θ_k as contraction, and that it has p connected components iff it can be contracted to $\overline{K}_p$. A class $\mathcal{G}$ of graphs is said to be *contraction-closed* if $H \in \mathcal{G}$ whenever $H \trianglelefteq G$ for some $G \in \mathcal{G}$. As a consequence of our main theorem and of the fact that each infinite subset of $\mathcal{A}_\theta$ or $\mathcal{A}_{\overline{K}}$ is an obstruction to be well-quasi-ordered, we have the following characterization.

Corollary 1. *A contraction-closed class $\mathcal{H}$ is well-quasi-ordered by $\trianglelefteq$ iff there are $k, p \in \mathbb{N}$ such that $\forall k' > k,\ \theta_{k'} \notin \mathcal{H}$ and $\forall p' > p,\ \overline{K}_p \notin \mathcal{H}$.*

In his study of infinite antichains for the (induced) subgraph relation, Ding [4] introduced the two following concepts. An antichain $\mathcal{A}$ of a quasi-order $(\mathcal{S}, \preceq)$ is said to be *canonical* if the following holds for every $\preceq$-closed subclass $\mathcal{J}$ of $\mathcal{S}$: $\mathcal{J}$ has an infinite antichain iff $\mathcal{J} \cap \mathcal{A}$ is infinite. Canonical antichains can be used to characterize the $\preceq$-closed subclasses of a quasi-order $(\mathcal{S}, \preceq)$ and also to describe the variety of its antichains.

The following result is a complete characterization of the canonical antichains of $\trianglelefteq$ in finite graphs, which extends the results of Ding on canonical antichains of simple graphs for the relations of subgraph and induced subgraph [4].

Theorem 2. *An antichain $\mathcal{A}$ of $\trianglelefteq$ is canonical iff $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}} \setminus \mathcal{A}$ is finite.*

In other words, an antichain $\mathcal{A}$ is canonical iff it contains all but finitely many graphs from $\mathcal{A}_\theta$ and $\mathcal{A}_{\overline{K}}$. As noted in the proof, this implies that a canonical antichain contains a finite number of graphs that do not belong to $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}$. In the course of studying antichains, we also obtained general results that are of independent interest. They are presented in the corresponding section.

Organization of the paper. The notions and tools that are used in this paper are introduced in Section 2. Theorem 1 is proved in Section 3 and results pertaining to antichains appear in Section 4.

Conclusion. This work settles the case of multigraph contractions in the study of well-quasi-ordered subclasses, a problem investigated by Damaschke for induced subgraphs [1], by Ding for subgraphs [2] and induced minors [3], and by Fellows et al. for several containment relations [5]. In particular, we give necessary and sufficient conditions for a class of (multi)graphs to be well-quasi-ordered by multigraph contractions. Furthermore, we characterize canonical antichains for this relation, in the continuation of Ding's results for subgraph and contraction relation in [4].

2 Preliminaries

We denote by $V(G)$ the set of vertices of a graph G and by $E(G)$ its multiset of edges. Given two adjacent vertices u, v of a graph G, $\mathrm{mult}_G(\{u, v\})$ stands for the number of parallel edges between u and v, called *multiplicity* of the edge $\{u, v\}$. We denote by $\mathcal{P}^{<\omega}(S)$ the class of finite subsets of a set S and by $\mathcal{P}(S)$ its power set.

Definition 1. A *model* of H in G (*H-model* for short) is a function $\mu \colon V(H) \to \mathcal{P}(V(G))$ such that:

(M1) $\mu(u)$ and $\mu(v)$ are disjoint whenever $u, v \in V(H)$ are distinct;

(M2) $\{\mu(u)\}_{u \in V(H)}$ is a partition of $V(G)$;

(M3) for every $u \in V(H)$, the graph $G[\mu(u)]$ is connected;

(M4) for every $u, v \in V(H)$, $\mathrm{mult}_H(\{u, v\}) = \sum_{(u',v') \in \mu(u) \times \mu(v)} \mathrm{mult}_G(\{u', v'\})$.

Remark that H is a contraction of G iff G has an H-model. Intuitively, an H-models indicates which connected subgraphs to contract in G in order to obtain a graph isomorphic to H.

Quasi-orders. A *quasi-ordered set* (*qoset* for short) is a pair $(\Sigma, \preceq)$ where Σ is a set and $\preceq$ is a binary relation on S which is reflexive and transitive. An *antichain* is a sequence of pairwise non-comparable elements. We say that $(\Sigma, \preceq)$ is a *well-quasi-order* (*wqo* for short), or that its elements are *well-quasi-ordered* by $\preceq$, if it has neither an infinite decreasing sequence, nor an infinite antichain.

Definition 2 (Cartesian production of qosets)**.** If $(\Sigma, \preceq_\Sigma)$ and $(\Sigma', \preceq_{\Sigma'})$ are two qosets, then their *Cartesian product* $(\Sigma \times \Sigma', \preceq_\Sigma \times \preceq_{\Sigma'})$ is the qoset defined by:

$$\forall (x, x'), (y, y') \in \Sigma \times \Sigma',\ (x, x') \preceq_\Sigma \times \preceq_{\Sigma'} (y, y') \text{ if } x \preceq_\Sigma y \text{ and } x' \preceq_{\Sigma'} y'.$$

Proposition 1 (Higman [6])**.** *If $(\Sigma, \preceq_\Sigma)$ and $(\Sigma', \preceq_{\Sigma'})$ are wqo, then so is $(\Sigma \times \Sigma', \preceq_\Sigma \times \preceq_{\Sigma'})$.*

Sequences. We use the notation $\Sigma^\star$ for the class of all finite sequences over the set Σ (including the empty sequence). For any qoset $(\Sigma, \preceq)$, we define the relation $\preceq^\star$ on $\Sigma^\star$ as follows: for every $r = (r_1, \dots, r_p)$ and $s = (s_1, \dots, s_q)$ of $\Sigma^\star$, we have $r \preceq^\star s$ if there is a increasing function $\varphi \colon \{1, p\} \to \{1, q\}$ such that for every $i \in \{1, p\}$ we have $r_i \preceq s_{\varphi(i)}$. This generalizes the subsequence relation. This order relation is extended to the class $\mathcal{P}^{<\omega}(\Sigma)$ of finite subsets of Σ as follows, generalizing the subset relation: for every $A, B \in \mathcal{P}^{<\omega}(\Sigma)$, we write $A \preceq^\star B$ if there is an injection $\varphi \colon A \to B$ such that $\forall x \in A,\ x \preceq \varphi(x)$.

Proposition 2 (Higman [6])**.** *If $(\Sigma, \preceq)$ is a wqo, then so is $(\Sigma^\star, \preceq^\star)$.*

Labeled graphs as defined below will allow us to focus on 2-connected graphs.

Definition 3 (labeled graph)**.** Let $(\Sigma, \preceq)$ be a qoset. A $(\Sigma, \preceq)$-labeled graph is a pair (G, λ) where $\lambda \colon V(G) \to \mathcal{P}^{<\omega}(\Sigma)$ is a function, referred to as the *labeling of the graph*. If $\mathcal{H}$ is a class of (unlabeled) graphs, $\mathrm{lab}_{(\Sigma, \preceq)}(\mathcal{H})$ denotes the class of $(\Sigma, \preceq)$-labeled graphs of $\mathcal{H}$.

For simplicity, we will use the same symbol G to refer to a labeled graph and its underlying unlabeled graph and we will denote by λ_G its labeling function. Remark that any unlabeled graph can be seen as a $\emptyset$-labeled graph.

The contraction relation is extended to labeled graphs by additionally allowing to relabel by l' any vertex labeled l whenever $l' \preceq l$. In terms of models, this corresponds to the following extra requirement for μ to be a model of H in G:

(M5) $\forall v \in V(H),\ \lambda_H(v) \preceq^\star \bigcup_{u' \in \mu(u)} \lambda_G(u')$.

3 Small bonds and well-quasi-ordering

We first show that we can focus on (labeled) 2-connected graphs.

Lemma 1. *Let $\mathcal{H}$ be a class of graphs and let $\mathcal{H}'$ denote the class of graphs of $\mathcal{H}$ that are connected. If $(\mathcal{H}', \trianglelefteq)$ is a wqo, then so is $(\mathcal{H}, \trianglelefteq)$.*

Proof. Every disconnected graph of $\mathcal{H}$ can be seen as a sequence of graphs of $\mathcal{H}'$. Then the result follows by an application of Higman's Lemma (Proposition 2). □

Lemma 2. *Let $\mathcal{H}$ be a class of connected graphs and let $\mathcal{H}^{(2)}$ be the subclass of 2-connected graphs of $\mathcal{H}$. If for every wqo $(\Sigma, \preceq)$, the qoset $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{H}^{(2)}), \trianglelefteq)$ is a wqo, then so is $(\mathcal{H}, \trianglelefteq)$.*

Proof. This proof is very similar to the induced minor case proved in [5]. We deal here with rooted graphs, that are graphs with a distinguished vertex called *root*. We denote the root of a rooted graph G by $\mathrm{root}(G)$. The contraction relation is extended to this setting by requiring roots to be contracted to roots. Formally, $\mathrm{root}(G) \in \mu(\mathrm{root}(H))$ for every model of H in G. Assuming that $(\mathcal{H}, \trianglelefteq)$ is not a wqo, we consider the class $\mathcal{H}_r$ of all graphs that can be obtained by choosing a root in a graph of $\mathcal{H}$. Clearly, this class is not well-quasi-ordered by $\trianglelefteq$.

In a sequence of graphs, (H, G) is a *good pair* if H appear before G and $H \trianglelefteq G$. We use the concept of *bad sequence*, that are infinite sequences with no good pair. The absence of bad sequences in a class of graphs is equivalent to this class being well-quasi-ordered (see [9]). Towards a contradiction, we suppose the existence of a bad sequence. We consider a minimal one, in the following sense. Let $(G_i)_{i\in\mathbb{N}}$ be a bad sequence of graphs of $\mathcal{H}_r$ such that for every $i \in \mathbb{N}$, there is no contraction G of G_i (distinct from G_i) such that a bad sequence starts with $G_0, \dots, G_{i-1}, G$.

For every $i \in \mathbb{N}$, let A_i be the maximal 2-connected subgraph of G_i which contains $\mathrm{root}(G_i)$. Let C_i the set of cutvertices of G_i that belong to A_i. For each cutvertex $c \in C_i$, let B^i_c be the connected component of $G_i \setminus (V(A_i) \setminus C_i)$ turned into a rooted graph by setting $\mathrm{root}(B^i_c) = c$. Note that we have $B^i_c \trianglelefteq G_i$. Let us denote by $\mathcal{B}$ the family of rooted graphs $\mathcal{B} = \{B^i_c \colon c \in C_i, i \in \mathbb{N}\}$. We will show that $(\mathcal{B}, \trianglelefteq)$ is a wqo. Let $(H_j)_{j\in\mathbb{N}}$ be an infinite sequence in $\mathcal{B}$ and for every $j \in \mathbb{N}$ choose an $i = \varphi(j) \in \mathbb{N}$ for which $H_j \trianglelefteq G_i$. Pick a j with smallest $\varphi(j)$, and consider the sequence $G_1, \dots, G_{\varphi(j)-1}, H_j, H_{j+1}, \dots$. By minimality of $(G_i)_{i\in\mathbb{N}}$ and by our choice of j, since $H_j \trianglelefteq G_{\varphi(j)}$ and $H_j \neq G_{\varphi(j)}$, this is not a bad sequence so it contains a good pair (G, G'). Now, if G is among the first $\varphi(j)-1$ elements, then as $(G_i)_{i\in\mathbb{N}}$ is bad we must have $G' = H_{j'}$ for some $j' \geq j$ and so we have $G_{i'} = G \trianglelefteq G' = H_{j'} \trianglelefteq G_{\varphi(j')}$, a contradiction. So there is a good pair in $(H_i)_{i\geq j}$ and hence the infinite sequence $(H_j)_{j\in\mathbb{N}}$ has a good pair, so $(\mathcal{B}, \trianglelefteq)$ is a wqo.

We will now find a good pair in $(G_i)_{i\in\mathbb{N}}$ to show a contradiction. The idea is to label the graph family $\mathcal{A} = \{A_i\}_{i\in\mathbb{N}}$ so that each cutvertex c of a graph A_i gets labeled by their corresponding connected component B_c^i, and the roots are preserved under this labeling. More precisely, for each A_i we define a labeling σ_i that assigns to every vertex $v \in V(G_i)$ a label $\{(\sigma_i^1(v), \sigma_i^2(v))\}$ defined as follows:

- $\sigma_i^1(v) = 1$ if $v = \text{root}(G_i)$ and $\sigma_i^1(v) = 0$ otherwise;
- $\sigma_i^2(v) = B_v^i$ if $v \in C_i$ and $\sigma_i^2(v)$ is the one-vertex rooted graph otherwise.

The labeling σ of $\mathcal{A}$ is then $\{\sigma_i : i \in \mathbb{N}\}$. Let us define a quasi-ordering $\preceq$ on the set of labels Σ assigned by σ. For two labels $(s_a^1, s_a^2), (s_b^1, s_b^2) \in \Sigma$ we define $(s_a^1, s_a^2) \preceq (s_b^1, s_b^2)$ iff $s_a^1 = s_b^1$ and $s_a^2 \trianglelefteq s_b^2$. Note that in this situation, s_a^2 and s_b^2 are rooted graphs, so $\trianglelefteq$ compares rooted graphs. Observe that since $(\mathcal{B}, \trianglelefteq)$ is wqo, then $(\Sigma, \preceq)$ is wqo. For every $i \in \mathbb{N}$, let A_i' be the $(\Sigma, \preceq)$-labeled rooted graph (A_i, σ_i). We now consider the infinite sequence $(A_i')_{i\in\mathbb{N}}$. By our initial assumption, $(\text{lab}_\Sigma(\mathcal{A}), \trianglelefteq)$ is wqo (as $\mathcal{A}$ consists only in 2-connected graphs), so there is a good pair (A_i', A_j') in the sequence $(A_i')_{i\in\mathbb{N}}$.

To complete the proof, we will show that $A_i' \trianglelefteq A_j' \Rightarrow G_i \trianglelefteq G_j$. Let μ be a model of A_i' in A_j'. Then for each cutvertex $c \in C_i$, $\mu(c)$ contains a vertex $d \in C_j$ with $B_c^i \trianglelefteq B_d^j$. Let μ_c denote a model of B_c^i onto B_d^i. We construct a model g as follows:

$$\nu\colon \begin{cases} V(G_i) & \to & \mathcal{P}(V(G_j)) \\ v & \mapsto & \mu(v) \text{ if } v \in A_i \setminus C_i \\ v & \mapsto & \mu_c(v) \text{ if } v \in B_c^i \setminus C_i \\ v & \mapsto & \mu(v) \cup \mu_v(v) \text{ if } v \in C_i \end{cases}$$

We now prove that ν is a model of G_i onto G_j. First note that by definition of μ and each μ_c, we have $\nu(u) \cap \nu(v) = \emptyset$ for any pair of distinct vertices u and v in G_i, and also every vertex of G_j is in the image of some vertex of G_i (items (M1) and (M2) in the definition of a model). If $u \in C_i$, then $\mu(u)$ contains a vertex $v \in C_j$ for which $B_u^i \trianglelefteq B_v^j$, and v is also contained in $\mu_v(v)$ since μ_v preserves roots. Thus, $G_j[\nu(u)]$ is connected when $u \in C_i$ (item (M3)). This is obviously true when $u \notin C_i$ again by the definitions of μ and each μ_c. Moreover, the endpoints of every edge of G_i belong either both to A_i, or both to B_c^i, so item (M4) follows from the properties of μ and each μ_c. Finally, as the labeling σ ensures that $\text{root}(G_j) \in \nu(\text{root}(G_i))$, we establish that $G_i \trianglelefteq G_j$. So $(G_i)_{i\in\mathbb{N}}$ has a good pair (G_i, G_j), a contradiction. $\square$

Then we show that removing a maximum bond in a 2-connected graphs yields a graphs with smaller maximum bonds. This will be used in the inductive proof of Theorem 1.

Lemma 3. *If $B \subseteq E(G)$ is a bond of maximum order in a 2-connected graph G, then none of the components of $G \setminus B$ has a bond of order $|B|$.*

Proof. By contradiction, let us assume that one component of $G \setminus B$ has a bond C with $|C| = |B|$. Let $\{X, Y, Z\}$ be a partition of $V(G)$ be such that $G[X]$ and $G[Y \cup Z]$ are the connected components of $G \setminus B$ and $G[Y]$ and $G[Z]$ are those of $G[Y \cup Z] \setminus C$, while no edge of C is incident with a vertex of X.
First case: B is incident with vertices of both Y and Z. Let B' be the set of the edges of B that are incident with vertices of Y. Then observe that $G[X \cup Z]$ is connected and is a connected component of $G \setminus (B' \cup C)$, the other one being $G[Y]$. Therefore, $B' \cup C$ is a cut and $|B' \cup C| > |B|$, which contradicts the maximality of B. Therefore this case is not possible.
Second case: B is incident to vertices of exactly one of Y and Z. Without loss of generality, let us assume that this set is Y. Let $x \in X$ and $y \in Z$. As G is 2-connected, there are two paths Q_1 and Q_2 connecting x to y and sharing only their endpoints. Let Q'_1 and Q'_2 be minimal subpaths of Q_1 and Q_2, respectively, containing exactly one edge from B and C. Let D be a minimum cut of $G[Y]$ separating the internal vertices of Q_1 from that of Q_2. As $G[Y]$ is connected, $D \neq \emptyset$. For every $i \in \{1, 2\}$, let Y_i be the connected component of $G[Y] \setminus D$ containing the internal vertices of Q'_i and let B_i (resp. C_i) be the edges of B (resp. C) that have an endpoint in Y_i. As $\{B_1, B_2\}$ is a partition of B, there is an $i \in \{1, 2\}$ such that $|B_i| \geq |B|/2$. Similarly, there is a $j \in \{1, 2\}$ such that $|C_j| \geq |C|/2 = |B|/2$. Let $B' = B_i \cup C_j \cup D$ and let us show that it is a bond. As $|B'| > |B|$, this would provide the desired contradiction to the maximality of B. If $i = j$, then B' separates Y_i from $X \cup Y_{3-i} \cup Z$. The subgraph $G[Y_i]$ is connected by definition of Y_i, and $G[X \cup Y_{3-i} \cup Z]$ is because of the path Q'_{3-i}. If $i \neq j$, then B' separates $Y_i \cup Z$ from $X \cup Y_{3-i}$. There vertex sets induce connected subgraphs thanks to the paths Q'_i and Q'_{3-i}, respectively. Therefore B' is a bond and we are done. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1. Our goal is to show that for every $p, k \in \mathbb{N}$, the class $\mathcal{G}_{p,k}$ is well-quasi-ordered by $\trianglelefteq$. Lemma 2 allows us to focus on labeled 2-connected graphs. We call $\mathcal{G}'_{p,k}$ the class of graphs of $\mathcal{G}_{p,k}$ that are 2-connected. Also, according to Lemma 1, if $\left(\mathcal{G}'_{1,k}, \trianglelefteq\right)$ is a wqo, then so is $\left(\mathcal{G}'_{p,k}, \trianglelefteq\right)$. Therefore we only need to consider the case where $p = 1$ (the case $p = 0$ being trivial).

The proof then goes by induction on k. When $k = 0$, then $\mathcal{G}'_{1,k}$ is empty, so $\mathrm{lab}_{(\Sigma,\preceq)}\left(\mathcal{G}'_{1,k}\right)$ is trivially well-quasi-ordered, for every wqo $(\Sigma, \preceq)$. Let us now assume that $k > 0$ and that for and every wqo $(\Sigma, \preceq)$, the class $\mathrm{lab}_{(\Sigma,\preceq)}\left(\mathcal{G}'_{1,k-1}\right)$ is well-quasi-ordered by $\trianglelefteq$ (induction hypothesis). By the remarks above, $\left(\mathrm{lab}_{(\Sigma,\preceq)}\left(\mathcal{G}_{p,k-1}\right), \trianglelefteq\right)$ is a wqo, for every $p \in \mathbb{N}$.

Let $(\Sigma, \preceq)$ be a wqo. By contradiction, we assume that $\mathrm{lab}_{(\Sigma,\preceq)}\left(\mathcal{G}_{1,k}\right)$ is not a well-quasi-order. Let $\{G_i\}_{i \in \mathbb{N}}$ be an infinite antichain in $\mathrm{lab}_{(\Sigma,\preceq)}\left(\mathcal{G}_{1,k}\right)$ such that, for some $k' \leq k$ and for every $i \in \mathbb{N}$, a largest bond B_i in G_i has order k'. As no graphs of $\mathcal{G}_{1,k}$ has a bond of order more than k, such antichain always exist.

We may also assume that, if we respectively denote by $\{x_i^1, \dots, x_i^{k'}\}$ and $\{y_i^1, \dots, y_i^{k'}\}$ the endpoints of the edges of B_i in the two connected components of $G_i \setminus B_i$, the bipartite graph between $\{x_i^1, \dots, x_i^{k'}\}$ and $\{y_i^1, \dots, y_i^{k'}\}$ is the same for every i. We mean here that $\mathrm{mult}_{G_i}(\{x_i^l, y_i^{l'}\}) = \mathrm{mult}_{G_j}(\{x_j^l, y_j^{l'}\})$ for every $l, l' \in \{1, k'\}$. This is possible as there is a finite number of bipartite graphs on k edges and at most $2k$ vertices.

For every $i \in \mathbb{N}$, let $G'_i = G_i \setminus B_i$. Let H_i be a copy of G'_i (with the same vertex set) that we label as follows. Let $v \in V(H_i)$. If $v = x_i^j$ (resp. $v = y_i^j$) for some $j \in \{1, k'\}$, then we set $\lambda_{H_i}(v) = (\lambda_{G'_i}(v), j)$ (resp. $\lambda_{H_i}(v) = (\lambda_{G'_i}(v), 2j)$), otherwise we set $\lambda_{H_i}(v) = (\lambda_{G'_i}(v), 0)$.

Let $\Sigma' = \Sigma \times \{0, 2k-1\}$, let $\preceq'$ be the Cartesian product of $\preceq$ and $=$ and notice that, as a Cartesian product of wqos, $(\Sigma', \preceq')$ is a wqo (Proposition 1). According to Lemma 3, the graph H_i belong to $\mathrm{lab}_{(\Sigma', \preceq')}(\mathcal{G}_{2,k-1})$. By induction hypothesis, $\{H_i\}_{i\in\mathbb{N}}$ is wqo by $\trianglelefteq$. Let i and j be distinct integers such that $H_i \trianglelefteq H_j$ and let μ be a model of H_i in H_j. Let us show that μ is also a model of G_i in G_j. The properties (M1), (M2), and (M3) follow from the definition of μ, as well as (M4) when u and v belong to the same connected component of H_i. Also, as $V(G_i) = V(H_i)$, μ seen as model of G_i in G_j satisfies (M5). Therefore we only need to prove that (M4) holds when u and v belong to distinct connected components of H_i.

Let $l \in \{1, k'\}$. Since $\lambda_{G_i}(x_i^l)$ is only comparable with the labels that have l as their second coordinate, and that x_j^l is the only vertex of G_j with that property, we get $x_j^l \in \mu(x_i^l)$. Similarly, $y_j^l \in \mu(y_i^l)$. The image of μ consists of disjoint subsets, thus we have that if $v \in V(G_i)$ is not of the form x_i^l or y_i^l for some $l \in \{1, k'\}$, then $\mu(v) \cap \bigcup_{l=1}^{k'} \{x_j^l, y_j^l\} = \emptyset$. As the possible edges in G_j between two vertices u, v that belong to distinct connected components of H_i are the edges of B_i, we deduce that if one of u, v is not of the form x_i^l or y_i^l for some $l \in \{1, k'\}$, then there is no edge between a vertex of $\mu(u)$ and one of $\mu(v)$. This proves (M4) in this case. The case where $u = x_i^l$ and $v = y_j^{l'}$ for some $l, l' \in \{1, k'\}$ follows from our choice of the antichain $\{G_i\}_{i\in\mathbb{N}}$ with the property that $\mathrm{mult}_{G_i}(\{x_i^l, y_i^{l'}\}) = \mathrm{mult}_{G_j}(\{x_j^l, y_j^{l'}\})$. $\square$

4 On canonical antichains

We present in this section general lemmas on antichains and prove Theorem 2. If S is the subset of a qoset $(\Sigma, \preceq)$, the $\preceq$-*closure* of S is defined as $\{x,\ x \preceq y$ for some $y \in S\}$. For every $A \subseteq \Sigma$, we define $\mathrm{Incl}_{\preceq}(A) = \{x \in \Sigma,\ x \prec a$ for some $a \in A\}$. An antichain A of $(\Sigma, \preceq)$ is said to be a *fundamental antichain* if $\mathrm{Incl}_{\preceq}(A)$ has no infinite antichains. This concept was introduced by [4] in its study of canonical antichains.

Lemma 4. *Let $(\Sigma, \preceq)$ be a qoset and let A be an antichain. If A is canonical, then A is fundamental.*

Proof. Let $C = \mathrm{Incl}_{\preceq}(A)$. As C is $\preceq$-closed, it contains infinite antichains iff $C \cap A$ is infinite. However, for every $x \in C$ there is a element $y \in A$ such that $x \prec y$.

Therefore, if $x \in C \cap A$, then A contains two distinct elements that are comparable. We deduce $C \cap A = \emptyset$. Therefore C has no infinite antichains: A is canonical. $\square$

A consequence of Lemma 4 and Theorem 2 is that every antichain $\mathcal{A}$ of $\trianglelefteq$ such that $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}} \setminus \mathcal{A}$ is finite is fundamental. In the following lemma, we formalize and extend observations on canonical antichains made in [4].

Lemma 5. *Let $(\Sigma, \preceq)$ be a qoset and let A, B be two infinite antichains:*

(i) if A and B are canonical, $A \setminus B$ and $B \setminus A$ are finite;

(ii) if A is canonical and $A \setminus B$ is finite then B is canonical and $B \setminus A$ is finite.

Proof. Proof of (i). We consider $D = (B \setminus A) \cup \mathrm{Incl}(B \setminus A)$, the $\preceq$-closure of $B \setminus A$. If $B \setminus A$ is infinite, then D contains an infinite antichain. Therefore, as A is canonical, $D \cap A$ is infinite. However, $A \cap \mathrm{Incl}(B)$ is finite because B, being canonical, is fundamental (Lemma 4). Hence $A \cap (B \setminus A)$ is infinite, a contradiction.
Proof of (ii). By definition, for every $\preceq$-closed class C, C has an infinite antichain iff $C \cap A$ is infinite. As $A \setminus B$ is finite, $C \cap A$ is infinite iff $C \cap A \cap B$ is infinite. This implies that $C \cap B$ is infinite. On the other hand, if $C \cap B$ is infinite, then C clearly has an infinite antichain. Therefore B is a canonical antichain. Applying (i) we get that $B \setminus A$ is finite. $\square$

Theorem 2 is a consequence of Lemma 5 applied with $\mathcal{A}$ and $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}$ which is, as a consequence of Corollary 1, a canonical antichain.

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