Multigraphs without large bonds are wqo by contraction

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Multigraphs without large bonds are wqo by contraction*

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Abstract

We show that the class of multigraphs with at most p connected components and bonds of size at most k is well-quasi-ordered by edge contraction for all positive integers p, k. (A *bond* is a minimal non-empty edge cut.) We also characterize canonical antichains for this relation and show that they are fundamental.

1 Introduction

A *well-quasi-order* (*wqo* for short) is a partial order which contains no infinite decreasing sequence, nor infinite collection of pairwise incomparable elements. The beginnings of the theory of well-quasi-orders go back to the 1950s and some early results on wqos include that of Higman on sequences from a wqo [7], Kruskal's Tree Theorem [9], as well as other (now standard) techniques, for example the *minimal bad sequence* argument of Nash-Williams [10].

A recent result on wqos and arguably one of the most significant results in this field is the theorem by Robertson and Seymour which states that graphs are well-quasi-ordered by the minor relation [13]. Later, the same authors also proved that graphs are well-quasi-ordered by the immersion relation [12].

Nonetheless, most of containment relations do not well-quasi-order the class of all graphs. For example, graphs are not well-quasi-ordered by subgraphs, induced subgraphs, or topological minors. Therefore, attention was naturally brought to classes of graphs where well-quasi-ordering for such relations exists. Damaschke proved that cographs are well-quasi-ordered by induced subgraphs [1] and Ding characterized subgraph ideals that are well-quasi-ordered by the subgraph relation [3]. Finally, Liu and Thomas recently announced that graphs excluding as topological minor any

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graph of a class called “Robertson chain” are well-quasi-ordered by the topological minor relation [8].

Another line of research is to classify non-wqo containment relations depending on the type of obstructions they contain. Ding introduced the concepts of *canonical antichain* and *fundamental antichain* aimed at extending the study of the existence of obstructions of being well-quasi-ordered in a partial order [5]. In particular, he proved that finite graphs do not admit a canonical antichain under the induced subgraph relation but they do under the subgraph relation.

In this paper, we consider finite graphs where parallel edges are allowed, but not loops. Graphs where no edges are parallel are referred to as *simple graphs*. An *edge contraction* is the operation that identifies two adjacent vertices and deletes the possibly created loops (but keeps multiple edges). A graph H is said to be a *contraction* of a graph G, denoted $H \trianglelefteq G$, if H can be obtained from G by a sequence of edge contractions. A *bond* is a minimal non-empty edge cut, i.e. a minimal set of edges whose removal increases the number of connected components (cf. Figure 1).

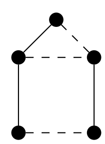

Figure 1: A bond of size 3 (dashed edges) in the house graph.

Let $\mathfrak{G}$ denote the class of finite graphs. The contraction relation defines a partial order on $\mathfrak{G}$. This order is not a wqo. An illustration of this fact is the infinite sequence of incomparable graphs $\langle \theta_i \rangle_{i \in \mathbb{N}}$, where θ_k is the graph with two vertices and k edges, for every positive integer k (cf. Figure 2).

An *antichain* is a sequence of pairwise incomparable elements of $(\mathfrak{G}, \trianglelefteq)$. Remark that a class of graphs is well-quasi-ordered by $\trianglelefteq$ iff it does not contain infinite antichains. Indeed, every decreasing sequence of graphs is finite since the edge contraction operation used to define $\trianglelefteq$ decreases the number of edges of a graph.

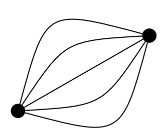

Figure 2: The graph θ_5.

For every $p, k \in \mathbb{N}$, let $\mathcal{G}_{p,k}$ be the class of graphs having at most p connected components and not containing a bond of order more than k. Our main result is the following.

Theorem 1. *For every $p, k \in \mathbb{N}$, the class $\mathcal{G}_{p,k}$ is well-quasi-ordered by $\trianglelefteq$.*

The complement of a simple graph G, denoted $\overline{G}$ is the graph obtained by replacing every edge by a non-edge and vice-versa in G. Remark that a graph has a bond of order k iff it contains θ_k as contraction, and that it has p connected components iff it can be contracted to $\overline{K}_p$. A class $\mathcal{G}$ of graphs is said to be *contraction-closed* if $H \in \mathcal{G}$ whenever $H \trianglelefteq G$ for some $G \in \mathcal{G}$. As a consequence of our main theorem and of the fact that each of $\{\theta_i\}_{i\in\mathbb{N}}$ and $\{\overline{K}_i\}_{i\in\mathbb{N}}$ is an obstruction to be well-quasi-ordered, we have the following results.

Corollary 1. *A class of graphs $\mathcal{H}$ is well-quasi-ordered by $\trianglelefteq$ iff there are $k, p \in \mathbb{N}$ such that for every $H \in \mathcal{H}$ we have $\forall k' > k,\ H \not\trianglelefteq \theta_{k'}$ and $\forall p' > p,\ H \not\trianglelefteq \overline{K}_p$.*

Corollary 2. *A contraction-closed class $\mathcal{H}$ is well-quasi-ordered by $\trianglelefteq$ iff there are $k, p \in \mathbb{N}$ such that $\forall k' > k,\ \theta_{k'} \notin \mathcal{H}$ and $\forall p' > p,\ \overline{K}_p \notin \mathcal{H}$.*

Figure 3 presents two infinite antichains for $(\mathfrak{G}, \trianglelefteq)$: the sequence of multiedges $\mathcal{A}_\theta = \{\theta_i\}_{i\in\mathbb{N}^*}$ and the sequence of cocliques $\mathcal{A}_{\overline{K}} = \{\overline{K_i}\}_{i\in\mathbb{N}}$. In his study of infinite antichains for the (induced) subgraph relation, Ding [5] introduced the two following concepts. An antichain $\mathcal{A}$ of a partial order $(\mathcal{S}, \preceq)$ is said to be *canonical* if it is such that every contraction-closed subclass $\mathcal{J}$ of $\mathcal{S}$ has an infinite antichain iff $\mathcal{J} \cap \mathcal{A}$ is infinite. If $\mathrm{Incl}(\mathcal{A}) = \{x \in \mathcal{S},\ x \prec a \text{ for some } a \in \mathcal{A}\}$ has no infinite antichains, then $\mathcal{A}$ is a *fundamental antichain*. Note that canonical antichains can be used to characterize the $\preceq$-closed subclasses of a partial order $(\mathcal{S}, \preceq)$ and also to describe the variety of its antichains.

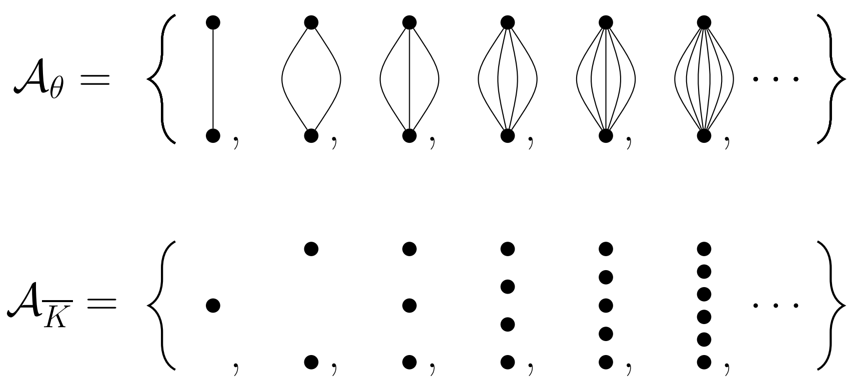

Figure 3: Two infinite antichains for contractions: multiedges and cocliques.

The following result is a complete characterization of the canonical antichains of $(\mathfrak{G}, \trianglelefteq)$, which extends the results of Ding on canonical antichains of simple graphs for the relations of subgraph and induced subgraph [5].

Theorem 2. *Every antichain $\mathcal{A}$ of $(\mathfrak{G}, \trianglelefteq)$ is canonical iff each of the following sets are finite:*

$$\mathcal{A}_\theta \setminus \mathcal{A}; \qquad \mathcal{A}_{\overline{K}} \setminus \mathcal{A};\ \textit{and} \qquad \mathcal{A} \setminus \{\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}\}.$$

In other words, an antichain $\mathcal{A}$ is canonical iff it contains all but finitely many graphs from $\mathcal{A}_\theta$, all but finitely many graphs from $\mathcal{A}_{\overline{K}}$ and a finite number of graphs that do not belong to $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}$. Two straightforward consequences are that $(\mathfrak{G}, \trianglelefteq)$ has infinite antichains and the following result.

Corollary 3. *Every canonical antichain of $(\mathfrak{G}, \trianglelefteq)$ is fundamental.*

Organization of the paper. The notions and tools that are used in this paper are introduced in Section 2, in particular notions related to well-quasi-ordering and to rooted graphs. Then, Section 3 deals with rooted graphs in order to build large wqos from small ones. Finally, Theorem 1 is proven in Section 4 and results on canonical antichains appear in Section 5.

Conclusion. This work settles the case of multigraph contractions in the study of well-quasi-ordered subclasses, a problem investigated by Damaschke for induced subgraphs [1], by Ding for subgraphs [3] and induced minors [4], and by Fellows et al. for several containment relations [6]. In particular, we give necessary and sufficient conditions for a class of (multi)graphs to be well-quasi-ordered by multigraph contractions. Furthermore, we characterize canonical antichains for this relation and show that they are fundamental, in the continuation of Ding's results for subgraph and contraction relation in [5].

2 Preliminaries

We denote by $\mathrm{V}(G)$ the set of vertices of a graph G and by $\mathrm{E}(G)$ its multiset of edges. Given two adjacent vertices u, v of a graph G, $\mathrm{mult}_G(\{u, v\})$ stands for the number of parallel edges between u and v, called *multiplicity* of the edge $\{u, v\}$. We denote by $\mathcal{P}^{<\omega}(S)$ the class of finite subsets of a set S, by $\mathcal{P}(S)$ its power set and by $[\![i, j]\!]$ the interval of integers $\{i, \dots, j\}$, for all integers $i \leq j$. A maximally 2-connected subgraph is called a *block*. In this paper, we will have to handle many objects with several indices, and we find more convenient to use the dot notation $A.b$, informally meaning "object b related to object A".

2.1 Tree-decompositions and models.

A *tree decomposition* of a graph G is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X}$ a family $(X_t)_{t\in \mathrm{V}(T)}$ of subsets of $\mathrm{V}(G)$ (called *bags*) indexed by elements of $\mathrm{V}(T)$ and such that:

(i) $\bigcup_{t\in \mathrm{V}(T)} X_t = \mathrm{V}(G)$;

(ii) for every edge e of G there is an element of $\mathcal{X}$ containing both ends of e;

(iii) for every $v \in \mathrm{V}(G)$, the subgraph of T induced by $\{t \in \mathrm{V}(T),\ v \in X_t\}$ is connected.

The *torso* of a bag X_t of a tree decomposition $(T, \{X_t\}_{t\in \mathrm{V}(T)})$ is the underlying simple graph of the graph obtained from $G[X_t]$ by adding all the edges $\{x, y\}$ such that $x, y \in X_t \cap X_{t'}$ for some neighbor t' of t in T.

A *model* of H in G (H-model for short) is a function $\mu\colon \mathrm{V}(H) \to \mathcal{P}(\mathrm{V}(G))$ such that:

M1: $\mu(u)$ and $\mu(v)$ are disjoint whenever $u, v \in \mathrm{V}(H)$ are distinct;

M2: $\{\mu(u)\}_{u\in \mathrm{V}(H)}$ is a partition of $\mathrm{V}(G)$;

M3: for every $u \in \mathrm{V}(H)$, the graph $G[\mu(u)]$ is connected;

M4: for every $u, v \in \mathrm{V}(H)$, $\mathrm{mult}_H(u,v) = \sum_{(u',v') \in \mu(u) \times \mu(v)} \mathrm{mult}_G(u',v')$.

Remark that H is a contraction of G iff G has a H-model. When μ is a H-model in G, we write $H \trianglelefteq^{\mu} G$.

For every $i \in \{2,3\}$ we denote by $\mathcal{H}_k^{(i)}$ the class of all i-connected graphs in a class $\mathcal{H}$. Now we state several results that we will use. The first one is a decomposition theorem for 2-connected graphs by Tutte.

Proposition 1 ([14], see also [2, Exercise 20 of Chapter 12])**.** *Every 2-connected simple graph has a tree-decomposition $(T, \mathcal{X})$ such that $|X_t \cap X_{t'}| = 2$ for every edge $\{t, t'\} \in T$ and all torsos are either 3-connected or a cycle.*

Proposition 2 ([11])**.** *For every $k \in \mathbb{N}$ there is an $\zeta_k \in \mathbb{N}$ such that every 3-connected simple graph of order at least ζ_k contains a wheel of order k or a $K_{3,k}$ as minor.*

2.2 Sequences, posets and well-quasi-orders

In this section, we introduce basic definitions and facts related to the theory of well-quasi-orders. In particular, we recall that being well-quasi-ordered is preserved by several operations including union, Cartesian product, and application of a monotone function.

A *sequence* of elements of a set A is an ordered countable collection of elements of A. Unless otherwise stated, sequences are finite. The sequence of elements $s_1, \dots, s_k \in A$ in this order is denoted by $\langle s_1, \dots, s_k \rangle$. We use the notation $A^\star$ for the class of all finite sequences over A (including the empty sequence).

A *partially ordered set* (*poset* for short) is a pair $(A, \preceq)$ where A is a set and $\preceq$ is a binary relation on S which is reflexive, antisymmetric and transitive. An *antichain* is a sequence of pairwise non-comparable elements. In a sequence $\langle x_i \rangle_{i \in I \subseteq \mathbb{N}}$ of a poset $(A, \preceq)$, a pair (x_i, x_j), $i, j \in I$ is a *good pair* if $x_i \preceq x_j$ and $i < j$. A poset $(A, \preceq)$ is a *well-quasi-order* (*wqo* for short), and its elements are said to be *well-quasi-ordered* by $\preceq$, if every infinite sequence has a good pair, or equivalently, if $(A, \preceq)$ has neither an infinite decreasing sequence, nor an infinite antichain. An infinite sequence containing no good pair is called an *bad sequence*.

Union and product. If $(A, \preceq_A)$ and $(B, \preceq_B)$ are two posets, then

- their union $(A \cup B, \preceq_A \cup \preceq_B)$ is the poset defined as follows:

  $$\forall x, y \in A \cup B,\ x \preceq_A \cup \preceq_B y \text{ if } (x, y \in A \text{ and } x \preceq_A y) \text{ or } (x, y \in B \text{ and } x \preceq_B y);$$

- their Cartesian product $(A \times B, \preceq_A \times \preceq_B)$ is the poset defined by:

  $$\forall (a,b), (a',b') \in A \times B,\ (a,b) \preceq_A \times \preceq_B (a',b') \text{ if } a \preceq_A a' \text{ and } b \preceq_B b'.$$

Remark 1 (union of wqos). If $(A, \preceq_A)$ and $(B, \preceq_B)$, are two wqos, then so is $(A \cup B, \preceq_A \cup \preceq_B)$. In fact, for every infinite antichain S of $(A \cup B, \preceq_A \cup \preceq_B)$, there is an infinite subsequence of S whose all elements belong to one of A and B (otherwise S is finite). But then one of $(A, \preceq_A)$ and $(B, \preceq_B)$ has an infinite antichain, a contradiction with our initial assumption. Similarly, every finite union of wqos is a wqo.

Proposition 3 (Higman [7]). *If $(A, \preceq_A)$ and $(B, \preceq_B)$ are wqo, then so is $(A \times B, \preceq_A \times \preceq_B)$.*

Sequences. For any partial order $(A, \preceq)$, we define the relation $\preceq^\star$ on $A^\star$ as follows: for every $r = \langle r_1, \ldots, r_p \rangle$ and $s = \langle s_1, \ldots, s_q \rangle$ of $A^\star$, we have $r \preceq^\star s$ if there is a increasing function $\varphi\colon [\![1, p]\!] \to [\![1, q]\!]$ such that for every $i \in [\![1, p]\!]$ we have $r_i \preceq s_{\varphi(i)}$. This generalizes the subsequence relation. This order relation is extended to the class $\mathcal{P}^{<\omega}(A)$ of finite subsets of A as follows, generalizing the subset relation: for every $B, C \in \mathcal{P}^{<\omega}(A)$, we write $B \preceq^\star C$ if there is an injection $\varphi\colon B \to C$ such that $\forall x \in B,\ x \preceq \varphi(x)$.

Proposition 4 (Higman [7]). *If $(A, \preceq)$ is a wqo, then so is $(A^\star, \preceq^\star)$.*

Corollary 4. *If $(A, \preceq)$ is a wqo, then so is $(\mathcal{P}^{<\omega}(A), \preceq^\star)$.*

In order to stress that domain and codomain of a function are posets, we sometimes use, in order to denote a function φ from a poset $(A, \preceq_A)$ to a poset $(B, \preceq_B)$, the following notation: $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$.

Monotonicity. A function $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$ is said to be *monotone* if it satisfies the following property:

$$\forall x, y \in A,\ x \preceq_A y \Rightarrow f(x) \preceq_B f(y).$$

A function $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$ is a *poset epimorphism* (*epi* for short) if it is surjective and monotone. We introduce poset epimorphisms because they have the following interesting property, which we will use to show that some posets are well-quasi-ordered.

Remark 2 (epi from a wqo). Any epi φ maps a wqo to a wqo. Indeed, for any pair x, y of elements of the domain of φ such that $f(x)$ and $f(y)$ are incomparable, x and y are incomparable as well (by monotonicity of φ). Therefore, and as φ is surjective, any infinite antichain of the codomain of φ can be translated into an infinite antichain of its domain.

Remark 3 (componentwise monotonicity). Let $(A, \preceq_A)$, $(B, \preceq_B)$, and $(C, \preceq_C)$ be three posets and let $f\colon (A \times B, \preceq_A \times \preceq_B) \to (C, \preceq_C)$ be a function. If we have both

$$\forall a \in A,\ \forall b, b' \in B,\ b \preceq_B b' \Rightarrow f(a, b) \preceq_C f(a, b') \tag{1}$$

$$\text{and } \forall a, a' \in A,\ \forall b \in B,\ a \preceq_A a' \Rightarrow f(a, b) \preceq_C f(a', b) \tag{2}$$

then f is monotone. Indeed, let $(a, b), (a', b') \in A \times B$ be such that $(a, b) \preceq_A \times \preceq_B (a', b')$. By definition of the relation $\preceq_A \times \preceq_B$, we have both $a \preceq a'$ and $b \preceq b'$. From (1) we get that $f(a, b) \preceq_C f(a, b')$ and from (2) that $f(a, b') \preceq_C f(a', b')$, hence $f(a, b) \preceq_C f(a', b')$ by transitivity of $\preceq_C$. Thus f is monotone. Note that this remark can be generalized to functions with more than two arguments.

2.3 Roots and labels

Labeled graphs. Let $(\Sigma, \preceq)$ be a poset. A $(\Sigma, \preceq)$-labeled graph is a pair (G, λ) where $\lambda\colon \mathrm{V}(G) \to \mathcal{P}^{<\omega}(\Sigma)$ is a function, referred to as the *labeling of the graph*. For simplicity, we will denote by G the labeled graph (G, λ) and by $G.\lambda$ its labeling function. If $\mathcal{H}$ is a class of (unlabeled) graphs, $\mathrm{lab}_\Sigma(\mathcal{H})$ denotes the class of Σ-labeled graphs of $\mathcal{H}$. Remark that any unlabeled graph can be seen as a $\emptyset$-labeled graph.

The contraction relation is extended to labeled graphs by additionally allowing to relabel by l' any vertex labeled l whenever $l' \preceq l$. In terms of model, this corresponds to the following extra requirement for μ to be a model of H in G:

$$\forall v \in \mathrm{V}(H),\ H.\lambda(v) \preceq^\star \bigcup_{u' \in \mu(u)} G.\lambda(u').$$

When such a requirement is met, the model μ is said to be *label-preserving*.

Rooted graphs. A *rooted graph* is a couple (G, r) where G is a graph and r is a vertex of G. Given two rooted graphs (G, r) and (H, r'), we say that (H, r') is a contraction of (G, r), what we denote by $(H, r') \trianglelefteq (G, r)$, if there is a model μ of H in G such that $r' \in \mu(r)$. Such a model is said to be *root-preserving*. For the sake of simplicity, we sometimes denote by G the rooted graph (G, r) and refer to its root by $G.r$. For every rooted graph G, we define $\mathrm{root}(G) = G.r$. If $\mathcal{H}$ is a class of graphs, we define its rooted closure, denoted $\mathcal{H}_r$ as the class of rooted graphs $\mathcal{H}_r = \{(G, v) : G \in \mathcal{H}, v \in G\}$. Note that $\mathcal{H}$ is wqo under $\trianglelefteq$ whenever $\mathcal{H}_r$ is wqo under $\trianglelefteq$.

We define a *2-rooted graph* in a very similar way. A 2-rooted graph is a triple (G, r, s) where G is a graph and r and s are two distinct vertices of G. Given two 2-rooted graphs $(G, r, s), (H, r', s')$, we say that (H, r', s') is a contraction of (G, r, s), what we denote by $(H, r', s') \trianglelefteq (G, r, s)$, if there is a model μ of H in G such that $r' \in \mu(r)$ and $s' \in \mu(s')$. For the sake of simplicity, we sometimes denote by G the 2-rooted graph (G, r, s) and refer to its first (respectively second) root by $G.r$ (respectively $G.s$). For every rooted graph G, we define $\mathrm{root}(G) = \{G.r, G.s\}$. A 2-rooted graph G is *edge-rooted* if $\{G.r, G.s\} \in \mathrm{E}(G)$.

The operation of *attaching* a 2-rooted graph H on the pair of vertices (u, v) of graph G, denoted $G \oplus_u^v H$, yields the graph rooted in $(G.r, G.s)$ obtained by identifying u with $H.r$ and v with $H.s$ in the disjoint union of G and H (see Figure 4 for an illustration). If both G and H are $(\Sigma, \preceq)$-labeled (for some poset $(\Sigma, \preceq)$), then the

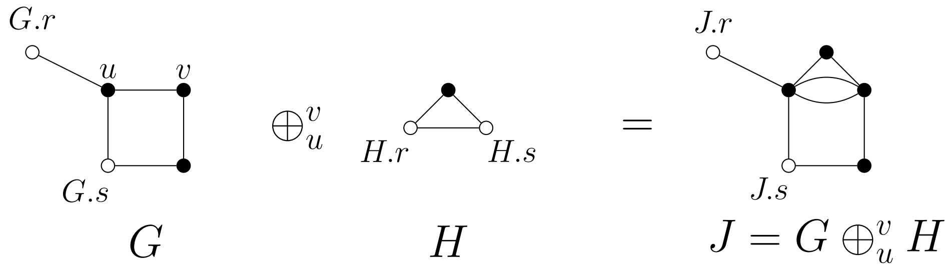

Figure 4: Attaching H to vertices (u, v) of G (roots are the white vertices).

labeling function λ of the graph $G \oplus_u^v H$ is defined as follows:

$$\lambda\colon \begin{cases} \mathrm{V}(G \oplus_u^v H) & \to & \mathcal{P}^{<\omega}(\Sigma) \\ w & \mapsto & G.\lambda(w) \quad \text{if } w \in \mathrm{V}(G) \setminus \{u, v\} \\ w & \mapsto & H.\lambda(w) \quad \text{if } w \in \mathrm{V}(H) \setminus \{H.r, H.s\} \\ w & \mapsto & G.\lambda(w) \cup H.\lambda(w) \quad \text{otherwise, i.e. when } w \in \{u, v\}. \end{cases}$$

3 Raising well-quasi-orders

This section is devoted to building larger wqos from smaller ones in classes of labeled graphs that are rooted by two vertices. Step by step, we will construct wqos that will be directly used in the proof of the main result. Labels will be used to reduce the study of (unlabeled) graphs to the case of 2-connected graphs with labels (by the virtue of Lemma 5), whereas roots enable us to construct graphs using the operation $\oplus$. In this section, $(\Sigma, \preceq)$ be any poset.

Lemma 1. *Let H, H', G, G' be four $(\Sigma, \preceq)$-labeled 2-rooted graphs. If $H \trianglelefteq H'$ and $G \trianglelefteq^{\mu} G'$, then for every distinct u, v in $\mathrm{V}(G)$ and $u' \in \mu(u)$, $v' \in \mu(v)$ we have*

$$G \oplus_u^v H \trianglelefteq G' \oplus_{u'}^{v'} H'.$$

Proof. Let $\mu_H\colon \mathrm{V}(H) \to \mathcal{P}(\mathrm{V}(H'))$ (respectively $\mu_G\colon \mathrm{V}(G) \to \mathcal{P}(\mathrm{V}(G'))$) be a model of H in H' (respectively of G in G'). We consider the following function:

$$\nu\colon \begin{cases} \mathrm{V}(G \oplus_u^v H) & \to & \mathcal{P}(\mathrm{V}(G' \oplus_{u'}^{v'} H')) & \\ v & \mapsto & \mu_H(v) & \text{if } v \in H \setminus \mathrm{root}(H) \\ v & \mapsto & \mu_G(v) & \text{if } v \in G \setminus \{u, v\} \\ v & \mapsto & \mu_H(v) \cup \mu_G(v) & \text{otherwise.} \end{cases}$$

Let us check that ν is a model of $G \oplus_u^v H$ in $G' \oplus_{u'}^{v'} H'$. First, observe that for every $x \in \mathrm{V}(G \oplus_u^v H)$, the subgraph induced in $G' \oplus_{u'}^{v'} H'$ by $\nu(x)$ is connected (M3): either $\nu(x) = \mu_H(x)$ or $\nu(x) = \mu_G(x)$ (and in these cases it follows from the fact that

μ_H and μ_G are models) or $\nu(x) = \mu_H(x) \cup \mu_G(x)$ (if $x \in \{u, v\}$) and $(G' \oplus_{u'}^{v'} H')[\nu(x)]$ is connected because both $\mu_H(x)$ and $\mu_G(x)$ induce a connected subgraph and both contain the root of H'. Furthermore, the images through ν of two distinct vertices are always disjoint (M1), and every vertex of $G' \oplus_{u'}^{v'} H'$ belongs to the image of a vertex (M2), again because μ_H and μ_G are models. Let us now show point M4. For every distinct $x, y \in \mathrm{V}(G \oplus_u^v H)$,

- either $x, y \in \mathrm{V}(H)$ and $\{x, y\} \neq \mathrm{root}(H)$ and then

$$\mathrm{mult}_{G \oplus_u^v H}(x, y) = \sum_{(x', y') \in \nu(x) \times \nu(y)} \mathrm{mult}_{G' \oplus_u^v H'}(x', y')$$

  as μ_H is a model (and symmetrically for the case $x, y \in \mathrm{V}(G)$ and $\{x, y\} \neq \{u, v\}$);

- or $x \in \mathrm{V}(H) \setminus \mathrm{root}(H)$ and $y \in \mathrm{V}(G) \setminus \{u, v\}$: there are no edges between x and y because every edge of $G \oplus_u^v H$ is either an edge of H or an edge of G, neither between $\nu(x)$ and $\nu(y)$ since $\nu(x) \subseteq \mathrm{V}(H) \setminus \mathrm{root}(H)$ and $\nu(y) \subseteq \mathrm{V}(G) \setminus \{u, v\}$, therefore we get

$$\mathrm{mult}_{G \oplus_u^v H}(x, y) = \sum_{(x', y') \in \nu(x) \times \nu(y)} \mathrm{mult}_{G' \oplus_u^v H'}(x', y') = 0;$$

- or $\{x, y\} = \{u, v\} = \mathrm{root}(H)$:

$$\begin{aligned}
\mathrm{mult}_{G \oplus_u^v H}(x, y) &= \mathrm{mult}_G(x, y) + \mathrm{mult}_H(x, y) \quad \text{(by definition of } \oplus\text{)} \\
&= \sum_{(x', y') \in \mu_G(x) \times \mu_G(y)} \mathrm{mult}_{G'}(x', y') + \sum_{(x', y') \in \mu_H(x) \times \mu_H(y)} \mathrm{mult}_{H'}(x', y') \\
&= \sum_{(x', y') \in \nu(x) \times \nu(y)} \mathrm{mult}_{G' \oplus_u^v H'}(x', y').
\end{aligned}$$

Besides, as a consequence that μ_G is root-preserving, ν also has this property. Last, let us check that ν is label-preserving. Let $x \in \mathrm{V}(G \oplus_u^v H)$. If $x \notin \{u, v\}$, then $(G \oplus_u^v H).\lambda(x) = G.\lambda(x)$ or $(G \oplus_u^v H).\lambda(x) = H.\lambda(x)$ (depending whether $x \in \mathrm{V}(G) \setminus \{u, v\}$ or $x \in H \setminus \mathrm{root}(H)$) and in these cases labels are preserved, since μ_G and μ_H are label-preserving. If $x \in \{u, v\}$, then, as μ_G and μ_H are label-preserving we have:

$$\begin{aligned}
(G \oplus_u^v H).\lambda(x) &= G.\lambda(x) \cup H.\lambda(y) \\
&\preceq^\star \bigcup_{x' \in \mu_G(x)} G'.\lambda(x') \cup \bigcup_{x' \in \mu_H(x)} H'.\lambda(x') \\
&\preceq^\star \bigcup_{x' \in \nu(x)} (G' \oplus_u^v H').\lambda(x')
\end{aligned}$$

and thus ν is label-preserving as well. We just proved that ν is a model of $G \oplus_u^v H$ in $G' \oplus_u^v H'$. Consequently, $G \oplus_u^v H \trianglelefteq G' \oplus_u^v H'$, as desired. $\square$

Corollary 5. *Let $l \in \mathbb{N}^*$, let J be a $(\Sigma, \preceq)$-labeled 2-rooted graph and $\langle (u_i, v_i) \rangle_{i \in [\![1,l]\!]}$ be a sequence of pairs of distinct vertices of J. Let $\mathcal{H}$ be a class of $(\Sigma, \preceq)$-labeled 2-rooted graphs, $\langle G_1, \ldots, G_l \rangle, \langle H_1, \ldots, H_l \rangle \in \mathcal{H}^l$ and let G (respectively H) be the graph constructed by attaching G_i (respectively H_i) to the vertices (u_i, v_i) of J, for every $i \in [\![1, l]\!]$.*

If $\langle H_1, \ldots, H_l \rangle \trianglelefteq^l \langle G_1, \ldots, G_l, \rangle$ then $H \trianglelefteq G$.

Proof. By induction on l. The case $l = 1$ follows from Lemma 1. If $l \geq 2$, then, let G' (respectively H') be the graph constructed by attaching G_i (respectively H_i) to the vertices (u_i, v_i) of J, for every $i \in [\![1, l-1]\!]$. By induction hypothesis, we have $H' \trianglelefteq G'$. Since H (respectively G) is isomorphic to $H' \oplus_{u_l}^{v_l} H_l$ (respectively $G' \oplus_{u_l}^{v_l} G_l$), and $H_l \trianglelefteq G_l$, by Lemma 1, we have $H \trianglelefteq G$ as desired. □

Lemma 2. *Let $\mathcal{H}$ be a family of $(\Sigma, \preceq)$-labeled 2-rooted connected graphs, let J be a $(\Sigma, \preceq)$-labeled 2-rooted graph, and let $\mathcal{H}_J$ be the class of $(\Sigma, \preceq)$-labeled 2-rooted graphs that can be constructed by attaching a graph $H \in \mathcal{H}$ to (u, v) for every $u, v \in \mathrm{V}(J)$. If $(\mathcal{H}, \trianglelefteq)$ is a wqo, then so is $(\mathcal{H}_J, \trianglelefteq)$.*

Proof. Let $(u_1, v_1), \ldots, (u_l, v_l)$ be an enumeration of all the pairs of distinct vertices of J. In this proof, we will design an epi that constructs graphs of $\mathcal{H}_{\mathcal{J}}$ from a tuple of l graphs of $\mathcal{H}$. Let $f \colon (\mathcal{H}^l, \trianglelefteq^l) \to (\mathcal{H}_J, \trianglelefteq)$ be the function that, given a tuple $(H_1, \ldots, H_l)$ of l graphs of $\mathcal{H}$, returns the graph constructed from J attaching H_i to (u_i, v_i) for every $i \in [\![1, l]\!]$. This function is clearly surjective. Let us show that it is monotone.

Let $(G_1, \ldots, G_l), (H_1, \ldots, H_l) \in \mathcal{H}^l$ be two tuples such that $(H_1, \ldots, H_l) \trianglelefteq^l (G_1, \ldots, G_l)$. According to Remark 3, it is enough to deal with the cases where these two sequences differ only in one coordinate. Since all parameters of f play a similar role, we only look at the case where $H_1 \trianglelefteq G_1$ and $\forall i \in [\![2, l]\!],\ H_i = G_i$. Let J' be the graph obtained from J by attaching G_i to (u_i, v_i), for every $i \in [\![2, l]\!]$. Remark that $f(H_1, \ldots, H_l)$ (respectively $f(G_1, \ldots, G_l)$) can be obtained by attaching H_1 (respectively G_1) to (u_1, v_1) in J'. By Lemma 1 and since $H_1 \trianglelefteq G_1$, we have $J \oplus_{u_1}^{v_1} H_1 \trianglelefteq J \oplus_{u_1}^{v_1} G_1$ and thus $f(H_1, \ldots, H_l) \trianglelefteq f(G_1, \ldots, G_l)$. Consequently, f is monotone and surjective: f is an epi. In order to show that $\mathcal{H}_J$ is a wqo, it suffices to prove that the domain of f is a wqo (cf. Remark 2). As a finite Cartesian product of wqos, $(\mathcal{H}^l, \trianglelefteq^l)$ is a wqo by Proposition 3. This concludes the proof. □

Lemma 3. *Let $\mathcal{H}$ be a family of $(\Sigma, \preceq)$-labeled 2-rooted connected graphs and let $\mathcal{H}_\circ$ be the class of $(\Sigma, \preceq)$-labeled graphs that can be constructed from a cycle by attaching a graph of $\mathcal{H}$ to either (u, v) or (v, u) for every edge $\{u, v\}$, after deleting the edge $\{u, v\}$. If $(\mathcal{H}, \trianglelefteq)$ is a wqo, then so is $(\mathcal{H}_\circ, \trianglelefteq)$.*

Proof. Again, this proof relies on the property of epimorphisms to send wqos on wqos: we will present a epi that maps sequences of graphs of $(\mathcal{H}, \trianglelefteq)$ to graphs of $(\mathcal{H}_\circ, \trianglelefteq)$. Let $\mathcal{H}' = \mathcal{H} \cup \{(H, s, r),\ (H, r, s) \in \mathcal{H}\}$, i.e. $\mathcal{H}'$ contains graphs of $\mathcal{H}$ with the roots possibly swapped. As the union of two wqos, $(\mathcal{H}', \trianglelefteq)$ is a wqo (Remark 1). We consider the function $f \colon (\mathcal{H}'^\star, \trianglelefteq^\star) \to (\mathcal{H}_\circ, \trianglelefteq)$ that, given a sequence $\langle H_1, \ldots, H_k \rangle$

of graphs of $(\mathcal{H}', \trianglelefteq)$ (for some integer $k \geq 2$), returns the graph obtained from the cycle on vertices $v_0, \dots, v_{k-1}$ (in this order) by deleting the edge $\{v_i, v_{(i+1) \mod k}\}$ and attaching H_i to $(v_i, v_{(i+1) \mod k})$, for all $i \in [\![1, k]\!]$. Observe that by definition of $\mathcal{H}_\circ$ and $\mathcal{H}'$, the function f is surjective. We now show that f is monotone. Let $G = \langle G_0, \dots, G_{k-1} \rangle$ and $H = \langle H_0, \dots, H_{l-1} \rangle \in \mathcal{H}'^\star$ be two sequences such that $G \trianglelefteq^\star H$. For the sake of readability, we will refer to the vertices of $f(G)$ (respectively $f(H)$) and of the graphs of G (respectively H) by the same names. By definition of the relation $\trianglelefteq^\star$, there is an increasing function $\rho\colon [\![0, k-1]\!] \to [\![0, l-1]\!]$ such that for every $i \in [\![0, k-1]\!]$, we have $G_i \trianglelefteq H_{\rho(i)}$.

A crucial remark here is that since the graphs of $\mathcal{H}'$ are connected, each of them can be contracted to an edge between its two roots. Therefore, for every graph H_i of the sequence H (for some $i \in [\![0, l-1]\!]$) we can first contract H_i to an edge in $f(H)$, and then contract this edge. That way we obtain a graph similar to $f(H)$ except that H_i has been deleted and its roots merged: this is the graph $f(\langle H_0, \dots, H_{i-1}, H_{i+1}, \dots, H_{l-1} \rangle)$. By applying this operation on every subgraph of $f(H)$ belonging to $\{H_i,\ i \in [\![1, l]\!] \setminus \rho([\![0, k]\!])\}$, we obtain the graph $f(\langle H_{\rho(i)} \rangle_{i \in [\![1,k]\!]})$, and we thus have $f(\langle H_{\rho(i)} \rangle_{i \in [\![1,k]\!]}) \trianglelefteq f(H)$. Now, recall that the function ρ is such that for every $i \in [\![0, k-1]\!]$, we have $G_i \trianglelefteq H_{\rho(i)}$. Furthermore, the graphs $f(G)$ and $f(\langle H_{\rho(i)} \rangle_{i \in [\![1,k]\!]})$ are both constructed by attaching graphs to the same graph (a cycle on k vertices). By Corollary 5, we therefore have $f(G) \trianglelefteq f(\langle H_{\rho(i)} \rangle_{i \in [\![1,k]\!]})$, hence $f(G) \trianglelefteq f(H)$ by transitivity of $\trianglelefteq$. We just proved that f is an epi. The domain of f is a wqo (as a set of finite sequences from a wqo, cf. Proposition 4), so its codomain $(\mathcal{H}_\circ, \trianglelefteq)$ is a wqo as well according to Remark 2, and this concludes the proof. □

Lemma 4. *Let $k \in \mathbb{N}$ and let $\mathcal{H}$ be a class of 2-rooted graphs, none of which having more than k edges between the two roots. Let $\mathcal{H}^-$ be the class of graphs of $\mathcal{H}$ where all edges between the two roots have been removed. If $(\mathcal{H}, \trianglelefteq)$ is a wqo, then so is $(\mathcal{H}^-, \trianglelefteq)$.*

Proof. Let us assume that $(\mathcal{H}, \trianglelefteq)$ is a wqo. For every $i \in [\![0, k]\!]$, let $\mathcal{H}_i$ be the subclass of graphs of $\mathcal{H}$ having exactly i edges between the two roots. Each class $\mathcal{H}_i$ ($i \in [\![0, k]\!]$) is a subclass of $\mathcal{H}$ which is well-quasi-ordered by $\trianglelefteq$, therefore it is well-quasi-ordered by $\trianglelefteq$ as well. Let f be the function that, given a 2-rooted graph G, returns a copy of G where all edges between the roots have been deleted. The rest of the proof draws upon the following remark.

Remark 4. Let G, H be two edge-rooted graphs where the edge between the roots has the same multiplicity. Then $H \trianglelefteq G \Leftrightarrow f(H) \trianglelefteq f(G)$ (every model of H in G is also a model of $f(H)$ in $f(G)$, and vice-versa).

Let $i \in [\![0, k]\!]$, let $\mathcal{H}_i^- = \{f(H),\ H \in \mathcal{H}_i\}$, and let $\langle f(G_i) \rangle_{i \in \mathbb{N}}$ be an infinite sequence of $\mathcal{H}_i^-$. By an observation above, $(\mathcal{H}_i, \trianglelefteq)$ is a wqo, hence $\langle G_i \rangle_{i \in \mathbb{N}}$ has a good pair (G_i, G_j) (with $i, j \in \mathbb{N},\ i < j$). According to Remark 4, $(f(G_i), f(G_j))$ is a good pair of $\langle f(G_i) \rangle_{i \in \mathbb{N}}$. Every infinite sequence of $(\mathcal{H}_i^-, \trianglelefteq)$ has a good pair, therefore this poset is a wqo. Remark that $(\mathcal{H}^-, \trianglelefteq)$ is the union of the $k+1$ wqos $\{(\mathcal{H}_i^-, \trianglelefteq)\}_{i \in [\![0,k]\!]}$, therefore it is a wqo as well (cf. Remark 1) and this concludes the proof. □

Lemma 5. *Let $\mathcal{H}$ be a class of connected graphs and let $\mathcal{H}^{(2)}$ be the subclass of 2-connected graphs of $\mathcal{H}$. If for every wqo $(\Sigma, \preceq)$, the poset $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{H}^{(2)}), \trianglelefteq)$ is a wqo, then so is $(\mathcal{H}, \trianglelefteq)$.*

Proof. This proof is very similar to induced minor case proved in [6] and we will proceed by induction. Assuming that $(\mathcal{H}, \trianglelefteq)$ is not a wqo, we will reach a contradiction by showing that its rooted closure $(\mathcal{H}_r, \trianglelefteq)$ is a wqo.

Let $\langle G_i \rangle_{i\in\mathbb{N}}$ be a bad sequence in $\mathcal{H}_r$ such that for every $i \in \mathbb{N}$, there is no $G \trianglelefteq G_i$ such that a bad sequence starts with $G_0, \ldots, G_{i-1}, G$ (a so-called *minimal bad sequence*). For every $i \in \mathbb{N}$, let A_i be the block of G_i which contains $\mathrm{root}(G_i)$. Let C_i the set of cutvertices of G_i that are included in A_i. For each cutvertex $c \in C_i$, let B^i_c the connected component in $G_i \setminus (V(A_i) \setminus C_i)$, and made into a rooted graph by setting $\mathrm{root}(B^i_c) = c$. Note that we have $B^i_c \trianglelefteq G_i$.

Let us denote by $\mathcal{B}$ the family of rooted graphs $\mathcal{B} = \{B^i_c \colon c \in C_i, i \in \mathbb{N}\}$. We will show that $(\mathcal{B}, \trianglelefteq)$ is a wqo. Let $\langle H_j \rangle_{j\in\mathbb{N}}$ be an infinite sequence in $\mathcal{B}$ and for every $j \in \mathbb{N}$ choose an $i = \varphi(j) \in \mathbb{N}$ for which $H_j \trianglelefteq G_i$. Pick a j with smallest $\varphi(j)$, and consider the sequence $G_1, \ldots, G_{\varphi(j)-1}, H_j, H_{j+1}, \ldots$. By minimality of $\langle G_i \rangle_{i\in\mathbb{N}}$ and by our choice of j, since $H_j \trianglelefteq G_{\varphi(j)}$ and $H_j \neq G_{\varphi(j)}$, this sequence is good so contains a good pair (G, G'). Now, if G is among the first $\varphi(j) - 1$ elements, then as $\langle G_i \rangle_{i\in\mathbb{N}}$ is bad we must have $G' = H_{j'}$ for some $j' \geq j$ and so we have $G_{i'} = G \trianglelefteq G' = H_{j'} \trianglelefteq G_{\varphi(j')}$, a contradiction. So there is a good pair in $\langle H_i \rangle_{i\geq j}$ and hence the infinite sequence $\langle H_j \rangle_{j\in\mathbb{N}}$ has a good pair, so $(\mathcal{B}, \trianglelefteq)$ is a wqo.

We will now find a good pair in $\langle G_i \rangle_{i\in\mathbb{N}}$ to show a contradiction. The idea is to label the graph family $\mathcal{A} = \{A_i\}_{i\in\mathbb{N}}$ so that each cutvertex c of a graph A_i gets labeled by their corresponding connected component B^i_c, and the roots are preserved under this labeling. More precisely, for each A_i we define a labeling σ_i that assigns to every vertex $v \in \mathrm{V}(G_i)$ a label $\{(\sigma^1_i(v), \sigma^2_i(v))\}$ defined as follows:

- $\sigma^1_i(v) = 1$ if $v = \mathrm{root}(G_i)$ and $\sigma^1_i(v) = 0$ otherwise;
- $\sigma^2_i(v) = B^i_v$ if $v \in C_i$ and $\sigma^2_i(v)$ is the one-vertex rooted graph otherwise.

The labeling σ of $\mathcal{A}$ is then $\{\sigma_i : i \in \mathbb{N}\}$. Let us define a quasi-ordering $\preceq$ on the set of labels Σ assigned by σ. For two labels $(s^1_a, s^2_a), (s^1_b, s^2_b) \in \Sigma$ we define $(s^1_a, s^2_a) \preceq (s^1_b, s^2_b)$ iff $s^1_a = s^1_b$ and $s^2_a \trianglelefteq s^2_b$. Note that in this situation, s^2_a and s^2_b are rooted graphs, so $\trianglelefteq$ compares rooted graphs. Observe that since $(\mathcal{B}, \trianglelefteq)$ is wqo, then $(\Sigma, \preceq)$ is wqo. For every $i \in \mathbb{N}$, let A'_i be the $(\Sigma, \preceq)$-labeled rooted graph (A_i, σ_i). We now consider the infinite sequence $\langle A'_i \rangle_{i\in\mathbb{N}}$. By our initial assumption, $(\mathrm{lab}_\Sigma(\mathcal{A}), \trianglelefteq)$ is wqo (as $\mathcal{A}$ consists only in 2-connected graphs), so there is a good pair (A'_i, A'_j) in the sequence $\langle A'_i \rangle_{i\in\mathbb{N}}$.

To complete the proof, we will show that $A'_i \trianglelefteq A'_j \Rightarrow G_i \trianglelefteq G_j$. Let μ be a model of A'_i in A'_j. Then for each cutvertex $c \in C_i$, $\mu(c)$ contains a vertex $d \in C_j$ with $B^i_c \trianglelefteq B^j_d$. Let μ_c denote a root-preserving model of B^i_c onto B^i_d. We construct a model g as follows:

$$\nu\colon \begin{cases} V(G_i) & \to & \mathcal{P}(V(G_j)) \\ v & \mapsto & \mu(v) \text{ if } v \in A_i \setminus C_i \\ v & \mapsto & \mu_c(v) \text{ if } v \in B^i_c \setminus C_i \\ v & \mapsto & \mu(v) \cup \mu_v(v) \text{ if } v \in C_i \end{cases}$$

We now prove that ν is a model of G_i onto G_j. First note that by definition of μ and each μ_c, we have $\nu(u) \cap \nu(v) = \emptyset$ for any pair of distinct vertices u and v in G_i, and also every vertex of G_j is in the image of some vertex of G_i (points M1 and M2 in the definition of a model). If $u \in C_i$, then $\mu(u)$ contains a vertex $v \in C_j$ for which $B^i_u \trianglelefteq B^j_v$, and v is also contained in $\mu_v(v)$ since μ_v preserves roots. Thus, $G_j[\nu(u)]$ is connected when $u \in C_i$ (point M3). This is obviously true when $u \notin C_i$ again by the definitions of μ and each μ_c. Moreover, the endpoints of every edge of G_i belong either both to A_i, or both to B^i_c, so point M4 follows from the properties of μ and each μ_c. Finally, as the labeling σ ensures that $\mathrm{root}(G_j) \in \nu(\mathrm{root}(G_i))$, we establish that $G_i \trianglelefteq G_j$. So $\langle G_i \rangle_{i\in\mathbb{N}}$ has a good pair (G_i, G_j), a contradiction. □

Proposition 1 provides an interesting description of the structure of 2-connected simple graphs. The two following easy lemmas show that it can easily be adapted to multigraphs.

Lemma 6. *Let G be a graph and let G' be its underlying simple graph. The graph G is 2-connected iff G' is 2-connected or $G = \theta_k$ for some integer $k \geq 2$.*

Proof. It is clear that G is 2-connected whenever G' is. Let us now assume that G is 2-connected but G' is not, and let $u, v \in \mathrm{V}(G')$ be two distinct vertices of G' such that there is no pair of internally disjoint paths from u to v in G'. Since G is 2-connected, there are two internally disjoint paths P and Q in G linking u to v. Remark that if P and Q are edge-disjoint, then the corresponding paths in G' are internally disjoint and link u to v, a contradiction with the choice of these two vertices. Therefore P and Q share an edge (which has multiplicity at least two). Since these paths are internally disjoint, their ends must be the ends of the edge that they share: $\{u, v\}$ is an edge with multiplicity at least two. Removing the edge $\{u, v\}$ in G yields two connected components, one, G_u, containing u and the other, G_v, containing v. Since every path from vertices of G_u to vertices of G_v in G contains u, the graph G_u contains only the vertex u (otherwise G is not 2-connected) and by symmetry $\mathrm{V}(G_v) = \{v\}$. Therefore $G = \theta_k$, for some integer $k \geq 2$, as required. □

Lemma 7 (extension of Proposition 1 to graphs)**.** *Every 2-connected graph has a a tree-decomposition $(T, \mathcal{X})$ such that $|X_t \cap X_{t'}| = 2$ for every edge $\{t, t'\} \in T$ and where every torso is either 3-connected or a cycle.*

Proof. Let G be a 2-connected graph and G' be its underlying simple graph. If G' is 2-connected, then by Proposition 1 it has a tree-decomposition $(T, \mathcal{X})$ such that $|X_t \cap X_{t'}| = 2$ for every edge $\{t, t'\} \in T$ and where every torso is either 3-connected, or a cycle. Noticing that $(T, \mathcal{X})$ is also a tree-decomposition of G concludes this case. If G' is not 2-connected, then by Lemma 6 we have $G = \theta_k$ for some integer $k \geq 2$.

If $k = 2$ the graph G is a cycle, and if $k > 2$ it is 3-connected, therefore it has a trivial tree-decomposition with one bag, which satisfies the properties required in the statement of the lemma. ☐

We call such a tree decomposition a *Tutte decomposition*.

4 Well-quasi-ordering graphs without big bonds

The main result is proved in three steps. First, we show that for every $k \in \mathbb{N}$, the class of labeled 2-connected graphs of $\mathcal{G}_{1,k}$ is well-quasi-ordered by $\trianglelefteq$. Then, we use Lemma 5 to extend this result to all graphs of $\mathcal{G}_{1,k}$, i.e. all connected graphs not containing a bond of size more than k. Last, we adapt this result to classes of graphs with a bounded number of connected components.

Lemma 8. *For every $k \in \mathbb{N}$, and for every wqo $(\Sigma, \preceq)$, the poset $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k}), \trianglelefteq)$ is a wqo.*

Proof. Let $k \in \mathbb{N}$, and let $(\Sigma, \preceq)$ be a wqo. By contradiction, let us assume that $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k}), \trianglelefteq)$ is not a wqo. We consider the edge-rooted closure $\mathcal{H}$ of $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k})$, i.e. the class of all edge-rooted graphs whose underlying non-rooted graphs belongs to $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k})$. Clearly, $(\mathcal{H}, \trianglelefteq)$ is not a wqo, as a consequence of our initial assumption. We will show that this leads to a contradiction.

Let $\{A_i\}_{i\in\mathbb{N}}$ be an infinite minimal (wrt. $\trianglelefteq$) bad sequence of $(\mathcal{H}, \trianglelefteq)$: for every $i \in \mathbb{N}$, A_i is a minimal graph (wrt. $\trianglelefteq$) such that there is an infinite bad sequence starting with $A_0, \dots, A_i$. For every $i \in \mathbb{N}$, A_i has a Tutte decomposition (Lemma 7) which has a bag containing the endpoints of the edge $\{A_i.r, A_i.s\}$ (because it is a tree decomposition). Let $A_i.X$ be the torso of some (arbitrarily chosen) bag in such a decomposition which contains $A_i.r$ and $A_i.s$.

For every edge $x, y \in \mathrm{V}(A_i.X)$, let $A_i.V_{x,y}$ be the vertex set of the (unique) block which contains both x and y in the graph obtained from A_i by deleting vertices $\mathrm{V}(A_i.X) \setminus \{x, y\}$ and adding the edge $\{x, y\}$ with multiplicity 2.

Let us consider graphs obtained by contracting all the edges of A_i that does not have both endpoints in $A_i.V_{x,y}$ in a way such that $A_i.r$ gets contracted to x and $A_i.s$ gets contracted to y. Remark that for fixed i and (x, y), these graphs differ only by the multiplicity of the edge between the two roots x and y. For every $i \in \mathbb{N}$ and $x, y \in \mathrm{V}(A_i.X)$, we denote by $A_i.C_{x,y}$ an arbitrarily chosen such graph. Eventually, we set $A_i.\mathcal{C} = \{A_i.C_{x,y},\ x, y \in \mathrm{V}(A_i.X)\}$. Remark that every graph of $A_i.\mathcal{C}$ belongs to $\mathcal{G}^{(2)}_{1,k}$ and is a contraction of A_i.

Claim 1. $\mathcal{C} = \cup_{i\in\mathbb{N}} A_i.\mathcal{C}$ is wqo by $\trianglelefteq$.

Proof. By contradiction, assume that $(\mathcal{C}, \trianglelefteq)$ has an infinite bad sequence $\{B_i\}_{i\in\mathbb{N}}$. By definition of $\mathcal{C}$, for every $i \in \mathbb{N}$ there is a $j = \varphi(i) \in \mathbb{N}$ such that $B_i \trianglelefteq A_j$. Let $i_0 \in \mathbb{N}$ be an integer with $\varphi(i_0)$ minimum. Let us consider the following infinite sequence:

$$A_0, \dots, A_{\varphi(i_0)-1}, B_{i_0}, B_{i_0+1}, \dots.$$

Remark that this sequence cannot have a good pair of the form $A_i \trianglelefteq A_j$, $0 \leq i < j < \varphi(i_0)$ (respectively $B_i \trianglelefteq B_j$, $i_0 \leq i < j$) since $\{A_i\}_{i\in\mathbb{N}}$ (respectively $\{B_i\}_{i\in\mathbb{N}}$) is an antichain. Let us assume that there is a good pair of the form $A_i \trianglelefteq B_j$, for some $i \in [\![0, \varphi(i_0) - 1]\!]$, $j \geq i_0$. Then we have $A_i \trianglelefteq B_j \trianglelefteq A_{\varphi(j)}$. By the choice of i_0 we have $\varphi(i_0) \leq \varphi(j)$, hence $i < \varphi(j)$ so $(A_i, A_{\varphi(j)})$ is a good pair of $\{A_i\}_{i\in\mathbb{N}}$, a contradiction. Therefore, this sequence is an infinite bad sequence of $(\mathcal{H}, \trianglelefteq)$ and we have $B_{i_0} \trianglelefteq A_{\varphi(i_0)}$ and $B_{i_0} \neq A_{\varphi(i_0)}$. This contradicts the minimality of $\{A_i\}_{i\in\mathbb{N}}$, therefore $(\mathcal{C}, \trianglelefteq)$ is a wqo. $\square$

Let $\mathcal{C}^-$ be the class of 2-rooted graphs obtained from graphs of $\mathcal{C}$ by deleting the edge between the roots. We set $\mathcal{C}^+ = \{H \oplus_{H.r}^{H.s} \theta_i,\ i \in [\![0, k]\!],\ H \in \mathcal{C}^-\}$. In other words C^+ is the class of graphs that can be constructed by possibly replacing the edge at the root of a graph of $\mathcal{C}$ by an edge of multiplicity i, for any $i \in [\![1, k]\!]$.

Remark 5. It follows from Lemma 4 that both $(\mathcal{C}^-, \trianglelefteq)$ and $(\mathcal{C}^+, \trianglelefteq)$ are wqos.

Notice that for every $i \in \mathbb{N}$ and $\{x, y\} \in \mathrm{E}(A_i.X)$, the graph $A_i[A_i.V_{x,y}]$ rooted in (x, y) belongs to $\mathcal{C}^+$. As explained thereafter, this property enables us to see A_i as a graph built from graphs of $\mathcal{C}^+$.

According to Lemma 7, for every $i \in \mathbb{N}$, the graph $A_i.X$ (which is the torso of a bag of a Tutte decomposition) is either a 3-connected graph (and thus $|\mathrm{V}(A_i.X)| < \zeta_k$ by Proposition 2), or a cycle (of any length). Therefore we can partition $\{A_i\}_{i\in\mathbb{N}}$ into at most ζ_k subsequences depending on the *type* of $A_i.X$, where this type can be either "cycle", or one type for each possible value of $|\mathrm{V}(A_i.X)|$ when $A_i.X$ is 3-connected. Let us show that each of these subsequences are finite.

First case: $\{A_i\}_{i\in\mathbb{N}}$ has an infinite subsequence $\{D_i\}_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$, $D_i.X$ is a cycle. Then each graph of $\{D_i\}_{i\in\mathbb{N}}$ can be constructed by attaching a graph of the wqo $(\mathcal{C}^+, \trianglelefteq)$ to each edge of a cycle after deleting this edge. By Lemma 3, these graphs are wqo by $\trianglelefteq$, a contradiction.

Second case: for some positive integer $n < \zeta_k$, $\{A_i\}_{i\in\mathbb{N}}$ has an infinite subsequence $\{D_i\}_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$, $|\mathrm{V}(D_i.X)| = n$. Then every graph of $\{D_i\}_{i\in\mathbb{N}}$ can be constructed by attaching a graph of the wqo $(\mathcal{C}^+, \trianglelefteq)$ to each pair of distinct vertices of $\overline{K}_n$. By Lemma 2, $\{D_i\}_{i\in\mathbb{N}}$ has a good pair, which is contradictory since it is an bad sequence.

We just proved that $\{A_i\}_{i\in\mathbb{N}}$ can be partitioned into a finite number of subsequences each of which is finite. Hence $\{A_i\}_{i\in\mathbb{N}}$ is finite as well, a contradiction. Therefore our initial assumption is false and $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k}), \trianglelefteq)$ is a wqo. $\square$

Corollary 6. *For every $k \in \mathbb{N}$, the class $\mathcal{G}_{1,k}$ is well-quasi-ordered by $\trianglelefteq$.*

Proof. According to Lemma 8, for every wqo $(\Sigma, \preceq)$, the class of Σ-labeled 2-connected graphs of $\mathcal{G}$ are wqo by $\trianglelefteq$. By Lemma 5, this implies that $(\mathcal{G}_{1,k}, \trianglelefteq)$ is a wqo and we are done. $\square$

Proof of Theorem 1. Let $p, k \in \mathbb{N}^*$. Let us consider the function defined as follows.

$$f \colon \left\{ \begin{array}{rcl} (\mathcal{G}_{1,k}^p, \trianglelefteq^p) & \to & (\mathcal{G}_{p,k}, \trianglelefteq) \\ (G_1, \ldots, G_p) & \mapsto & \bigcup_{i=1}^p G_i \end{array} \right.$$

Given a tuple of p connected graphs not having a bond of size more than k (possibly containing the graph with no vertex), the function f returns their disjoint union. Clearly, the resulting graph has at most p connected components and do not contain a bond of size more than k. Conversely, let $G \in \mathcal{G}_{p,k}$ and let $G_1, \ldots, G_i$, $(i \leq p)$ be an enumeration of its connected components taken in an arbitrary order. For every $j \in [\![i+1, p]\!]$, let G_j be the graph with no vertices. Remark that G is isomorphic to $f(G_1, \ldots, G_p)$. Therefore f is surjective. Furthermore, for every pair of tuples $(G_1, \ldots, G_p)$ and $(H_1, \ldots, H_p)$ such that $(G_1, \ldots, G_p) \trianglelefteq^p (G_1 \ldots, G_p)$, we clearly have $f((H_1, \ldots, H_p)) \trianglelefteq f(G_1, \ldots, G_p)$: f is monotone.

We just proved that f is an epi. Its domain is a wqo since it is the Cartesian product of the wqo $(\mathcal{G}_{1,k}, \trianglelefteq)$ (cf. Proposition 3 and Corollary 6), therefore its codomain is a wqo as well, by the virtue of Remark 2.

□

5 Canonical antichains of $(\mathfrak{G}, \trianglelefteq)$

This section is devoted to the proof of the two results related to antichains of $(\mathfrak{G}, \trianglelefteq)$ and stated in Section 1: Theorem 2 and Corollary 3. The *closure* of a graph class $\mathcal{G}$ is defined as the class $\{H,\ H \trianglelefteq G \text{ for some } G \in \mathcal{G}\}$. Notice that any closure is contraction-closed.

Remark 6. Every canonical antichain of $(\mathfrak{G}, \trianglelefteq)$ is infinite.

Proof of Theorem 2. "$\Rightarrow$": Let $\mathcal{A}$ be a canonical antichain of $(\mathfrak{G}, \trianglelefteq)$ and let us assume for contradiction that $\mathcal{B} = \mathcal{A}_\theta \setminus \mathcal{A}$ (respectively $\mathcal{B} = \mathcal{A}_{\overline{K}} \setminus \mathcal{A}$) is infinite. Let $\mathcal{B}^+$ be the closure of $\mathcal{B}$ and remark that $\mathcal{B}^+ = \mathcal{B} \cup \{K_1\}$ (respectively $\mathcal{B}^+ = \mathcal{B}$). Then the contraction-closed class $\mathcal{B}^+$ has finite intersection with $\mathcal{A}$ whereas it contains the infinite antichain $\mathcal{B}$. This is a contradiction with the fact that A is canonical, hence both $\mathcal{A}_\theta \setminus \mathcal{A}$ and $\mathcal{A}_{\overline{K}} \setminus \mathcal{A}$ are finite.

Let us now assume that $\mathcal{C} = \mathcal{A} \setminus \{\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}\}$ is infinite and let $\mathcal{C}^+$ be the closure of $\mathcal{C}$. Being a subset of an antichain, $\mathcal{C}$ is an antichain as well and consequently $\mathcal{C}^+$ is a contraction-closed class that is not well-quasi-ordered. By Corollary 2, $\mathcal{C}^+$ contains infinitely many elements of $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}$. Notice that besides being infinite, $\mathcal{C}^+ \cap (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$ is also disjoint from $\mathcal{A} \cap (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$, otherwise $\mathcal{A}$ would contain an element from $\mathcal{C}$ contractible to an element of $\mathcal{A} \cap (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$. But then one of $\mathcal{A}_\theta \setminus \mathcal{A}$ and $\mathcal{A}_{\overline{K}} \setminus \mathcal{A}$ is infinite, a contradiction with our previous conclusion. Therefore $\mathcal{C}$ is finite.

"$\Leftarrow$": Let $\mathcal{A}$ be an antichain such that each of $\mathcal{A}_\theta \setminus \mathcal{A}$, $\mathcal{A}_{\overline{K}} \setminus \mathcal{A}$, and $\mathcal{A} \setminus \{\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}\}$ is finite, and let us show that $\mathcal{A}$ is canonical. Let $\mathcal{F}$ be a contraction-closed class of $\mathfrak{G}$. If $\mathcal{F} \cap \mathcal{A}$ is infinite, then $\mathcal{F}$ trivially contains the infinite antichain $\mathcal{F} \cap \mathcal{A}$. On the other hand, if $\mathcal{F} \cap \mathcal{A}$ is finite then by Corollary 2 the class $\mathcal{F}$ is well-quasi-ordered, hence by definition it does not contain an infinite antichain. Consequently, $\mathcal{A}$ is canonical, as required. □

Proof of Corollary 3. Let $\mathcal{A}$ be a canonical antichain of $(\mathfrak{G}, \trianglelefteq)$. Observe that we have the following:

$$\mathrm{Incl}(\mathcal{A}) = \mathrm{Incl}(\mathcal{A} \cap \mathcal{A}_\theta) \cup \mathrm{Incl}(\mathcal{A} \cap \mathcal{A}_{\overline{K}}) \cup \mathrm{Incl}(\mathcal{A} \setminus (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})).$$

Now, is is easy to notice that:

- $\mathrm{Incl}(\mathcal{A} \cap \mathcal{A}_\theta) \subseteq \mathrm{Incl}(\mathcal{A}_\theta) = \{K_1\}$;
- $\mathrm{Incl}(\mathcal{A} \cap \mathcal{A}_{\overline{K}}) \subseteq \mathrm{Incl}(\mathcal{A}_{\overline{K}}) = \emptyset$;
- $\mathrm{Incl}(\mathcal{A} \setminus (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}))$ is finite, because $\mathcal{A} \setminus (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$ is finite by Theorem 2 and since $\mathcal{A}$ is canonical.

Therefore, $\mathrm{Incl}(\mathcal{A})$ is finite as well and hence cannot contain an infinite antichain; this proves that $\mathcal{A}$ is fundamental. □

References

[1] Peter Damaschke. Induced subgraphs and well-quasi-ordering. *Journal of Graph Theory*, 14(4):427–435, 1990.

[2] Reinhard Diestel. Graph theory. 2005. *Graduate Texts in Mathematics*, 2005.

[3] Guoli Ding. Subgraphs and well-quasi-ordering. *Journal of Graph Theory*, 16(5):489–502, 1992.

[4] Guoli Ding. Chordal graphs, interval graphs, and wqo. *Journal of Graph Theory*, 28(2):105–114, 1998.

[5] Guoli Ding. On canonical antichains. *Discrete Mathematics*, 309(5):1123 – 1134, 2009.

[6] Michael R. Fellows, Danny Hermelin, and Frances A Rosamond. Well-quasi-ordering bounded treewidth graphs. In *Proceedings of IWPEC*, 2009.

[7] Graham Higman. Ordering by divisibility in abstract algebras. *Proceedings of the London Mathematical Society*, s3-2(1):326–336, 1952.

[8] Chun Hung Liu. *Graph Structures and Well-Quasi-Ordering*. PhD thesis, Georgia Tech, 2014.

[9] J. B. Kruskal. Well-quasi-ordering, the tree theorem, and Vazsonyi's conjecture. *Transactions of the American Mathematical Society*, 95:210–225, 1960.

[10] C. St. J. A. Nash-Williams. On well-quasi-ordering finite trees. *Proceedings of the Cambridge Philosophical Society*, 59:833–835, 1963.

[11] Bogdan Oporowski, James Oxley, and Robin Thomas. Typical subgraphs of 3- and 4-connected graphs. *Journal of Combinatorial Theory, Series B*, 57(2):239–257, March 1993.

[12] Neil Robertson and Paul Seymour. Graph Minors XXIII. Nash-Williams' immersion conjecture. *Journal of Combinatorial Theory, Series B*, 100(2):181–205, March 2010.

[13] Neil Robertson and Paul D. Seymour. Graph Minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325 – 357, 2004.

[14] W. T. Tutte. A theory of 3-connected graphs. *Indagationes Mathematicae*, 23(1961):441–455, 1961.