The Star-Topology: a topology for image analysis

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Abstract: *Our previous works in image analysis dealing with image representations by means of adjacency or boundary graphs, led us to the need for a coherent representation model.*

In fact, classical approaches to this problem seem to be unsufficient or even uncoherent; for example they are unclear with the well-known connectivity paradox or with the border description of regions.

Different works like those of Kovalevsky, Herman and Malandain pointed out the advantages of cellular complex based topologies [Kovalevsky89, Herman90, Malandain93]. But no one of them suggested a formalism that can be applied to any type of image.

In this work we propose a topological representation for any type of image, colour or grey-level of whatever dimension. It is based on convex complexes, and looks closely at the elements realizing the connectivity within complexes and later within regions. Futhermore it remains coherent with pixel and voxel only based representations.

An important feature is still maintaining a direct correspondance with the classical $\mathbb{R}^n$ *topology. Finally, we suggest a characterization of regions, their borders and boundaries which is useful as a basic tool for segmentation.*

Keywords : topology on finite sets, image processing, convex complex, adjacency and connectivity.

1 Introduction

In image analysis, when trying to deal with the representation of images, a first fact appears: the uncoherences (and unsufficiency) of the traditionnal representation models. Apart from some particular cases, they all end up in the connectivity paradox due to the non conformity with Jordan's theorem, and to unstable geometric representations of images' objects (regions).

In fact, classical approaches tend to describe objects by using a pixel only 4, 8-adjacency based topology. Even in binary image such choices may be uncoherent and contradict some geometric evidences:

- How can we define the perimeter for one pixel large objects ?
- Objects with an "apparent" (we cannot say "real") perimeter proportional to a constant k don't have a number of border pixels in the same proportion.
- It is difficult to have a clear definition for the boundary between objects.
- How can we distinguish a one pixel large region of an edge ?

In a set of previous works ([Danielsson82, Ahronovitz85]), the so called interpixel approach was developed. It allows a clearer description, much more coherent with basic geometric properties. We can summarize its characteristics as follows:

- The boundary between objects is more "natural" (a linked path between objects) as is the perimeter definition: number of border interpixel links.
- One pixel large objects are not a particular case and don't need any special development.
- It is currently used in 3D images: surfaces are determined from the voxels' faces [Rosenfeld91].

A very important extension to this approach was made afterwards ([Charnier95]) and allows to deal with grey level and colour images in the same way.

Moreover we have to note that in all these works we have developed linear algorithms, which process in one image scan in order to get a segmented representation efficiently.

The study describe hereafter can be looked at as an extension of these results, an attempt to go deeper into the topological bases in order to build theoretical and formal basis. It lies upon the cellular complex notion. When applied to images, it consists in determining the open sets of the topology, starting with base elements called *cells*. These are respectively pixels, edges (line elements or linels), vertices (point elements or pointels) in 2D and voxels, faces (surface elements or surfels), linels and pointels in 3D (see Figure 1).

We show that it is very important to take all these elements into account if we want to cope with the connectivity paradox. The formalism that we use is based on Kovalevsky's works [Kovalevsky89], and we adapt it to all types of images. We then suggest characterizations up to the region, its border and its boundaries.

The formalism of Malandain [Malandain93] is suitable only to binary (multidimensionnal) images and shows again that two differents connectivities are necessary, one for objects and the other for the background. From our point of view, two different connectivities are not necessary and we suggest a model based on a connectivity closely linked to the dimension of the cell actually achieving the connectivity.

Finally after the justification of these choices for connectivity and adjacency we suggest a new look at regions and at images in general, the latters being just regions of $\mathbb{R}^n$ after all.

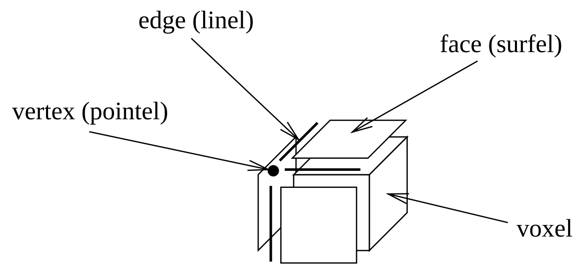


Figure 1: Cells for 3D images.

2 The convex complexes

Images are finite sets composed of pixels or voxels. A non trivial topology on a finite set is defined out of an order relation and unfortunaly no natural order exists for sets of pixels (or voxels) ([Kovalevsky89, Herman90]). So Kovalevsky proposed to add new elements to the set of pixels to obtain an ordered set where links between pixels are well described. In this work, we propose a variant of Kovalevsky's work more adapted to our geometric context.

A complex is a topological space with a particular partition whose sets are called cells. Kovalevsky used abstract cellular complexes. We propose convex complexes whose cells are convex polytopes.

Definition 2.1 *A convex complex $\mathcal{C}$ is a finite family of open polytopes P_i of $\mathbb{R}^n$ called the cells of the complex such that :*

- *If $P_i \in \mathcal{C}$, then every face of $P_i \in \mathcal{C}$.*
- *If $P_i \in \mathcal{C}$ and $P_j \in \mathcal{C}$,$i \neq j$, then $\overline{P_i} \bigcap \overline{P_j}$ is empty [1] or is a common face of P_i and P_j.*

By definition, the dimension of a complex[2] is the max of the dimensions of the cells of the complex.

A polytope of dimension 0 is called a point (or a pointel), one of dimension 1 is called a vertex (or a linel), and one of dimension 2 is called a polygone (a surfel).

The notion of a polyhedral set given by the Definition 2.2 will allow us to work with a topology induced by the topology of $\mathbb{R}^n$ on discrete structure such as complexes.

Definition 2.2 *Let E be a family of cells. We call the polyhedral set of E, and we denote $|E|$ the union of the elements of cells of E.*

Example 1 *: The pixels of the digital image.*

In this case dim $(K) = 2$ and the maximal polytopes are all equal squares.

[1] We denote $\overline{E}$ the closure of the set E. Here the considered topology is induced by the classical topology of $\mathbb{R}^n$.

[2] For now on, we often will use the term complex instead of convex complex.

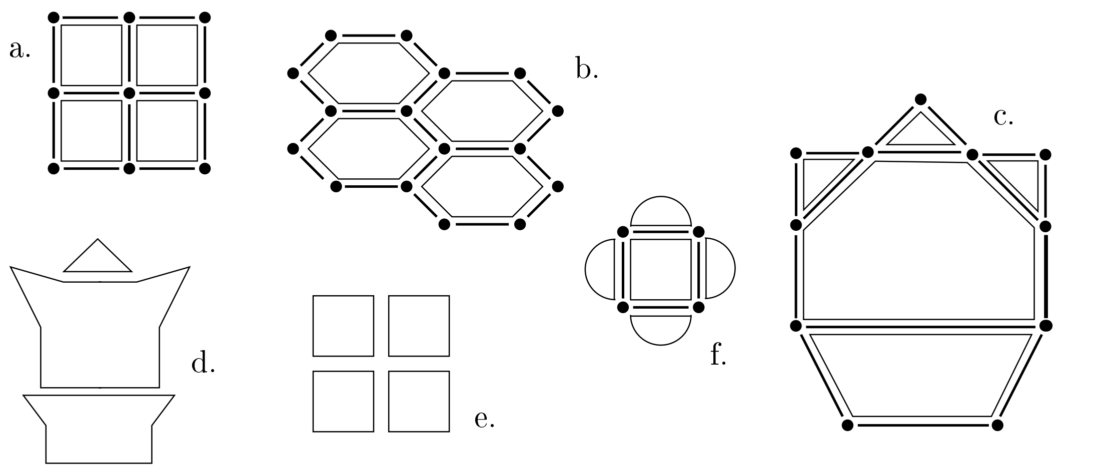


Figure 2: Example (a. b. c.) and counter-examples (d. e. f.) of convex complexes.

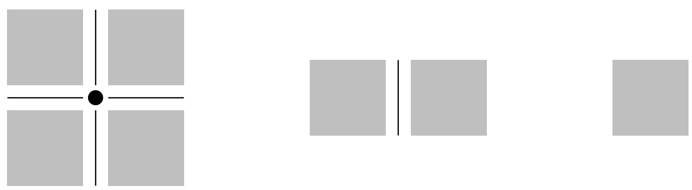


Figure 3: Open star of a pointel, linel and pixel.

Example 2 *: The voxels*
In this case dim(K) = 3 and the maximal polytopes are all equal cubes.

In Figure 2, we can see other examples and counter-examples of convex complexes.

Definition 2.3 (boundary and incidence relations) *Given two polytopes P_1 and P_2, we write that P_1 bounds P_2 and we denote $P_1 \leqslant_B P_2$ if P_1 is a face of P_2 or if $P_1 = P_2$. If P_1 bounds P_2 or P_2 bounds P_1, we write that P_1 and P_2 are incident.*

As seen above, an order relation allows to define a topology on a finite space. So the bounding relation allows us to define a topology called the star-topology on a convex complex.

Definition 2.4 *The open star of a cell c, is the set of cells of K bounded by c. We denote star(c) the open star of c (see for example for 2D images Figure 3).*

Definition 2.5 *The star-topology is the topology define on the complex K whose one basis of open sets is the set of the open stars of all the cells of K.*

We get the properties :

- A set E of cells is an open set for the star-topology if and only if : if a cell $c \in E$ then $star(c) \subseteq E$.
- A set E of cells is a closed set for the star-topology if and only if : if a cell $c \in E$, and if $c \leqslant_B c'$ then $c' \in E$;

We can now characterize the star-topology.

Theorem 2.5.1 (fundamental theorem) *Let $|K|$ be a convex complex. The star-topology is the quotient topology of the topology induced on the polyhedral set $|K|$ by the classical topology of $\mathbb{R}^n$ by means of the relation cell: two points of $|K|$ x and y are said cell-equivalent if and only if they are in the same cell.*

Thus the topology of the finite space $|K|$ can be described out of the classical topology of $\mathbb{R}^n$ whose properties are well known. The power of this theorem comes from the ability to deduce results on star-topology from classical results of algebraic topology. So we give some definitions and get again here some results that are compatible with the relation *cell*.

Definition 2.6 (K-curve) *A K-curve is a finite sequence $c_0, ..., c_{2n}$ of cells which are alternatively points and vertices so that c_0 and c_{2n} are points and that cells of successive indices are incident. If all the cells are distinct, the K-curve is said to be simple, if $c_0 = c_{2n}$, then the K-curve is said to be closed.*

Definition 2.7 (connected set) *A set E of cells of a complex is connected if for whatever cells c and c' their exists a sequence $c = c_0, \ldots, c' = c_{2n}$ of cells of E such that two cells of successive indices are incident.*

We can now, with the fundamental theorem and the above definitions, give a theorem of Jordan for the K-curves and also topological properties of complexes that can be deduced from those of homotopy, covers and so on. We give the Jordan's theorem for K-curves.

Theorem 2.7.1 (Jordan's theorem) *A closed, simple K-curve cuts a two dimensionnal complex into two connected component : the interior and the exterior of the K-curve.*

(see [Ahronovitz95]).

3 Star-Topology and Image analysis

It is clear that taking into account such cells as surfels, linels or pointels makes the problem of image analysis more difficult to cope with. In fact, pixels or voxels represent already a big amount of data, and adding a lot of new informations can overload too much the processing. We should not forget that in image analysis, efficient run time is an important characteristic of applications. So increasing data can only slow down the computations. Moreover the informations given by linels, pointels or surfels are not always relevant. Consequently our approach consists in working mainly with the highest dimension cells,

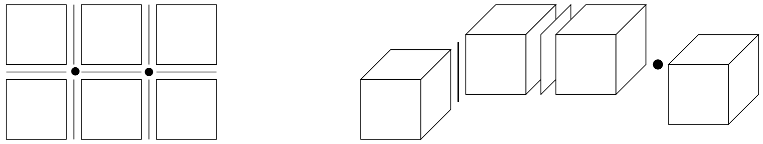


Figure 4: Two by two adjacent n-cells

i.e. pixels or voxels[3], but we paying attention to the necessary "less-than-n-dimension" cells. These cells are only border cells. The star-topology invites us to look closely at them if we want to avoid adjacency or connectivity misunderstanding.

So we will introduce in this section how and why to adjust the previous notions to the image analysis point of view. As a result we will be able to define a region. Then we will introduce a problem due to this new image analysis point of view: the adjacency between regions.

3.1 Connectivity in image analysis

Usually two pixels or voxels are said to be adjacent if they share an edge, vertex or face in the case of voxels. In the 2D case this leads to the 4, 8-connectivity. If we extend in a straight forward manner this notion to the Star-Topology, we get:

Definition 3.1 *Two n-cells are said to be adjacent if and only if there exists a third cell which bounds the first two (see Figure 4).*

But usually the connectivity is also directly deduced from the adjacency notion. Unfortunately such a connectivity does not meet our requirements. In fact if we look at Figure 5,

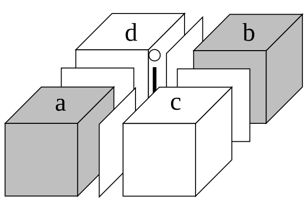


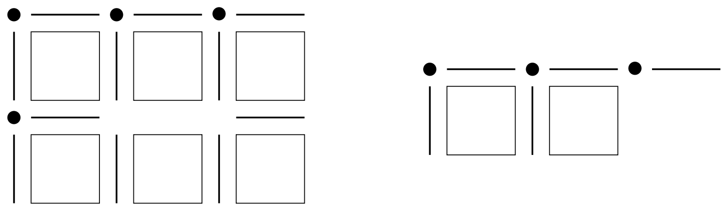
Figure 5: Connectivity

because of the middle linel, the grey voxels are adjacent and, because of the pointel, the white voxels are adjacent too. We are falling again into the "connectivity paradox" type of problems. So having a particular definition of the adjacency in order to allow a suitable definition of the connectivity is essential. Be carefull, we don't deny the adjacency of Definition 3.1, but we only give an other, led by practical image analysis considerations.

In fact an "*adjacency in a set*" is defined which will allow us to decide if two n-cells belonging to the same set are adjacent or not. According to this adjacency a connectivity set is defined. The idea is the following: the grey voxels a and b in Figure 5 are "more adjacent" because they are linked by a linel whereas the white voxels c and d are linked only by a pointel. So the idea is to use a *junction-cell* to study how are realized the adjacency and the connectivity within the image analysis scope.

Definition 3.2 (junction) *Let e_1, e_2 be two adjacent n-cells; we call Junction(e_1, e_2) the cell with the highest dimension among those bounding e_1 and e_2.*

[3] We will call these cells, *n-cells*, where n is the highest dimension, i.e. 2 for pixels and 3 for voxels



a. a hole in region   b. a region with its border

Figure 6: Regions and cells of dimension less that n.

Let E be a set of cells the new way in which we define adjacency and connectivity between n-cells is:

Definition 3.3 (adjacency in E) *Two n-cells* $e_1, e_2 \in E$ *are said to be* adjacent *in* E *if and only if* e_1 *and* e_2 *are adjacent and* $Junction(e_1, e_2) \in E$

Definition 3.4 (connectivity between n-cells of E) *two n-cells* $e, e' \in E$ *are connected if and only if there exists a sequence of n-cells of* E*,* $e = e_0, \ldots, e_p = e'$*, such that* $\forall i = 1 \ldots p,\ e_i$ *and* e_{i-1} *are adjacent in* E*.*

3.2 The region concept

Traditionnaly a region is a connected set of pixels or voxels. With the formalism of our approach , it leads to define a region as a set of connected n-cells.

But if we look at Figure 6.a, the missing pointels and linel are to be considered as a region. Even if this is correct from the Star-Topology point of view, in terms of image analysis it would not be reasonable to let such regions possible. The problem presented in Figure 6.b is different: it is due to the fact that we want the regions to have a border, and the boundary between two regions to be shared among them. But the definition of a region must not authorize such "strangeness" as Figure 6.b. We can verify with Lemmas 3.5.1 and 3.5.2 that the conditions imposed by the following Definition 3.5 prohibit cases like those presented in Figure 6.

Definition 3.5 (region) *We call* region *a set* R *of connected n-cells such that:*

$$\overset{\circ}{\overline{\mathrm{R}}} \subseteq \mathrm{R} \subseteq \overline{\overset{\circ}{\mathrm{R}}}$$

Lemma 3.5.1 $x \in \overset{\circ}{\overline{\mathrm{R}}} \Rightarrow star(x) \subseteq \overset{\circ}{\overline{\mathrm{R}}}$

Lemma 3.5.2 $\forall\, x \in \overset{\circ}{\overline{\mathrm{R}}},\ dim(x) = n \Rightarrow x \in \mathrm{R}$

For a complete proof of these two lemmas, see [Ahronovitz95].

3.3 Image analysis

Our works in image analysis deal with image segmentation. We will first remark that the classical segmentation definition as gived by Pavlidis ([Zucker76, Horowitz74]) stays with our formalism:

Definition 3.6 (segmentation) *Let* P *be a logical predicate defined on a set of contiguous picture points. Then a segmentation can be defined as a partition of the image* I *into disjoint subsets (regions)* $R_1, \ldots, R_k$ *such that:*

1. $\mathrm{I} = \bigcup_{i=1}^{k} R_i$
2. $R_i \bigcap R_j = \emptyset,\ i, j \in \{1, \ldots, k\}, i \neq j$
3. $\forall\ i = 1, \ldots, k$ P(R_i) *is true.*
4. P$(R_i \bigcup R_j)$ *is false* $\forall\ i \neq j$*, where* R_i *and* R_j *are adjacent.*

On the other hand we will remark that the definitions of the outside, border and boundaries are identical to those of the Euclidean topology in $\mathbb{R}^n$. Let $\mathcal{C}$ be a complex containing the image, we have:

Outside: $\mathrm{Ext}(R) = (\mathcal{C} \overset{\circ}{\setminus} \mathcal{R})$

Border: $\mathrm{Bd}(R) = R \setminus \overset{\circ}{R}$

Boundary: $\mathcal{F}r(R) = \mathrm{Bd}(R) \bigcup \mathrm{Bd}(\mathcal{C} \setminus \mathcal{R})$

Let us now look at the problem of the adjacency between regions. In fact, a usual representation used in image analysis is the region adjacency graph where vertices are regions and edges denote the adjacency relation. What about our formalism ? Commonly, two regions are adjacent if they have a common boundary. If we look at Figure 7 we can see that the three regions are adjacent two by two. Since the region adjacency graph is an abstract representation of the image, we will only work on it during the segmentation process. The merging of two regions will only be realized between related vertices (linked by an edge). But if we look closely at Figure 7 we can see that merging the grey regions a and c involves the division of white region b. And this information can not be found in the graph! This is why we think that we need a new adjacency that we call the *strict adjacency*. In fact the whole problem is related to the middle pointel. We will call such cells *linking-cells*.

Definition 3.7 (linking-cells) *Let be* $e \in \mathrm{Bd}(R)$*;* e *is a linking-cell of* R *if and only if there exists two n-cells* $e_1, e_2 \in R$ *such that* $e = Junction(e_1, e_2)$*.*

A linking-cell can be seen as a *Junction* at the region border, or also as a cell ensuring the connectivity. Indeed if this cell is removed, the connectivity property is lost.

Definition 3.8 (strict adjacency) *Two regions* R_1 *and* R_2 *are said to be stricly adjacent if and only if there exists two n-cells* $e_1 \in R_1,\ e_2 \in R_2$ *such that there exists one cell* $e = Junction(e_1, e_2)$ *which is not a linking-cell of a third region.*

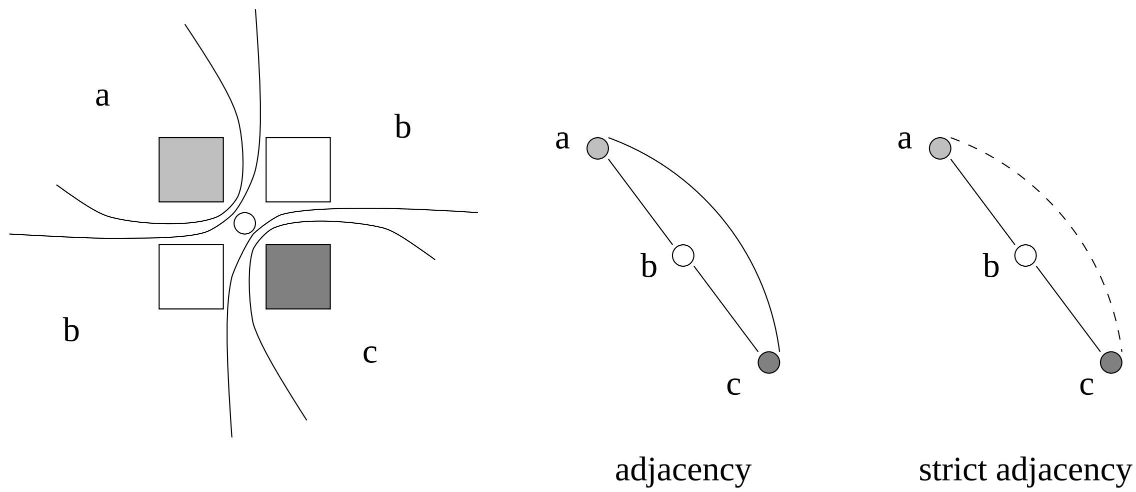


Figure 7: Adjacency and strict adjacency between regions.

4 Conclusion and prospects

The formalism that we introduced allows us to work with a coherent representation model for images. It lies upon the cellular complex notion, remains closely linked to the $\mathbb{R}^n$ classical topology and yet takes into account constraints yielded by image analysis.

The so-called Star-Topology, generated by the open stars, allows us to get back results from the real algebraic topology and to transpose them to our cells. Hence, curves and surfaces in $\mathbb{R}^n$ are also "naturally" transposed. Thus we can suggest a Jordan's theorem conformant with the star-topology.

When moving to image analysis, we adopt that topology and refine it in order to cope with specificities of this domain: the need to define regions, borders and boundaries. But we had to refine notions like adjacency and connectivity before coming to that of region. In fact, whether a region can be thought of as a homogeneous set of connected pixels or voxels, the Star-Topology tells us to have a closer look at the intermediate elements of lower dimension, achieving the connectivity.

Such an approach can seem heavy, as it requests a study of non-maximal dimension cells. Actually, these cells are studied only when they are related to the regions' borders. Thus, the new constraints that we add do not overload the segmentation process.

Our current works tend to the geometrical transformations of objects as described here. We are aware that the notion of region is still unsufficient: any object, even the simplest one is far beyond the region. It is rather a union of regions, represented by a sub-graph in the image representation. So our future works have an important connection with sub-graph morphisms.

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