Coverability and multi-scale coverability on infinite pictures

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Abstract

A word is *quasiperiodic* (or *coverable*) if it can be covered with occurrences of another finite word, called its *quasiperiod*. A word is *multi-scale quasiperiodic* (or *multi-scale coverable*) if it has infinitely many different quasiperiods. These notions were previously studied in the domains of text algorithms and combinatorics of right infinite words.

We extend them to infinite pictures (two-dimensional words). Then we compare the regularity properties (uniform recurrence, uniform frequencies, topological entropy) of quasiperiodicity with multi-scale quasiperiodicity, and we also compare each of them with its one-dimensional counterpart.

We also study which properties of quasiperiods enforce properties on the quasiperiodic words.

Keywords: Infinite pictures; combinatorics; quasiperiodicity; multi-scale

1. Introduction

At the beginning of the 1990's, in the area of text algorithms, Apostolico and Ehrenfeucht introduced the notion of *quasiperiodicity* [1]. Their definition is as follows: "a string w is quasiperiodic if there is a second string $u \neq w$ such that every position of w falls within some occurrence of u in w". The word w is also said to be u-quasiperiodic, and u is called a *quasiperiod* (or a *cover*) of w. For instance, the string:

$$abababababaabababababaababa$$

is aba-quasiperiodic and $ababa$-quasiperiodic.

☆This is an extended version of [8], by the same authors, published in the proceedings of the LATA2015 conference.

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In 2004, Marcus extended this notion to right-infinite words and observed some basic facts about this new class. He opened several questions [13], most of them related to Sturmian words and the factor complexity. First answers were given in [10]. A characterization of right-infinite quasiperiodic Sturmian words was given in [11] and extended to episturmian words in [9]. More details on their complexity function are given in [14, 15].

In [14], Marcus and Monteil showed that quasiperiodicity is independent from several other classical notions of symmetry in combinatorics on words. They also introduced a stronger notion, namely *multi-scale quasiperiodicity*, with better properties.

Finally, in [5], a two-dimensional version of quasiperiodicity was introduced. In particular, a linear-time algorithm computing all square quasiperiods of a square matrix of letters was given.

Warning. Note that in some contexts, most notably in the fields of sub-shifts, symbolic dynamics and tilings, "quasiperiodic" means "uniformly recurrent". Since we intend to move towards these areas in the future, the risk of collision is very high. Hence, from now on, we refer to quasiperiodic words as *coverable* words; each quasiperiod is a *cover* (or *covering pattern*).

In [8], we continued the study of two-dimensional coverability by generalizing the results from [14] to infinite pictures. In particular, we have shown some dependence and independence results between coverability (and multi-scale coverability) and aperiodicity, uniform recurrence, uniform frequencies, and topological entropy.

Our idea was (and still is) that coverability is a local rule. Hence a natural question, related to dynamical systems and tilings, is: does this local rule enforce some global order? (For a broader study of this question in the general context of tilings, see e.g. [6]). Independence results are negative answers to this question: coverability, which is a local rule, does not imply global properties. This is why we focus on a stronger notion, multi-scale coverability, in the last part of this paper.

Our preliminary results (from [8]) are summarized in the following table. Here, $\perp$ means "independent", ? means "not treated yet" and $\Longrightarrow$ means "implies something about".

	Aperiodicity	Uniform recurrence	Frequencies	Entropy
Coverability	$\perp$	?	?	?
Multi-scale	$\perp$	$\Longrightarrow$	?	$\Longrightarrow$

In this article, we extend these results in various ways. We complete our independence (and dependence) results with coverability. We also show that multi-scale coverability implies the existence of uniform frequencies. As a summary, we have get the following table.

	Aperiodicity	Uniform recurrence	Frequencies	Entropy
Coverability	$\perp$	$\perp$	$\perp$	$\Longrightarrow$
Multi-scale	$\perp$	$\Longrightarrow$	$\Longrightarrow$	$\Longrightarrow$

The paper is structured as follows.

In Section 2, we recall notations, definitions, and classical properties of symmetry on pictures, notably uniform recurrence, uniform frequencies and topological entropy. Then we adapt an elementary proof from the one-dimensional case to show that, in two dimensions, coverability is independent from these properties. This proof relies on a very specific cover; we conclude this section by showing that, for many other covers, coverability implies zero topological entropy. This new result might seem surprising, as it is different from dimension one. We conjecture that coverability may imply zero topological entropy, except for a very specific class of covers (which are essentially one-dimensional words). This would make things very different from the one-dimensional case.

In Section 3, we characterize the covers that are independent from aperiodicity, uniform recurrence and uniform frequencies. (Our previous work only did this for aperiodicity).

Finally in Section 4, we study relations between multi-scale coverability and topological entropy, uniform recurrence and uniform frequencies. Multi-scale coverability is a good notion of symmetry in one dimension, as it implies uniform recurrence, uniform frequencies and zero topological entropy. In our preliminary work, we have studied links between multi-scale coverability and uniform recurrence and topological entropy in two dimensions. We present these results, along with a new one: in two dimensions, multi-scale coverability also implies the existence of frequencies. This proof is purely combinatorial and does not involve ergodic theory.

2. Independence and Dependence Results

2.1. Definitions and Notations

In this section, we give all notations and definitions we will use afterwards. We will occasionally use classical notations and well-known results from the one-dimensional case, i.e. combinatorics on words; for these, see [12].

Let Σ be a finite alphabet. An *infinite picture* (or *two-dimensional word*, or $\mathbb{Z}^2$-word) is a function from $\mathbb{Z}^2$ to Σ. Unless otherwise stated, those functions are assumed to be total; otherwise, we note $\mathrm{dom}(\mathbf{w})$ the domain of $\mathbf{w}$, i.e. the set of coordinates where it has defined letters.

A *finite picture*, or *block*, is a function w such that $\mathrm{dom}(w) = \{i, \ldots, i+n-1\} \times \{j, \ldots, j+m-1\}$, for $i, j \in \mathbb{Z}$ and $n, m \in \mathbb{N}$. In that case, let $\mathrm{width}(w) = n$ and $\mathrm{height}(w) = m$. Moreover, the function w such that $\mathrm{dom}(w) = \emptyset$ is *the empty block*; it is considered as a block and has width and height equal to 0. Conversely, any block which has either width or height equal to 0 is the empty block. The set of blocks of dimension $n \times m$ is denoted by $\Sigma^{n\times m}$. More generally, if u is a block, then $u^{n\times m}$ denotes the rectangle formed of n rows and m columns of copies of u. Thus $u^{n\times m}$ has size $n\,\mathrm{width}(u) \times m\,\mathrm{height}(u)$.

If u and v are blocks, then let $|u|_v$ denote the number of occurrences of v in u. Let $u[x, y]$ denote the image of (x, y) by u. If $\mathbf{w}$ is an infinite picture,

$\mathbf{w}[(x,y),\cdots,(x+w-1,y+h-1)]$ denote the restriction of $\mathbf{w}$ to the rectangle $\{x,\ldots,x+w-1\}\times\{y,\ldots,y+h-1\}$, for $x,y\in\mathbb{Z}$ and $w,h\in\mathbb{N}$. If either w or h equal 0, then this denotes the empty block.

We will sometimes need to see finite blocks as one-dimensional words whose alphabets are "columns" or "lines". Let $\mathcal{C}_{\Sigma,n}$ (respectively $\mathcal{L}_{\Sigma,m}$) denote the set of n-columns (respectively m-lines) over Σ, i.e. $1\times n$-blocks (respectively $m\times 1$-blocks) over Σ. Concatenation in $\mathcal{C}$ and $\mathcal{L}$ is done respectively horizontally or vertically.

In what follows, let $\mathbf{w}$ be an infinite picture and let u,v be blocks. We recall some classical notions from combinatorics on words, adapted to the two-dimensional case.

By definition, $\mathbf{w}$ has a *vector of periodicity* $(k,\ell)\in\mathbb{Z}^2\setminus\{(0,0)\}$ if, for all positions $(x,y)\in\mathbb{Z}^2$, we have $\mathbf{w}(x,y)=\mathbf{w}(x+k,y+\ell)$. Moreover, we say that $\mathbf{w}$ is *periodic* if it has at least *two non-colinear* vectors of periodicity.

We say that u is a *cover* (or a *covering pattern*) of $\mathbf{w}$ if, for all $(x,y)\in\mathbb{Z}^2$, there exists $(i,j)\in\mathbb{N}^2$ with $0\le i<\mathrm{width}(u)$ and $0\le j<\mathrm{height}(u)$ such that $\mathbf{w}[(x-i,y-j)\ldots(x-i+\mathrm{width}(u)-1,y-j+\mathrm{height}(u)-1])$ is equal to u. Intuitively, u is a cover of $\mathbf{w}$ when each position of $\mathbf{w}$ belongs to an occurrence of u. If $\mathbf{w}$ has at least one cover, then it is *coverable*.

The picture $\mathbf{w}$ is *uniformly recurrent* if, for all $k\in\mathbb{N}$, there exists some $\ell\in\mathbb{N}$ such that all $k\times k$-blocks of $\mathbf{w}$ appear in all $\ell\times\ell$-blocks of $\mathbf{w}$. Intuitively, this means that any block of $\mathbf{w}$ appears infinitely often with bounded gaps.

Let $c_{\mathbf{w}}(n,m)$ be the number of different $n\times m$-blocks of $\mathbf{w}$. Note that $c_{\mathbf{w}}$ is known as the *block complexity function* of $\mathbf{w}$, and links between periodicity and block complexity are currently investigated (see e.g. [4]). In this paper, we will focus on the *topological entropy* of $\mathbf{w}$, which is the following quantity:

$$H(\mathbf{w})=\lim_{n\to\infty}\frac{\log_{|\Sigma|}c_{\mathbf{w}}(n,n)}{n^2}$$

Intuitively, if $c_{\mathbf{w}}(n,n)\simeq|\Sigma|^{\varepsilon n^2}$, then $H(\mathbf{w})\simeq\varepsilon$. In other words, when the complexity function of $\mathbf{w}$ is polynomial, $\mathbf{w}$ has zero entropy. This is a classical regularity property on words, often used in the context of dynamical systems. Note that there are several kinds of entropy, in addition to topological entropy. However, topological entropy is the canonical notion when studying finite-type sub-shifts, our area of interest.

Finally, the *frequency* of u in $\mathbf{w}$ is the following quantity:

$$f_u(\mathbf{w})=\lim_{n\to\infty}\frac{|\mathbf{w}[(-n,-n)\ldots(+n,+n)]|_u}{n^2}$$

if it exists. If $f_u(\mathbf{w})$ exists for all blocks u of $\mathbf{w}$, then u is said to *have frequencies* (or *have uniform frequencies*). This is another common regularity property coming from dynamical systems, where it is more often called *unique ergodicity.*

2.2. Coverability is Independent from Classical Notions of Symmetry

Now let us warm up with an easy independence result, already known in one-dimension (see [14]).

First, recall that Σ is a finite alphabet and let h denote a function from Σ to $\Sigma^{n\times m}$, for $n, m \in \mathbb{N}$. We extend h to a function from $\Sigma^{\mathbb{Z}^2}$ to $\Sigma^{\mathbb{Z}^2}$ as follows: if $\mathbf{w}$ is a picture (either finite or infinite), then $h(\mathbf{w})$ is the picture $\mathbf{w}'$ defined by $\mathbf{w}'[(xn, ym), \cdots, (xn+n-1, ym+m-1)] = h(\mathbf{w}[x,y])$ for all $x, y \in \mathbb{N}$. When convenient, we similarly extend h to finite pictures. We say that h is a *morphism on pictures*. We will use such a morphism in the next proof.

Proposition 1. *On infinite pictures, coverability is independent from uniform recurrence, existence of frequencies and topological entropy.*

Proof. For uniform recurrence, observe that $q = \begin{smallmatrix} b & b & a \\ a & b & b \end{smallmatrix}$ is a cover of the non-uniformly recurrent word displayed on Figure 1. With the same value of q, the q-periodic infinite picture is uniformly recurrent.

Let $\mathbf{w}$ be an infinite picture over $\{a, b\}$ with polynomial (respectively exponential, respectively. double-exponential) complexity. Consider the following function:

$$\nu(a) = ababaaba$$
$$\nu(b) = abaababa$$

The image $\nu(\mathbf{w})$ has polynomial with the same degree (respectively exponential, respectively double-exponential) complexity and is *aba*-coverable (viewing *aba* as a 3×1-block). Therefore, we can get either zero or positive topological entropy for coverable words.

Finally, the word $\nu(a^{\mathbb{Z}^2})$ has frequencies for all its blocks. By contrast, if $\mathbf{w}$ is a word having no frequencies for any block, then $\nu(\mathbf{w})$ has no frequencies either. □

Proposition 1 is a direct generalization of the one-dimensional case; it shows the independence between coverability and various notions. However, the proof involved a very specific quasiperiod, which was an 8×1-rectangle; i.e., we reproduced the behaviour of one-dimensional coverability on each line of an infinite picture. In the following, we use more specifically the two dimensions and show that a reasonable condition on the cover q can enforce a global property of q-coverable words.

2.3. Topological Entropy of Coverable Pictures

Let q and u be finite blocks such that $q \neq u$ and q is not empty. We say that u is *a border of* q when u occurs in two opposite corners of q. Note that it is possible to have either $\text{width}(b) = \text{width}(q)$ (which we call a *full-width border*) or $\text{height}(b) = \text{height}(q)$ (a *full-height border*), but not both. If neither case applies, we call b a *diagonal border*, following the terminology from [5]. We are going to show that, if q has a corner without any (non-empty) border, then all q-coverable pictures have zero topological entropy.

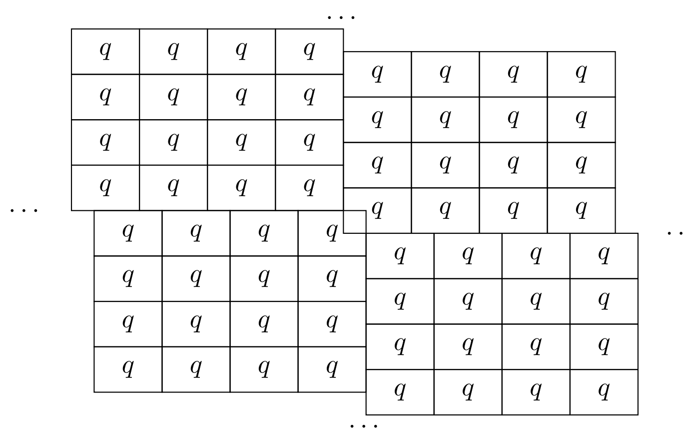

Figure 1: A coverable, non-uniformly recurrent word.

This is not a contradiction with Proposition 1, as we impose a condition on the cover q. However, there are no equivalent results in one dimension: no non-trivial condition on q can force topological entropy to be 0 on right-infinite words. (The following theorem could be reformulated in one dimension, but the hypothesis on q would be the trivial condition that q is border-free.) Therefore this is a striking contrast with the one-dimensional case: even though coverability and topological entropy are independent, the latter is forced to be zero for a large class of covers. Hence a global order might arise from coverability if the cover is sufficiently well-chosen.

Theorem 2. *Let q be a finite picture of size $w \times h$ and $\mathbf{w}$ an infinite q-coverable picture. If q has a corner without any non-empty border, then $\mathbf{w}$ has zero topological entropy.*

The end of this subsection is devoted to the proof of this theorem.

Suppose, without loss of generality, that all the borders are in the top left-hand corner of q. Moreover, q has no full-width nor full-height borders.

In what follows, "*occurrence*" denotes an occurrence of q in $\mathbf{w}$—unless otherwise stated. Moreover, $\mathrm{Occ}(x, y)$ denotes the occurrence of q which covers the letter at coordinates (x, y). If there are several such occurrences, we choose the leftmost one among the lowest ones. The *coordinates* of an occurrence (or a block) are the coordinates of its bottom, left-hand corner.

Let o be an occurrence whose domain is $\{(x, y), \cdots, (x+w-1, y+h-1)\}$. We denote $\mathrm{Right}(o) = \mathrm{Occ}(x+w, y+h-1)$. If $\mathrm{Right}(o)$ has coordinates (x', y'), then $d_R(o) = y' - y$. In particular, the coordinates of $\mathrm{Right}(o)$ are $(x+w, y+d_R(o))$ (since there are no borders in this corner, see Figure 2). Similarly, we denote $\mathrm{Above}(o) = \mathrm{Occ}(x + w - 1, y + h)$. If $\mathrm{Above}(o)$ has coordinates (x'', y''), then $d_A(o) = x'' - x$. The coordinates of $\mathrm{Above}(o)$ are $(x + d_A(o), y + h)$ (for the same reasons).

Observe that if both $d_A(o) > 0$ and $d_R(o) > 0$, then $(d_A(o), d_R(o))$ is the size of a border of q. So, there are only three possible cases, illustrated by Figure 2:

1. $d_R(o) = 0$ and $0 \leq d_A(o) < w$;
2. $d_A(o) = 0$ and $0 \leq d_R(o) < h$; or
3. $(d_A(o), d_R(o))$ is the size of a border.

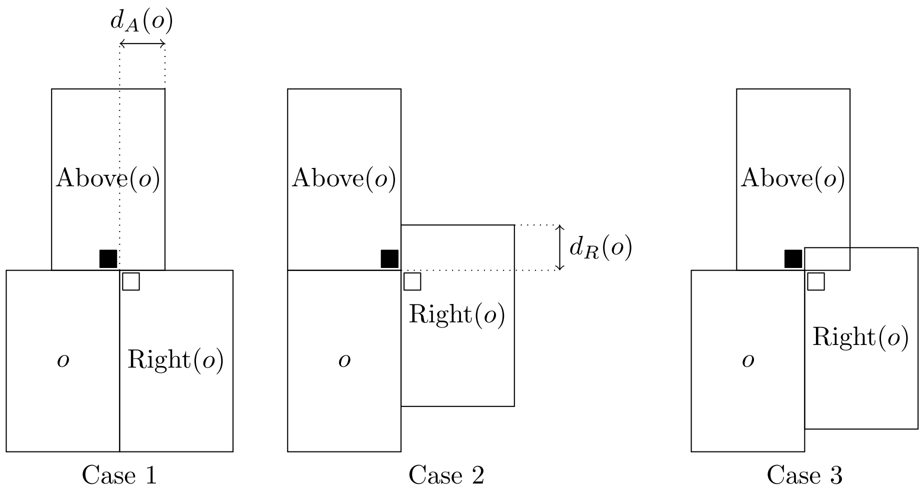

Figure 2: Illustration of Above, Right, d_A and d_R.
The squares □ and ■ show which letter Right(o) and Above(0) should cover, respectively.

Proposition 3. *Let* $\mathbf{w}$ *and* q *be as in Theorem 2.*

Let o *be an occurrence with coordinates* (x, y)*. Then we have:*

$$\text{Right}(\text{Above}(o)) = \text{Above}(\text{Right}(o)) = \text{Occ}(x + w + d_A(o), y + h + d_R(o))$$

In particular, this occurrence is uniquely determined by $o, d_A(o)$ *and* $d_R(o)$*.*

Proof. As in theorem 2, we suppose without loss of generality that the borders are on the top left-hand corner of q.

By definitions, we have:

$$\text{Above}(\text{Right}(o)) = \text{Above}(\text{Occ}(x+w, y+d_R(o)) = \text{Occ}(x+2w-1, y+h+d_R(o))$$

and:

$$\text{Right}(\text{Above}(o)) = \text{Right}(\text{Occ}(x+d_A(o), y+h)) = \text{Occ}(x+w+d_A(o), y+2h-1)$$

Consider $o' = \text{Occ}(x + w + d_A(o), y + h + d_R(o))$: its coordinates must be $(x + w + d_A(o), y + h + d_R(o))$, otherwise, it would overlap in an incorrect way (that is, on the upper right-hand corner) with either Above(o) or Right(o) (see Figure 3). So the top right-hand corner of o' must be at coordinates $(x + 2w +$

$d_A(o)-1, y+2h+d_R(o)-1)$. We can see that both $(x+2w-1, y+h+d_R(o))$ and $(x+w+d_A(o), y+2h-1)$ are covered by o' with some coordinates-checking:

$$x+w+d_A(o) \leq x+2w-1 \leq x+2w+d_A(o)-1$$
$$y+h+d_R(o) \leq y+2h-1 \leq y+2h+d_R(o)-1$$

Moreover, o' is the lowest and the leftmost occurrence covering those letters (any other one would overlap either Right(o) or Above(o) in an incorrect way, see Figure 3 again). Hence, by definition of Occ, o' must be the correct one. $\square$

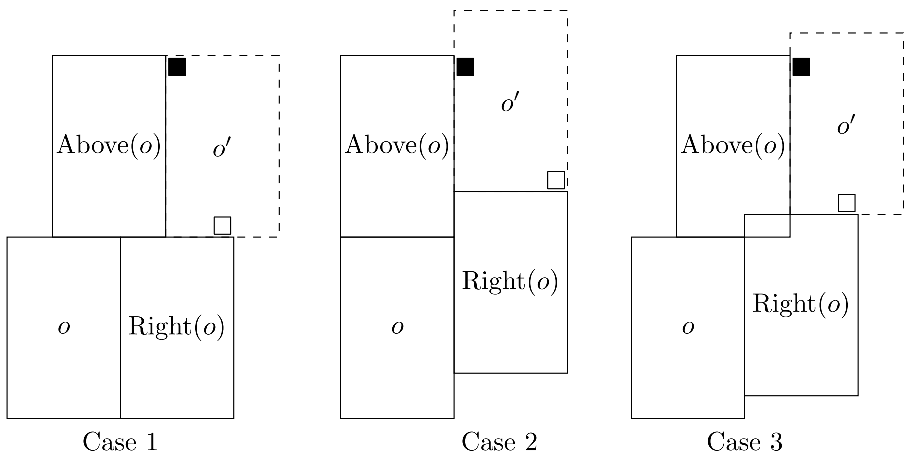

Figure 3: Illustration for proof of Proposition 3. The $\blacksquare$ indicates $(x+w+d_A(o), y+2h-1)$ and $\square$ indicates $(x+2w-1, y+h+d_R(o))$.

Proof of Theorem 2. Consider an arbitrary $n \times n$-block B of $\mathbf{w}$ whose leftmost bottom coordinate if (x, y). Since $\mathbf{w}$ is q-coverable, all letters of B are covered by occurrences of q. Call $u_1, \cdots, u_k$ the occurrences of q covering $B[(x,y), \cdots, (x+n-1, y)]$ (the bottom frontier of B), from left to right. Likewise, call $v_1, \cdots, v_\ell$ the occurrences of q covering $B[(x,y), \cdots, (x, y+n-1)]$ (the left-hand frontier of B), from bottom to top. Observe that $u_1 = v_1$.

By Proposition 3, B is uniquely determined by $u_1, \ldots, u_k$, $v_1, \ldots, v_k$ and the coordinates of B relative to u_1. Let us bound the number of possible such B's.

For all $1 \leq i < k$, we have either $u_{i+1} = \text{Right}(u_i)$ or $\text{Above}(u_{i+1}) = \text{Right}(u_i)$. Therefore, the number of possible sequences for $u_1, \ldots, u_k$, given u_1, is bounded by $2^k \leq 2^n$. Likewise, for $1 \leq j < \ell$, we have $v_{j+1} = \text{Above}(v_i)$ or $\text{Right}(v_{i+1}) = \text{Above}(v_i)$. Therefore, the number of possible sequences for $v_1, \ldots, v_k$, given v_1, is bounded by $2^\ell \leq 2^n$. Finally, the number of possible B is bounded by $|q| \times 2^{2n}$.

Observe that:

$$\lim_{n\to\infty} \frac{\log(|q| \times 2^{2n})}{n^2} = \lim_{n\to\infty} \frac{2n \times \log|q|}{n^2} \to 0$$

therefore, $\mathbf{w}$ has zero topological entropy. □

As we saw in the above proof, coverability when there is no border reduces to questions of tiling the plane with rectangles. The different possible overlaps could yield different polyomino-shaped tiles. Tiling of the plane by rectangles and polyominoes was investigated in [2].

By contrast with Theorem 2, any block q with only full-width (or only full-height) borders has either q-coverable pictures with positive entropy, or only periodic q-coverable pictures. We conjecture that, in the only remaining case, the entropy is zero.

Conjecture 4. *If q is a block without full-height or full-width borders, but with borders in all corners, the topological entropy of the q-coverable words is zero.*

Observe that, in the previous remark, we had to exclude the "only periodic coverable pictures" case. This is a common problem with coverability, which we will address in the next section.

3. Aperiodic Coverings

Here is a natural question about coverability: let q be a block with some property; do all q-coverable infinite pictures get some other property? In other terms, can a property on a cover enforce another property to all coverable pictures? For instance, we could imagine that some property on q would force each q-coverable picture to have uniform frequencies.

Theorem 2 is already a partial answer to this question; in this section, we give a more exhaustive answer. We show that there exist aperiodic q-coverable pictures if and and only if the smallest block which tesselates q does not overlap itself. This is an answer to the previous question in the case of periodicity. This is specifically important, as we generally want to exclude the periodic case when working with coverability.

Finally, we observe that this condition is also necessary and sufficient to force the existence of coverable non-uniformly recurrent pictures, and even coverable non-uniform frequencies pictures. Therefor, except for a trivial class of covers (trivial in the sense that they only allow periodic coverable pictures), coverability does not force any interesting property but (possibly) zero topological entropy.

3.1. A Condition for Simpler Cases

First, let us consider the question on one-dimensional words (either right-infinite or bi-infinite, it does not matter). Let q be a finite word. Is there a condition on q forcing all q-coverable one-dimensional words to be periodic?

Recall that, in the context of one-dimensional words, a *border* is a proper factor of q which is both a prefix and a suffix of q. (A word u is a proper factor of v if it is a factor of v and $u \neq v$). Moreover, the *primitive root* r of q is the shortest word such that, for some $k \in \mathbb{N}$, we have $q = r^k$. We might have $q = r$ and $k = 1$ for some words.

Now let us get some intuition about what is next. Let q be a finite (one-dimensional) word and u be the shortest nonempty border of q, so that we have $q = uvu$ for some $v \in \Sigma^*$. Then, the word $uvuv(uvu)^\omega$ is q-coverable (it is covered with occurrences of uvu) does not seem to be periodic ($uvuvuv$ only appears visually once). However, this is not quite true: if there exists some $r \in \Sigma^*, k, \ell \in \mathbb{N}$ such that $u = r^k$ and $v = r^\ell$, then $uvuv(uvu)^\omega = r^\omega$, which is periodic. This is why the condition is as follows.

Proposition 5. *Let q be a finite one-dimensional word. There exists an aperiodic q-coverable $\mathbb{N}$-word if and only if the primitive root of q has a non-empty border.*

Proof. First, suppose that q is a cover of an aperiodic infinite word $\mathbf{w}$. Call r the primitive root of q; observe that r is also a cover of $\mathbf{w}$. Suppose by contradiction that r does not have any non-empty borders; then, two occurrences of r never overlap. Hence $\mathbf{w}$, which is r-covered, only consists in concatenations of r. Therefore, $\mathbf{w}$ is r-periodic: a contradiction.

Conversely, write $q = r^k$ with r primitive and $k \geq 1$. Suppose that r has a non-empty border and let u be the smallest one, i.e. $r = uvu$ for some non-empty word v. Let h be the morphism defined by $h(a) = (uvu)^k$ and $h(b) = u(vu)^k$. Both $h(a)$ and $uvu \cdot h(b)$ are q-coverable, so the image of any word beginning with a by h is q-coverable. Moreover, since $r = uvu$ is a primitive word, $u(vu)^k \neq (uvu)^\ell$ for all $k, \ell \in \mathbb{N}$. Therefore, h is injective, so the image of any aperiodic word by h is also aperiodic. □

Now, let us shift to $\mathbb{Z}^2$-words. This shift is mainly motivated by the study of tilings, which is why we choose $\mathbb{Z}^2$-words instead of $\mathbb{N}^2$-words. We need some definitions before getting to prove one of our main theorem.

3.2. Preliminaries for the $\mathbb{Z}^2$-Case

Let q and r be blocks. In this context, r is a *root* of q if $q = r^{n \times m}$, for some positive integers n and m. If q has no roots except itself, it is said to be *primitive*. These notions initially came from combinatorics on one-dimensional words. The following lemma is a classical result about roots in one dimension: it shows that any one-dimensional finite word has a smallest root, called its *primitive root*.

Lemma 6. (See, e.g., [12], Prop. 1.3.1 and 1.3.2.)
Given any finite one-dimensional words u and v, the following statements are equivalent:

1. *there exist integers $n, m \geq 0$ with $(n, m) \neq (0, 0)$, such that $u^n = v^m$;*
2. *there exist a word t and positive integers k and ℓ such that $u = t^k$ and $v = t^\ell$;*
3. *$uv = vu$.*

Let us show that primitive roots are also well-defined on finite pictures.

Lemma 7. *Let q be a finite picture. Suppose that q has two distinct roots r_1 and r_2. Then there exists a finite picture r such that r is a root of both r_1 and r_2.*

Proof. Let r_1^k (respectively r_2^k) denote k occurrences of r_1 (respectively r_2) concatenated vertically. Since r_1 and r_2 are roots of q, there exist integers n and m such that both r_1^n and r_2^m are roots of q, with $\text{height}(q) = \text{height}(r_1^n) = \text{height}(r_2^m)$. Consider q, r_1^n and r_2^m as words over $\mathcal{C}_{\Sigma,\text{height}(q)}$; by Lemma 6, there exists a word c over $\mathcal{C}_{\Sigma,\text{height}(q)}$ such that c is a root of both r_1^n and r_2^m.

Let r_3 (respectively r_4) be the horizontal prefix of r_1 (respectively r_2) of length $\text{width}(c)$. Both r_3 and r_4 are prefixes of q, hence $r_3^n = r_4^m$ (the power is still taken for vertical concatenation). Now view r_3 and r_4 as words over $\mathcal{L}_{\Sigma,\text{width}(c)}$. By Lemma 6, there exists a word r over $\mathcal{L}_{\Sigma,\text{width}(c)}$ which is a common root of r_3 and r_4.

As r_1 (respectively r_2) is obtained by horizontal concatenations of occurrences of r_3 (respectively r_4), we deduce that r is a root of r_1 and of r_2. $\square$

The *primitive root* of a block q is the minimal root. By Lemma 7, it is the only root of q which is primitive. Note that q might be its own primitive root.

3.3. Blocks Covering Aperiodic Infinite Pictures

Now we can state the condition under which a block can be the covering pattern of a non-periodic $\mathbb{Z}^2$-word.

Theorem 8. *Let q be a finite block. Then there exists a q-coverable, non-periodic $\mathbb{Z}^2$-word if and only if the primitive root of q has a non-empty diagonal border.*

This subsection is entirely dedicated to the proof of Theorem 8.

Proof of the "only if" part. First, suppose that $\mathbf{w}$ is a $\mathbb{Z}^2$-word which is both q-coverable and non-periodic. There exists at least two overlapping occurrences of q in $\mathbf{w}$ (otherwise, $\mathbf{w}$ would be q-periodic). Moreover, the overlapping part is not a power of the primitive root of q: if all overlappings were powers of some root r of q, then $\mathbf{w}$ would be r-periodic. Therefore, q must have at least one border which is not a power of its primitive root. Hence its primitive root has a non-empty border.

Proof of the "if" part. Suppose that q's primitive root has a non-empty diagonal border. Let us build an infinite $\mathbb{Z}^2$-word which is q-coverable, but not periodic.

Let r be the primitive root of q and b be a non-empty diagonal border of r. (Thus b is potentially a picture and not a single letter.) Consider the four tiles α, β, δ and γ displayed on Figure 4. Each rectangle is an occurrence of q. The overlapping zones are all occurrences of the border b and the shifts on tile borders are sized accordingly. If the border b is on the opposite corner, all tiles are built symmetrically.

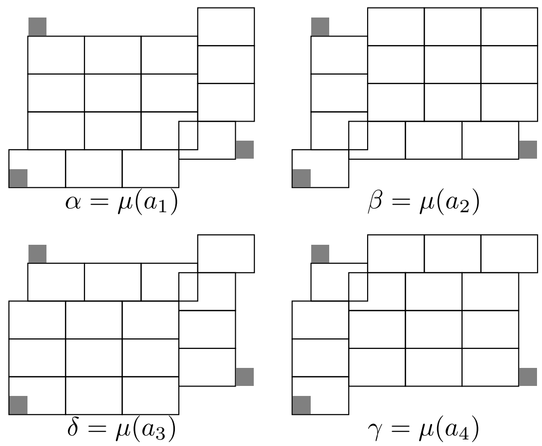

Figure 4: Four tiles to build a q-coverable word. Each rectangle is an occurrence of q.

Let $A = \{a_1, a_2, a_3, a_4\}$ and μ be the function from $A^{\mathbb{Z}^2}$ to $\Sigma^{\mathbb{Z}^2}$, defined by $\mu(a_1) = \alpha$, $\mu(a_2) = \beta$, $\mu(a_3) = \delta$ and $\mu(a_4) = \gamma$. If its input is regular enough, μ behaves more or less like a morphism, with the following concatenation rules.

On Figure 4, each tile has three *anchors*, i.e. letters marked by a small square. Concatenate two tiles horizontally by merging the right-anchor of the first one with the left-anchor of the second one. Concatenate two tiles vertically by merging the bottom-anchor of the first one with the top-anchor of the second one.

More formally, we have:

$$\mu(a_i \cdot u) = \mu(a_i) \cup S_{(4\,\mathrm{width}(q);\mathrm{height}(b))} \circ \mu(u)$$
$$\mu \begin{pmatrix} u \\ v \end{pmatrix} = \mu(v) \cup S_{(\mathrm{width}(b);4\,\mathrm{height}(q))} \circ \mu(u)$$

where $S_{(x,y)}$ denotes the translation (shift) by the vector (x, y) and the operator $\cup$ denotes the superposition of two finite words. Recall that we view two-dimensional words as (possibly partial) functions from $\mathbb{Z}^2$ to the alphabet. These functions have domains which may be strictly included in $\mathbb{Z}^2$. If w_1 and w_2 are two words with disjoints domains, then $(w_1 \cup w_2)[x, y] = w_1[x, y]$ where w_1 is defined and $w_2[x, y]$ where w_2 is defined. In what follows, we will only consider superpositions where no position (x, y) is defined in both $w_1[x, y]$ and $w_2[x, y]$.

If u is a block, the leftmost bottom anchor of $\mu(u[i, j])$ has coordinates:

$$(i \times 4 \times \mathrm{width}(q) + j \times \mathrm{width}(b);\ j \times 4 \times \mathrm{height}(q) + i \times \mathrm{height}(b))$$

in $\mu(u)$. Figure 5 gives an example of how μ works.

An infinite picture over A is *suitable* when it satisfies the following conditions:

1. each line is either on alphabet $\{a_1, a_2\}$ or on alphabet $\{a_3, a_4\}$;
2. each column is either on alphabet $\{a_1, a_3\}$ or on alphabet $\{a_2, a_4\}$.

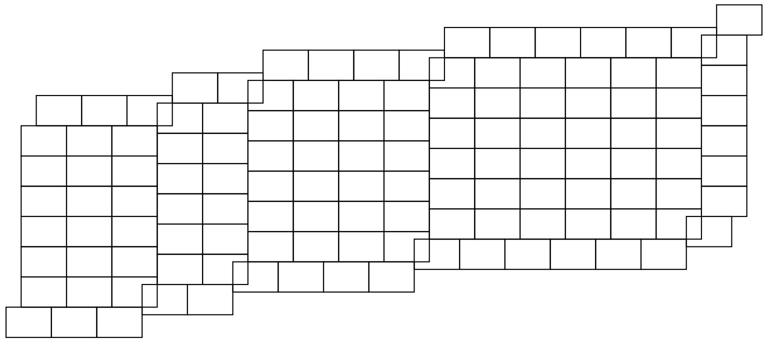

Figure 5: $\mu\left(\begin{smallmatrix} a_3 & a_4 & a_4 & a_3 \\ a_1 & a_2 & a_2 & a_1 \end{smallmatrix}\right)$, each rectangle is an occurrence of q

Let us check that if $\mathbf{w}$ is suitable, then each letter of $\mu(\mathbf{w})$ belongs to the image of exactly one letter of $\mathbf{w}$. This essentially means that all tiles "fit together" with no overlaps.

By construction, tiles α and δ fit together vertically, and tiles β and γ fit as well. Hence $\mu(\begin{smallmatrix} a_1 \\ a_3 \end{smallmatrix})$ and $\mu(\begin{smallmatrix} a_2 \\ a_4 \end{smallmatrix})$ are well-defined. Likewise, tiles α and β fit together horizontally, and tiles δ and γ fit as well. Hence $\mu(a_1a_2)$ and $\mu(a_3a_4)$ and are well-defined. Iterating this argument, we deduce that the image of any suitable word is well-defined.

Moreover, we let readers check that $\mu(\mathbf{w})$ has no "holes". More precisely, if if $\mathbf{w}$ is a suitable block, $\mu(w)$ satisfies the following weak convexity properties:

- for all $i, j, j_1, j_2 \in \mathbb{N}$ with $j_1 \leq j \leq j_2$, if (i, j_1) and (i, j_2) are in $\mathrm{dom}(\mu(\mathbf{w}))$, then (i, j) is in $\mathrm{dom}(\mu(\mathbf{w}))$ as well;
- for all $i, j, i_1, i_2 \in \mathbb{N}$ with $i_1 \leq i \leq i_2$, if (i_1, j) and (i_2, j) are in $\mathrm{dom}(\mu(\mathbf{w}))$, then (i, j) is in $\mathrm{dom}(\mu(\mathbf{w}))$ as well.

As a consequence, the definition of μ can be extended to suitable $\mathbb{Z}^2$-words. If $\mathbf{w}$ is a suitable $\mathbb{Z}^2$-word, then $\mu(\mathbf{w})$ is a well-defined $\mathbb{Z}^2$-word as well.

Now let us prove that if $\mathbf{w}$, an infinite picture, is aperiodic, then so is $\mu(\mathbf{w})$. First, we need a technical lemma about our tiles.

Lemma 9. *Let x and y be different tiles from $\{\alpha, \beta, \gamma, \delta\}$. Then an occurrence of x and an occurrence of y cannot overlap when their anchor points coincide.*

This essentially means that situations from Figure 6 cannot occur.

Proof. There are six possibilities for the set $\{x, y\}$. All proofs are the same, up to some symmetry, so we only provide a proof when $x = \alpha$ and $y = \beta$ (illustrated by the top left-hand case of Figure 6). In what follows, q refers to the block used for the construction of the tiles, r to its primitive root and b to a diagonal border of r.

There are three occurrences of q, named q_1, q_2 and q_3, such that q_1 is covered by q_2 and q_3 and all three are horizontally aligned. In other words, q_1 is an

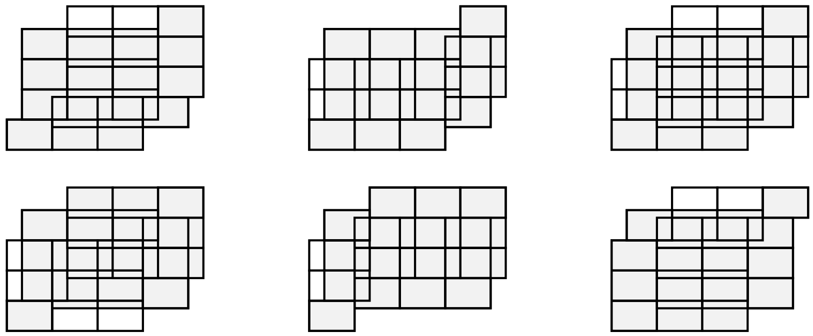

Figure 6: All other possible overlappings.

occurrence of q in $q_2 \cdot q_3$ viewed as words over $\mathcal{L}_{\Sigma,\mathrm{width}(q)}$. (See for instance the top second column of q's in the figure). View q_1, q_2 and q_3 as one-dimensional words over the alphabet $\mathcal{L}_{\Sigma,\mathrm{width}(q)}$. There exist words x and x' over $\mathcal{L}_{\Sigma,\mathrm{width}(q)}$ such that $q_1 = xx'$ and $q_2 = q_3 = x'x$ (where words are concatenated from bottom to top).

By Lemma 6, x and x' (and q) are powers of the same word s over $\mathcal{L}_{\Sigma,\mathrm{width}(q)}$. Notice that $\mathrm{height}(x') = \mathrm{height}(b)$ and $\mathrm{height}(x) = \mathrm{height}(q) - \mathrm{height}(b)$. It follows that $\mathrm{height}(s)$ divides $\mathrm{height}(x)$ and $\mathrm{height}(q) - \mathrm{height}(b)$.

Observe that s is a vertical prefix of both q and x. Thence one can find three occurrences of s, named s_1, s_2 and s_3, such that s_1 is covered by s_2 and s_3 and all three are vertically aligned. In other terms, s_q is an occurrence of q in $s_2 \cdot s_3$ viewed as a word over $\mathcal{C}_{\Sigma,\mathrm{height}(q)}$. (See for instance the second line of q's in the figure).

Now view s as a one-dimensional word on the alphabet $\mathcal{C}_{\Sigma,\mathrm{height}(s)}$. There exist words y, y' such that $s_1 = yy'$, $s_2 = s_3 = y'y$ and $\mathrm{width}(y') = \mathrm{width}(b)$. By Lemma 6, we deduce that there exists a word t over $\mathcal{C}_{\Sigma,\mathrm{height}(s)}$ such that y and y' (and s) are powers of t.

Let $k \leq 1$ be the integer such that $q = s^k$ (for vertical concatenation) and let $\ell \geq 1$ be the integer such that $s = t^\ell$ (for horizontal concatenation). We have that $q = t^{\ell \times k}$. Therefore t is a root of q such that $\mathrm{width}(t) \leq width(y') = width(b)$ and $\mathrm{height}(t) = \mathrm{height}(s) \leq \mathrm{height}(b)$. Thus $\mathrm{width}(s) \times \mathrm{height}(s) \leq \mathrm{width}(b) \times \mathrm{height}(b)$ which is a contradiction with the definition of b. Indeed, recall that b is a border (hence a proper block) of the primitive root of q, which is the smallest (in number of letters) root of q. $\square$

In the next proof, Lemma 9 helps to establish a correspondence between the letters of the $\mathbb{Z}^2$-word $\mu(\mathbf{w})$ and the "tiling" consisting of occurrences of α, β, δ and γ. We need this correspondence to prove that some $\mu(\mathbf{w})$ can always be made aperiodic.

Lemma 10. *Let q be a block, r its primitive root and b one non-empty diagonal border of r. Let $\mathbf{w}$ be an aperiodic, suitable $\mathbb{Z}^2$-word. Then $\mu(\mathbf{w})$ is an aperiodic, q-coverable $\mathbb{Z}^2$-word.*

Proof. By construction, $\mu(\mathbf{w})$ is q-coverable for all $\mathbf{w}$. Suppose that $\mu(\mathbf{w})$ has a non-zero vector of periodicity $\overrightarrow{p} \in \mathbb{Z}^2$. Let us prove that, under this assumption, $\mathbf{w}$ is periodic.

Let $a \in \mathbb{Z}^2$ be the coordinates of the anchor point of some tile in $\mu(\mathbf{w})$. For any $i \in \mathbb{Z}$, let $t_i = a + i \times \overrightarrow{p}$. Since tiles have at most $16 \times \mathrm{width}(q) \times \mathrm{height}(q)$ letters, by the pigeonhole principle, there are two pairs of coordinates t_i and t_j which have the same offset to the anchor points of their respective tiles (i.e. the tiles covering their respective positions). Hence the difference between these anchor points is a multiple of the vector of periodicity $\overrightarrow{p}$.

Let T_i (resp. T_j) be the tile covering position t_i (resp. t_j). Since T_i is the $(j-i) \times \overrightarrow{p}$-translation of T_j, they are both occurrences of a same tile. Moreover, the right-neighbours of T_i and T_j are both occurrences of a same tile, otherwise we would have a configuration forbidden by Lemma 9. Likewise, the top-neighbour, bottom-neighbour and left-neighbour of T_i and T_j are also equal. By iterating this argument over the neighbours' neighbours, and so on, we conclude that the tiling itself is periodic. Hence, $\mathbf{w}$ is periodic. □

This ends the proof of Theorem 8. From any block q with at least one non-empty diagonal border in its primitive root, we can build $\mu(\mathbf{w})$ for any aperiodic, suitable $\mathbb{Z}^2$-word $\mathbf{w}$. The picture $\mu(\mathbf{w})$ will be q-coverable and aperiodic.

3.4. Uniform Recurrence, Uniform Frequencies and Coverability

In this subsection, we extend Theorem 8 (which characterizes periodicity in terms of covers) to uniform recurrence and uniform frequencies. To do so, we exploit Theorem 8 itself as well as the function μ from its proof. Note that these results are direct generalizations of the one-dimensional case discussed in [14].

Let q be a block whose primitive root has a non-empty diagonal border. Let μ be as in the proof of Theorem 8. Recall that μ takes as an argument a picture over the alphabet $\{a_1, a_2, a_3, a_4\}$. Consider the word $\mathbf{w}$ such that:

- $\mathbf{w}[0,0] = a_4$;
- $\mathbf{w}[0,j] = a_2$ for $j \in \mathbb{Z}^*$;
- $\mathbf{w}[i,0] = a_3$ for $i \in \mathbb{Z}^*$;
- $\mathbf{w}[i,j] = a_1$ for $i \in \mathbb{Z}^*$ and $j \in \mathbb{Z}^*$.

Observe that $\mathbf{w}$ is suitable, hence $\mu(\mathbf{w})$ exists. Moreover, $\mu(\mathbf{w})$ is not uniformly recurrent: any block which contains $\mu(a_4)$ only occurs once. Therefore $\mu(\mathbf{w})$ is a q-coverable picture not uniformly recurrent.

Now consider t a $\mathbb{Z}-$word (one-dimensional) over $\{1, 2\}$, such that no factor of t has uniform frequencies in t. Then define $\mathbf{w}'$ an infinite picture over $\{a_1, a_2, a_3, a_4\}$ by $\mathbf{w}'[i,j] = a_{t[i]}$, for all $i, j \in \mathbb{Z}$. Observe that, in $\mu(\mathbf{w}')$, the frequency of any block containing exactly one occurrence of $\mu(a_2)$ is the frequency of a_2 in t. Therefore, those factors do not have uniform frequencies.

Proposition 11. *Let q be a block. There exists a q-coverable infinite picture which is not uniformly recurrent (respectively has no uniform frequencies) if and only if the primitive root of q has a non-empty diagonal border.*

Proof. If q does not have a primitive root with a diagonal border, then by Theorem 8 all q-coverable pictures are periodic and therefore uniformly recurrent (respectively have uniform frequencies). Otherwise, the constructions above gives $\mu(\mathbf{w})$ (respectively $\mu(\mathbf{w}'))$ a non-uniformly recurrent, and not even recurrent (respectively without uniform frequencies) q-coverable word. □

This remark generalizes a one-dimensional result from [14]. It is a negative answer to our initial question. As soon as the cover q we choose is non-trivial (has at least one non-periodic coverable word), we have q-coverable word without frequencies and without uniform recurrence. By contrast, we conjectured in Section 2.3 that, for almost any q, all q-coverable words have zero topological entropy. To sum up, coverability is independent from uniform recurrence and uniform frequencies for all "non-trivial" covers, and it likely implies zero topological entropy for "almost all" covers. Not only coverability implies very little symmetry properties on words, but the covers themselves bear very little information.

As a consequence, we move on a stronger notion of coverability, based on the same idea, but with (hopefully) better properties.

4. Multi-Scale Coverability in Two Dimensions

In [14], Monteil and Marcus called *multi-scale quasiperiodicity* any $\mathbb{N}$-word having infinitely many quasiperiods. In our context, we want to exclude cases where coverability is obtained on groups of one-dimensional lines (or columns) packed all over $\mathbb{Z}^2$. Hence we call a $\mathbb{Z}^2$-word *multi-scale coverable* if, for each $n \in \mathbb{N}$, it has a cover of size $k \times \ell$ with both $k \geq n$ and $\ell \geq n$. This is actually a generalization of one-dimensional multi-scale quasiperiodicity (in 1D, all quasiperiods are prefixes of the multi-scale quasiperiodic word, so the quasiperiods must be longer and longer).

An example of multi-scale coverable picture is given on Figure 1. No matter the value of q, this infinite picture admits $q^{n\times n}$ as a cover for any positive integer n. The widths and heights of $q^{n\times n}$ grow to infinity as required.

In [14], Monteil and Marcus prove that multi-scale coverable right-infinite words have zero topological entropy, uniform frequencies, and are uniformly recurrent. In this section, we generalize implications of topological entropy and uniform frequencies to the two-dimensional case. Then we see that uniform recurrence is a bit more subtle.

4.1. Topological Entropy

Let $\mathbf{w}$ be a $\mathbb{Z}^2$-word. Recall that $c_{\mathbf{w}}(n,m)$ is the number of different of blocks of size $n \times m$ which occur in $\mathbf{w}$ and that the topological entropy of $\mathbf{w}$ is the following quantity:

$$H(\mathbf{w}) = \lim_{n\to+\infty} \frac{\log_{|\Sigma|} c_w(n,n)}{n^2} \tag{1}$$

Proposition 12. *Any multi-scale coverable, $\mathbb{Z}^2$-word $\mathbf{w}$ has zero topological entropy.*

Proof. Consider a covering pattern q of $\mathbf{w}$ with size $n \times m$. Suppose without loss of generality that $n \leq m$. Let s be a $m \times m$-square of $\mathbf{w}$. The square s is covered with occurrences of q (which may spill out of s). The relative position of s and of the occurrences of q completely defines s.

We need at most $4m$ occurrences of q to define a covering of s. Indeed, each occurrence of q must have at least one of its corners in s. If some occurrence of q has its bottom right-hand corner in s, then no other occurrence of q may have their bottom right-hand corners on the same line of s. Otherwise, one of these occurrences would supersede the other one, which would be "useless" in the covering. Proceed the same way for the other corners and deduce that at most $4m$ occurrences of q (4 per line) uniquely define s.

Each of these occurrences is uniquely determined by its position of its corner on a line of s. There are at most m possibilities for each. Therefore, there are at most m^{4m} q-coverings which define at most m^{4m} squares of size $m \times m$.

This bound on $c_{\mathbf{w}}(m,m)$ allows us to compute the entropy of $\mathbf{w}$. Observe that:

$$\lim_{m\to\infty} \frac{\log m^{4m}}{m^2} = \lim_{m\to\infty} \frac{4m \log m}{m^2} \to 0 \tag{2}$$

Since there are infinitely many covering patterns of $\mathbf{w}$ with growing sizes, there are infinitely many integers m such that $c_{\mathbf{w}}(m,m) \leq m^{4m}$. Hence equation (2) shows that then topological entropy of $\mathbf{w}$ converges to zero. $\square$

Note that since the Kolmogorov complexity is bounded by the topological entropy (see [3]), this result also shows that the Kolmogorov complexity of multi-scale coverable words is zero as well.

4.2. Uniform Frequencies

In this subsection, we prove the following theorem:

Theorem 13. *Multi-scale coverable pictures have uniform frequencies.*

This answers an open question from our preliminary work on the subject [8], and we generalize a result from infinite words to infinite pictures. We believe that this proof is easily adaptable to higher dimensions (3-dimensional words, etc.). Moreover, the proof on words in [14] was expressed in terms of subshifts and used ergodic theory. By contrast, our proof uses purely combinatorial means, and hence is accessible to readers unfamiliar with this theory. For a survey about ergodic theory in the context of one-dimensional words, see [7].

First, let us recall some notation. If u and v are finite blocks, then $|u| = \text{width}(u) \times \text{height}(u)$ and $|v|_u$ is the number of occurrences of u in v. As for infinite words, we note $f_u(v)$ the *frequency* of u in v, which is the following quantity:

$$f_u(v) = \frac{|v|_u}{|v|}$$

Let $\mathbf{w}$ be an infinite picture. Define $B_n(\mathbf{w})$ as $\mathbf{w}[(-n,-n),\cdots,(+n,+n)]$ and $f_u(\mathbf{w})$ as:

$$f_u(\mathbf{w}) = \lim_{n\to+\infty} f_u(B_n)$$

if this quantity exists.

We say that $\mathbf{w}$ *has frequencies* (or has *uniform frequencies*, or is *uniquely ergodic*) if $f_u(\mathbf{w})$ exists for each block u of $\mathbf{w}$. Our purpose is to show that any multi-scale coverable picture has frequencies. This purpose is not void, because some infinite pictures do not have uniform frequencies. For instance, any picture containing arbitrarily large squares of 0's and arbitrarily large squares of 1's has no frequencies for 0 nor for 1.

Let us give a general idea of the proof, which is structured in several lemmas. Lemmas 14, 15 and 18 are technicalities for later calculations. Lemma 16 states that, if $\mathbf{w}$ is a multi-scale coverable picture without frequencies, then there exist either:

1. infinitely many covers with "high" frequencies and infinitely many blocks with "low" frequencies; or
2. infinitely many covers with "low" frequencies and infinitely many blocks with "high" frequencies.

(Here "low" and "high" are used to describe the ideas of the proof. Their formal meaning will be given later.) Lemma 17 states that, if a picture has infinitely many covers with high frequencies and infinitely many blocks with low frequencies, then it has infinitely many blocks with *even lower* frequencies. Similarly, Lemma 19 states that if a picture has infinitely many covers with low frequencies and infinitely many blocks with high frequencies, then it has infinitely many blocks with *even higher* frequencies. Finally, the proof of Theorem 13 is as follows: by Lemma 16, any multi-scale picture without frequencies is either in case 1 or in case 2. In case 1, apply Lemma 17 many times, until getting blocks with frequencies higher than 1: a contradiction. In case 2, apply Lemma 19 many times, until getting blocks with frequencies lower than 0: a contradiction again.

Let us start.

If $\mathbf{w}$ is an infinite picture, let $L_{\geq K}(\mathbf{w})$ the set of blocks of $\mathbf{w}$ whose width is larger than K and whose height is also larger than K.

For the purposes of the proof, we shall need to extend definitions of $|v|$ and $|v|_u$ to cases where v is a finite union of blocks (instead of a single block). In that case, $|v|$ is the number of letters in v and $|v|_u$ the number of complete occurrences of u in v.

Lemma 14. *Let v be a finite block which can be decomposed into v_1 and v_2, two unions of blocks such that* $\operatorname{dom} v_1 \cap \operatorname{dom} v_2 = \emptyset$*. Let u be a block. Then,*

$$f_u(v_1) \times \frac{|v_1|}{|v|} \leq f_u(v_1) \times \frac{|v_1|}{|v|} + f_u(v_2) \times \frac{|v_2|}{|v|} \leq f_u(v) \leq f_u(v_1) \times \frac{|v_1|}{|v|} + \frac{|v_2|}{|v|}$$

Proof. This is equivalent to the following inequation:

$$|v_1|_u \leq |v_1|_u + |v_2|_u \leq |v|_u \leq |v_1|_u + |v_2|$$

The first two inequality follows from $\operatorname{dom} v_1 \subseteq \operatorname{dom} v$ and $\operatorname{dom} v_2 \subseteq \operatorname{dom} v$. For the last inequality, consider that each occurrence of u in v has its top right-hand corner which is either in v_1 or in v_2. The number of occurrences with their corner in v_1 is precisely $|v_1|_u$. In the "worst" case, there is an occurrence of u per letter of v_2. □

We shall also need to express the relation between width, height and areas of bigger and bigger blocks.

Lemma 15. *Let x, y, z, t be real numbers such that $x > 0, y > 0$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of finite pictures such that, for all i,* $\operatorname{width}(u_i) > i$ *and* $\operatorname{height}(u_i) > i$*. Then, there exists an integer N such that, for all $n > N$:*

$$t < \frac{\operatorname{width}(u_n) \times \operatorname{height}(u_n)}{x \operatorname{width}(u_n) + y \operatorname{height}(u_n) + z}$$

Proof. Observe that, since $x > 0$ and $y > 0$, the following functions are non-decreasing:

$$w \mapsto \frac{wh}{xw + yh + z} \qquad\qquad h \mapsto \frac{wh}{xw + yh + z}$$

Hence, as for all i, $\operatorname{width}(u_i) > i$ and $\operatorname{height}(u_i) > i$, we have for arbitrarily large n:

$$\frac{n^2}{(x+y) \times n + z} < \frac{n \times \operatorname{height}(u_n)}{n + y \times \operatorname{height}(u_n) + z} < \frac{\operatorname{width}(u_n) \times \operatorname{height}(u_n)}{x \operatorname{width}(u_n) + y \operatorname{height}(u_n) + z}$$

The lemma follows from the fact that the function $n \mapsto n^2/(n \times (x+y) + z)$ has no upper bound. □

We are now ready for the proof that each multi-scale coverable picture has uniform frequencies. We section it into three technical Lemmas: 16, 17 and 19.

Lemma 16. *Let $\mathbf{w}$ be a multi-scale coverable picture. Suppose that u is a block of $\mathbf{w}$ and $(f_u(B_n))_{n\in\mathbb{N}}$ does not converge. Then there exists:*

- *a real number $\varepsilon > 0$;*
- *a real number $t \in [0; 1]$;*

- *an infinite set $\mathcal{V}$ of blocks of* $\mathbf{w}$*;*
- *an infinite set $\mathcal{Q}$ of covers of* $\mathbf{w}$*;*

such that either $f_u(v) < t - \varepsilon < t < f_u(q)$ for all $q \in \mathcal{Q}, v \in \mathcal{V}$, or $f_u(v) > t + \varepsilon > t > f_u(q)$ for all $q \in \mathcal{Q}, v \in \mathcal{V}$. Moreover, $L_{\geq K}(\mathbf{w}) \cap \mathcal{V}$ and $L_{\geq K}(\mathbf{w}) \cap \mathcal{Q}$ are non-empty, for all K.

Proof. Observe that $(f_u(B_n))$ takes its values in $[0; 1]$, a compact set. Since it does not converge, it has two subsequences converging to two different limits, say ℓ_1 and ℓ_2 with $\ell_1 < \ell_2$. Set $t = (\ell_1 + \ell_2)/2$ and $\varepsilon = (\ell_2 - \ell_1)/4$. Define:

$$B^- = \{v \mid \ v \text{ is a block of } \mathbf{w} \text{ and } f_u(v) < t\}$$
$$B^+ = \{v \mid \ v \text{ is a block of } \mathbf{w} \text{ and } f_u(v) \geq t\}$$

Remark that $\{B^-, B^+\}$ is a partition of the set of blocks of $\mathbf{w}$ and that both B^- and B^+ are infinite (thanks to existence of the subsequences). Moreover, B^- and B^+ even contain squares of arbitrarily large sizes (still thanks to the subsequences).

By the pigeonhole principle, there is either infinitely many covers in B^- or infinitely many covers in B^+. Suppose there are infinitely many covers in B^- and call $\mathcal{Q}$ the set of these covers. Then we have to set $\mathcal{V}$. By our previous remarks, there is an infinite subsequence of (B_n), call it $B_{\alpha(n)}$, such that $f_u(B_{\alpha(n)})$ converges to $\ell_2 = t + 2\varepsilon$. By definition of convergence, there exists N such that for all $n > N$, we have $f_u(B_{\alpha(n)}) > t + \varepsilon$. Let $\mathcal{V} = \{B_{\alpha(n)} | n > N\}$.

Symmetrically, suppose there are infinitely many covers in B^+ and call $\mathcal{Q}$ the set of these covers. To set $\mathcal{V}$, observe that there is an infinite sequence of blocks with arbitrarily large widths and heights whose frequencies converge to $t - 2\varepsilon$. By definition of convergence, there is an infinite sequence of blocks with arbitrarily large widths and heights whose frequencies are less than $t - \varepsilon$. Call the set of images of this sequence $\mathcal{V}$. ☐

Lemma 17. *Let $\mathbf{w}$ denote a multi-scale coverable picture and u a block of $\mathbf{w}$ without frequency. Suppose there exists some $t \in [0; 1]$ and some $\varepsilon > 0$ such that:*

$$\forall K \in \mathbb{N},\ \exists v \in L_{\geq K}(\mathbf{w}) \textit{ and } f_u(v) \leq t - \varepsilon$$

and that:

$$\forall K \in \mathbb{N},\ \exists q \in L_{\geq K}(\mathbf{w}) \textit{ and } q \textit{ is cover of } \mathbf{w} \textit{ and } f_u(q) > t$$

Then we have:

$$\forall K' \in \mathbb{N},\ \exists v' \in L_{\geq K'}(\mathbf{w}) \textit{ and } f_u(v') \leq t - \frac{11}{10}\varepsilon \tag{3}$$

Proof. We assume the hypotheses of the lemma and we have to prove that relation (3) holds. Let K' be some integer; let us find an appropriate block v'. The proof is in five steps. First we give a summary of the steps.

1. Use Lemma 15 to choose $q \in \mathcal{Q}$ and $v \in \mathcal{V}$ "big enough" for the following steps to work well.
2. Let β be a block such that $v \subseteq \beta$ and that width(β) is a multiple of $2\,\mathrm{width}(q)$ and height(q) is a multiple of $2\,\mathrm{height}(q)$. By hypothesis, v has a "low" frequency; we check that β has a "low" frequency as well.
3. Cut β into blocks of equal size $2\,\mathrm{width}(q) \times 2\,\mathrm{height}(q)$. Since the frequency of β is "low", one of the small blocks (call it b) must have a "low" frequency as well.
4. As b has dimensions $2\,\mathrm{width}(q) \times 2\,\mathrm{height}(q)$, it must contain a full occurrence of q. Cut b in two parts: q and m. By hypothesis, q has a "high" frequency, while b as a whole has a "low" frequency. Therefore, m must have a "very low" frequency. However, m is not a square, so we extract a "big enough" square with a "very low" frequency.
5. Do the final calculations to check that everything before is correct.

Step one. Let q be a cover of $\mathbf{w}$ with $f_u(q) > t$, and let $B = 2K' \times (\mathrm{width}(q) + \mathrm{height}(q))$. By hypothesis, we can choose such covers with arbitrarily large widths and heights, thus use Lemma 15 to choose q such that:

$$\mathrm{width}(q) > 10K' \tag{4}$$
$$\mathrm{height}(q) > 10K' \tag{5}$$

$$\frac{|q|}{B} > max(10, \frac{1}{\frac{t}{t-\varepsilon/10} - 1})$$

Therefore, q satisfies the following relations:

$$t \times |q| > (t - \frac{\varepsilon}{10})(|q| + B) \tag{6}$$
$$|q| > 10 \times B \tag{7}$$

We will use Equations (4), (5), (6) and (7) in Step four.

Now let v be a block of $\mathbf{w}$ with $f_u(v) \leq t - \varepsilon$, and let $A = 2\,\mathrm{width}(q) \times \mathrm{height}(v) + 2\,\mathrm{height}(q) \times \mathrm{width}(v) + 4|q|$. Use Lemma 15 to choose v such that:

$$\frac{|v|}{A} > \frac{10}{\varepsilon} \tag{8}$$

We will use Equation (8) in Step two.

Step two. Let β be one smallest block of $\mathbf{w}$ such that:

- v occurs in the bottom left-hand corner of β;
- $\mathrm{width}(\beta) = n \times 2\,\mathrm{width}(q)$ for some $n \in \mathbb{N}$;
- $\mathrm{height}(\beta) = m \times 2\,\mathrm{height}(q)$ for some $m \in \mathbb{N}$.

Since β is minimal, we have $\mathrm{width}(\beta) - \mathrm{width}(v) < 2\,\mathrm{width}(q)$ and $\mathrm{height}(\beta) - \mathrm{height}(v) < 2\,\mathrm{height}(q)$ (see Figure 7). So we have $|v| \leq |\beta| \leq |v| + A$ (on Figure 7, A is the maximal size of the gray area). In particular, $|\beta|_u \leq |v|_u + A$ by Lemma 14, , hence (by $f_u(v) < t - \varepsilon$ and Equation (8)):

$$\begin{aligned} f_u(\beta) &\leq f_u(v) \times \frac{|v|}{|\beta|} + \frac{A}{|\beta|} \\ &\leq f_u(v) + \frac{A}{|v|} \\ &\leq t - \varepsilon + \frac{A}{|v|} \\ &\leq t - \varepsilon + \frac{\varepsilon}{10} \\ &= t - \frac{9\varepsilon}{10} \end{aligned}$$

So we have:

$$f_u(\beta) \leq t - \frac{9\varepsilon}{10} \tag{9}$$

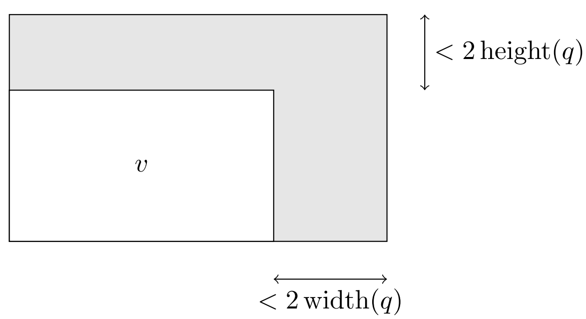

Figure 7: Anatomy of β.

Step three. The width and height of β are multiples of twice the width and height of q, respectively. So we can cut β into blocks of size $2\,\mathrm{width}(q) \times 2\,\mathrm{height}(q)$. Call $b_{i,j}$ those blocks, for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Since $f_u(\beta) \leq t - \frac{9\varepsilon}{10}$ (Equation (9)), there exists some i and j such that $f_u(b_{i,j}) \leq t - \frac{9\varepsilon}{10}$. Suppose not. Then, by Lemma 14 applied several times:

$$f_u(\beta) \geq \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} \frac{|b_{i,j}|}{|\beta|} f_u(b_{i,j}) > (t - \frac{9\varepsilon}{10})(\sum_i \sum_j \frac{|b_{i,j}|}{|\beta|}) = t - \frac{9\varepsilon}{10} \geq f_u(\beta)$$

a contradiction (one inequality is strict). From now on, note i and j some integers such that: $f_u(b_{i,j}) \leq t - \frac{9\varepsilon}{10}$.

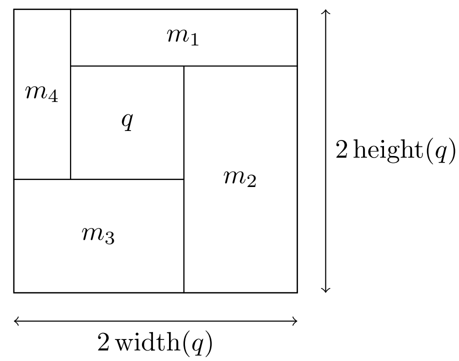

Figure 8: Anatomy of $b_{i,j}$.

Step four. Decompose $b_{i,j}$ into five parts: an occurrence of q and four blocks m_1, m_2, m_3, m_4 as on Figure 8.

By definition of the m_i's and Equations (4) and (5), we have:

$$\begin{aligned}
\text{width}(m_1) &\geq \text{width}(q) > 10K' \\
\text{width}(m_3) &\geq \text{width}(q) > 10K' \\
\text{width}(m_2) + \text{width}(m_4) &= \text{width}(q) > 10K' \\
\text{height}(m_2) &\geq \text{height}(q) > 10K' \\
\text{height}(m_4) &\geq \text{height}(q) > 10K' \\
\text{height}(m_1) + \text{height}(m_3) &= \text{height}(q) > 10K'
\end{aligned}$$

Let m_0 denote the empty block. We might have either $\text{width}(m_2) < K'$ or $\text{width}(m_4) < K'$, but not both. If $\min(\text{width}(m_2), \text{width}(m_4)) < K'$, let σ be the unique value in $\{2, 4\}$ satisfying $\text{width}(m_\sigma) < K'$; otherwise, let $\sigma = 0$. Likewise, we might have $\text{height}(m_1) < K'$ or $\text{height}(m_3) < K'$, but not both. If $\min(\text{height}(m_1), \text{height}(m_3)) < K'$, let τ be the unique value in $\{1, 3\}$ satisfying $\text{height}(m_\tau) < K'$; otherwise, let $\tau = 0$.

Observe that $|q'| = |q| + |m_\sigma| + |m_\tau| \leq |q| + 2\,\text{width}(q) \times K' + 2\,\text{height}(q) \times K'$. So we have by Equation (7):

$$|q| \leq |q'| \leq |q| + B \leq \frac{11}{10}|q|$$

and as $|q|/|b_{i,j}| = 1/4$:

$$\frac{1}{4} \leq \frac{|q'|}{|b_{i,j}|} \leq \frac{11}{40} \tag{10}$$

Moreover, we have $|q'|_u \geq |q|_u$ (as q' contains q). Therefore, by $f_u(q) > t$ and Equation (6):

$$f_u(q') \geq f_u(q) \times \frac{|q|}{|q'|} > t\frac{|q|}{|q| + B} > t - \frac{\varepsilon}{10} \tag{11}$$

Step five. We decompose $b_{i,j}$ into $q', m'_1, m'_2, m'_3, m'_4$ as follows. If $\text{width}(m_i) > K'$ and $\text{height}(m_i) > K'$, then $m'_i = m_i$; otherwise, m'_i is the empty block. Without loss of generality, suppose that m'_1 is not empty and is minimal for f_u among the non-empty m'_i's. Let $M = |m'_1| + |m'_2| + |m'_3| + |m'_4|$. Then we have, by Lemma 14 applied 4 times and minimality of m'_1 for f_u:

$$|b_{i,j}| = |q'| + M \tag{12}$$

$$f_u(b_{i,j}) \geq \frac{M}{|b_{i,j}|} \times f_u(m'_1) + \frac{|q'|}{|b_{i,j}|} \times f_u(q') \tag{13}$$

Keep these relations in mind and recall that we have from previous steps that:

$$\begin{aligned}
f_u(b_{i,j}) &\leq t - \frac{9\varepsilon}{10} && \text{(Step three)}\\
f_u(q') &> t - \frac{\varepsilon}{10} && \text{(Equation (11))}\\
\frac{1}{4} &\leq \frac{|q'|}{|b_{i,j}|} \leq \frac{11}{40} && \text{(Equation (10))}\\
\frac{29}{40} &\leq \frac{M}{|b_{i,j}|} \leq \frac{3}{4} && \text{(Equations (10) and (12))}
\end{aligned}$$

Now, assume by contradiction that $f_u(m'_1) \geq t - \frac{11\varepsilon}{10}$ and recall that $\varepsilon > 0$. Then we have:

$$\begin{aligned}
t - \frac{9\varepsilon}{10} \geq f_u(b_{i,j}) &\geq \frac{M}{|b_{i,j}|} \times (t - \frac{11\varepsilon}{10}) + \frac{|q'|}{|b_{i,j}|} \times (t - \frac{\varepsilon}{10})\\
&\geq t \times \frac{M + |q'|}{|b_{i,j}|} - \varepsilon(\frac{3}{4} \times \frac{11}{10} + \frac{11}{40} \times \frac{1}{10})\\
&\geq t - \frac{341\varepsilon}{400} \gg t - \frac{360\varepsilon}{400} = t - \frac{9\varepsilon}{10}
\end{aligned}$$

a contradiction. Hence we get $f_u(m'_1) < t - \frac{11\varepsilon}{10}$. Set $v' = m'_1$ and the lemma is proved. $\square$

Now we need a very similar lemma, but with "bigger and bigger" frequencies instead of "smaller and smaller". However, before this, we need an additional technical fact about frequencies.

Lemma 18. *Let u denote a finite block. Let v be a finite block which can be decomposed into disjoint blocks $v_1, \ldots, v_n$ such that* $\text{width}(v_i) > \text{width}(u)$ *and* $\text{height}(v_i) > \text{height}(u)$ *for all $1 \leq i < n$. Then we have:*

$$f_u(v) \leq \sum_{i=1}^{n} f_u(v_i) \times \frac{|v_i|}{|v|} + \frac{\text{width}(v_i) \times \text{height}(u) + \text{height}(v_i) \times \text{width}(u)}{|v|}$$

Proof. Each occurrence of u in v has its bottom left-hand corner in some v_i. It is either entirely contained in this v_i, or it overlaps it. There are respectively $|v_i|_u$ and at most $\text{width}(v_i) \times \text{height}(u) + \text{height}(v_i) \times \text{width}(u)$ of these. Divide by $|v|$ to obtain the result. $\square$

The next lemma is almost identical to Lemma 17, except that we get bigger and bigger frequencies instead of lower and lower frequencies. Since the proofs are very similar (we reverse inequalities and adapt everything to make it work), we only highlight the differences with the proof of Lemma 17.

Lemma 19. *Let* $\mathbf{w}$ *denote a multi-scale coverable picture and and* u *a block of* $\mathbf{w}$ *without frequency. Suppose there exists some* $t \in [0;1]$ *and some* $\varepsilon > 0$ *such that:*

$$\forall K \in \mathbb{N},\ \exists v \in L_{\geq K}(\mathbf{w}) \text{ and } f_u(v) \geq t + \varepsilon$$

and that:

$$\forall K \in \mathbb{N},\ \exists q \in L_{\geq K}(\mathbf{w}) \text{ cover of } \mathbf{w} \text{ and } f_u(q) < t$$

Then we have:

$$\forall K' \in \mathbb{N},\ \exists v' \in L_{\geq K'}(\mathbf{w}) \text{ and } f_u(v') \geq t + \frac{11}{10}\varepsilon \tag{14}$$

Proof. The five big steps are the same as in the proof of Lemma 17. We basically have to reverse each inequality, and replace $t - x\varepsilon$ with $t + x\varepsilon$ for all values of x. We use upper bounds instead of lower bounds, which sometimes slightly changes the details of the calcuations.

Step one. Let $B = 2K' \times (\mathrm{width}(q) + \mathrm{height}(q))$, $C = 2\,\mathrm{width}(q)\,\mathrm{height}(u) + 2\,\mathrm{height}(q)\,\mathrm{width}(u)$ and $D = 10(\mathrm{width}(q) + \mathrm{height}(q)) \times |u|$. Use Lemma 15 to choose q such that:

$$\begin{aligned}
\mathrm{width}(q) &> 10K' \\
\mathrm{height}(q) &> 10K' \\
\frac{|q|}{C} &> \frac{1000}{\varepsilon} \\
\frac{|q|}{B} &> \frac{\varepsilon}{10} \\
\frac{|q|}{D} &> \frac{100}{\varepsilon}
\end{aligned}$$

Therefore q satisfies the following relations:

$$\frac{C}{|q|} < \frac{\varepsilon}{1000} \tag{15}$$

$$\frac{B}{|q|} < \frac{10}{\varepsilon} \tag{16}$$

$$\frac{D}{4|q|} < \frac{\varepsilon}{400} \tag{17}$$

Equations (15), (16) and (17) will be used in steps three, four and five, respectively.

Let $A = 2\,\mathrm{width}(q) \times \mathrm{height}(v) + 2\,\mathrm{height}(q) \times \mathrm{width}(v) + 4|q|$. Use Lemma 15 again to choose v such that we have:

$$\begin{aligned}
\mathrm{width}(v) &> 2\,\mathrm{width}(q)\\
\mathrm{height}(v) &> 2\,\mathrm{height}(q)\\
\frac{|v|}{A} &> \frac{t + 9\varepsilon/10}{\varepsilon/10}
\end{aligned}$$

Therefore, v satisfies the following relation:

$$(t+\varepsilon) \times |v| > (t + \frac{9\varepsilon}{10})(|v| + A) \tag{18}$$

Equation (18) will be used in step two. Observe that $A < 3|v|$.

Step two. Let β be one smallest block of $\mathbf{w}$ such that:

- v occurs in the bottom left-hand corner of β;
- $\mathrm{width}(\beta) = 2n\,\mathrm{width}(q)$ for some $n \in \mathbb{N}$;
- $\mathrm{height}(\beta) = 2m\,\mathrm{height}(q)$ for some $m \in \mathbb{N}$.

As in proof of Lemma 17, $|\beta| \leq |v| + A$ and so (recall that $A < 3|v|$) we have $|\beta| < 4|v|$.

As $v \subseteq \beta$, we have $|\beta|_u \geq |v|_u$. Hence, by $f_u(v) \geq t + \varepsilon$ and Equation (18):

$$f_u(\beta) \geq f_u(v) \times \frac{|v|}{|\beta|} \geq (t+\varepsilon) \times \frac{|v|}{|v| + A} > t + \frac{9\varepsilon}{10} \tag{19}$$

Step three. Decompose β into blocks $b_{i,j}$, such that $\mathrm{width}(b_{i,j}) = 2\,\mathrm{width}(q)$ and $\mathrm{height}(b_{i,j}) = 2\,\mathrm{height}(q)$, for $1 \leq i \leq n$ and $1 \leq j \leq m$. In particular, $nm = |\beta|/4|q|$. Since $f_u(\beta) > t + \frac{9\varepsilon}{10}$, there exists i, j such that $f(b_{i,j}) > t + \frac{89\varepsilon}{100}$. Suppose not. Then, by Lemma 18:

$$\begin{aligned}
f_u(\beta) \leq& \sum_{i=1}^{n}\sum_{j=1}^{m} f_u(b_{i,j}) \times \frac{|b_{i,j}|}{|\beta|} + \frac{C}{|\beta|}\\
\leq& \sum_{i=1}^{n}\sum_{j=1}^{m} (t + \frac{89\varepsilon}{100}) \times \frac{1}{nm} + \frac{C}{|v|}\\
\leq& t + \frac{89\varepsilon}{100} + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{C}{|v|}\\
\leq& t + \frac{89\varepsilon}{100} + \frac{|\beta|}{4|q|} \times \frac{C}{|v|}\\
\leq& t + \frac{89\varepsilon}{100} + \frac{4|v|}{4|q|} \times \frac{C}{|v|}\\
<& t + \frac{9\varepsilon}{10} && \text{(Equation (15))}
\end{aligned}$$

a contradiction, as $f_u(\beta) > t + \frac{9\varepsilon}{10}$. From now on, let i and j denote some integers such that:

$$f_u(b_{i,j}) > t + \frac{89\varepsilon}{100} \tag{20}$$

Step four. Decompose $b_{i,j}$ into q, m_1, m_2, m_3, m_4 and define q' as in the proof of Lemma 17. We have $|q'| \geq |q|$. Then, by Lemma 14, $f_u(q) < t$ and Equation (16):

$$f_u(q') < f_u(q) \times \frac{|q|}{|q'|} + \frac{B}{|q'|} < t \times \frac{|q|}{|q'|} + \frac{B}{|q|} < t + \frac{\varepsilon}{10} \tag{21}$$

Moreover, Equation (10) is still valid.

Step five. Decompose $b_{i,j}$ into $q', m_1', m_2', m_3', m_4'$ as in Lemma 17. Let $M = |m_1'| + |m_2'| + |m_3'| + |m_4'|$. We have:

$$|b_{i,j}| = 4|q| = |q'| + M \tag{22}$$

Without loss of generality, suppose m_1' is not empty and is maximal for f_u among the m_i''s.

Now we argue that:

Fact.

$$|b_{i,j}|_u \leq |q'|_u + |m_1'|_u + |m_2'|_u + |m_3'|_u + |m_4'|_u + D$$

Proof. The previous equation means that D is an upper bound on the number of occurrences of u in $b_{i,j}$ which overlap over several components of $b_{i,j}$.

First, view $b_{i,j}$ as decomposed over $\{q, m_1, m_2, m_3, m_4\}$ as on Figure 8. Then the total length of horizontal frontiers is $\mathrm{width}(m_4) + \mathrm{width}(m_2) + 2\,\mathrm{width}(q)$, and the total length of vertical frontiers is $\mathrm{height}(m_1) + \mathrm{height}(m_3) + 2\,\mathrm{height}(q)$. Hence, with this decomposition, the number of occurrences of u overlapping over several components is bounded by

$$\begin{aligned}
&\mathrm{height}(u)(\mathrm{width}(m_4) + \mathrm{width}(m_2) + 2\,\mathrm{width}(q)) + \\
&\mathrm{width}(u)(\mathrm{height}(m_1) + \mathrm{height}(m_3) + 2\,\mathrm{height}(q)) \\
=\, &\mathrm{height}(u) \times 3\,\mathrm{width}(q) + \mathrm{width}(u) \times 3\,\mathrm{height}(q)
\end{aligned}$$

Recall that the decomposition $\{q', m_1', m_2', m_3', m_4'\}$ is $\{q, m_1, m_2, m_3, m_4\}$ where some components have been merged, and others turned to the empty set. Hence the sum of lengths of frontiers is shorter. Hence, the number of overlapping occurrences of u is smaller. So the given bound is also a bound for the number of overlapping occurrences of u for $b_{i,j}$ decomposed as $\{q', m_1', m_2', m_3', m_4'\}$. $\square$

To end the proof of Lemma 17, recall that $M = |m_1'| + |m_2'| + |m_3'| + |m_4'|$. We can deduce from the previous fact that:

$$f_u(b_{i,j}) \leq f_u(m_1') \times \frac{M}{|b_{i,j}|} + f_u(q') \times \frac{|q'|}{|b_{i,j}|} + \frac{D}{|b_{i,j}|} \tag{23}$$

Keep Equations (22) and (23) in mind, and recall from previous steps that:

$$\begin{aligned}
f_u(b_{i,j}) &> t + \frac{89\varepsilon}{100} && \text{(Equation (20))}\\
f_u(q') &< t + \frac{\varepsilon}{10} && \text{(Equation (21))}\\
\frac{1}{4} \leq \frac{|q'|}{|b_{i,j}|} &\leq \frac{11}{40} && \text{(Equation (10))}\\
\frac{29}{40} \leq \frac{M}{|b_{i,j}|} &\leq \frac{3}{4} && \text{(Equation (10) and (12))}\\
\frac{D}{|b_{i,j}|} &< \frac{\varepsilon}{400} && \text{(Equation (17))}
\end{aligned}$$

Finally assume by contradiction that $f_u(m'_1) \leq t + \frac{11\varepsilon}{10}$ and recall that $\varepsilon > 0$. Then:

$$\begin{aligned}
t + \frac{89\varepsilon}{100} < f_u(b_{i,j}) &< (t + \frac{11\varepsilon}{10}) \times \frac{M}{|b_{i,j}|} + (t + \frac{\varepsilon}{10}) \times \frac{|q'|}{|b_{i,j}|} + \frac{D}{|b_{i,j}|}\\
&< t \times \frac{M + |q'|}{|b_{i,j}|} + \varepsilon(\frac{11}{10} \times \frac{3}{4} + \frac{1}{10} \times \frac{11}{40} + \frac{1}{400})\\
&< t + \frac{342\varepsilon}{400} \ll t + \frac{356\varepsilon}{400} = t + \frac{89\varepsilon}{100},
\end{aligned}$$

which is a contradiction. Hence we get $f_u(m'_1) > t + \frac{11\varepsilon}{10}$. Set $v' = m'_1$ and the lemma is proved. □

We are now ready for the main proof.

Proof of Theorem 13. Suppose that $\mathbf{w}$ is a multi-scale coverable picture and that u is a block of $\mathbf{w}$ without frequencies. By Lemma 16 there exist real numbers t and ε, an infinite set of blocks $\mathcal{V}$ and an infinite set of covers $\mathcal{Q}$ such that either $f(v) > t+\varepsilon > t > f(q)$ for all $q \in \mathcal{Q}$, $v \in \mathcal{V}$, or $f(v) < t < t-\varepsilon < f(q)$ for all $q \in \mathcal{Q}$, $v \in \mathcal{V}$. Moreover, $\mathcal{V} \cap L_{\geq K}(\mathbf{w})$ and $\mathcal{Q} \cap L_{\geq K}(\mathbf{w})$ are non-empty for all $K \in \mathbb{N}$.

Suppose we have $f(v) < t - \varepsilon < t < f(q)$ for all $q \in \mathcal{Q}$, $v \in \mathcal{V}$. In this situation, Lemma 17 states that there exists an infinite set of blocks $\mathcal{V}'$ such that $f(v') \leq t - \frac{11}{10}\varepsilon$ for all $v' \in \mathcal{V}'$ (and $\mathcal{V}' \cap L_{\geq K}(\mathbf{w})$ is non-empty for all $K \in \mathbb{N}$). We can apply the same lemma again and get blocks whose frequency is $\leq t - (\frac{11}{10})^2\varepsilon$. Then we can apply the lemma again and again, until we get $f(v') \leq t - (\frac{11}{10})^n\varepsilon < 0$ for some n. At that point, we get blocks with negative frequencies: a contradiction.

If $f(v) > t + \varepsilon > t > f(q)$ for all $v \in \mathcal{V}, q \in \mathcal{Q}$, the proof follows the same idea, except that we use Lemma 19. We get blocks with higher and higher frequencies, until we find a block with a frequency bigger than 1: a contradiction again. □

4.3. Uniform Recurrence

Recall that a $\mathbb{Z}^2$-word $\mathbf{w}$ is *uniformly recurrent* when all its blocks occur infinitely often with bounded gaps. In $\mathbb{N}$-words, multi-scale coverability implies uniform recurrence. However, this result does not hold for $\mathbb{Z}$-words. Here is an example of a $\mathbb{Z}$-word which is multi-scale coverable, but not uniformly recurrent:

$$^{\omega}(ab)a(ab)^{\omega} = \dots bababab a\, a\, bababab a \dots$$

Any word matching the $aba(ba)^*$ regular expression is a covering pattern of this word. However, the pattern aa only occur once, hence it is not uniformly recurrent.

It is easy to generalize these results to pictures.

Proposition 20. *Any multi-scale, $\mathbb{N}^2$-word $\mathbf{w}$ is uniformly recurrent.*

Proof. This is an adaptation of the proof from [14]. Consider a finite picture r occurring in $\mathbf{w}$. Since $\mathbf{w}$ has arbitrarily large covering patterns and all these patterns occur at the origin, one of these patterns contains r entirely. Hence r occurs whenever the covering patterns occurs, and the latter occurs infinitely many times with bounded gaps. $\square$

Now let us see an example of coverable $\mathbb{Z}^2$-words which is not uniformly recurrent (or even recurrent). Consider $q = \begin{smallmatrix} b & b & a \\ a & b & b \end{smallmatrix}$ and the word displayed on Figure 1. The central block:

$$\begin{matrix} b & b & b & b & a \\ b & b & a & b & b \\ a & b & b & b & b \end{matrix}$$

occurs only once, hence this picture is not uniformly recurrent.

To get uniform recurrence back, we propose a notion of *strong* multi-scale coverability. A word (or a picture) is *strongly* multi-scale coverable if and only if any factor (or block) occuring also occurs in a cover. Observe that multi-scale coverability and its strong counterpart are equivalent on $\mathbb{N}$-words.

Proposition 21. *An infinite picture $\mathbf{w}$ is strongly multi-scale coverable if and only if it is multi-scale coverable and uniformly recurrent.*

Proof. Strong multi-scale coverability implies multi-scale coverability: each occurring block must occur within a cover, and there are arbitrarily large (in width and height) blocks, so there must be arbitrarily large (in width and height) covers.

Strong multi-scale coverability also implies uniform recurrence, almost by definition: any block occurs in a cover, which in turn occurs infinitely often with bounded gaps.

Finally, multi-scale coverability and uniform recurrence imply strong multi-scale coverability. Indeed, let $\mathbf{w}$ be a multi-scale coverable picture which is also uniformly recurrent, and let B be a block of $\mathbf{w}$. Since $\mathbf{w}$ is uniformly recurrent, there exists $n \in \mathbb{N}$ such that any $n \times n$-block of $\mathbf{w}$ contain an occurrence of B. By definition of multi-scale coverability, there are covers which are bigger than $2n \times 2n$ (both in width and in height); such covers must contain an occurrence of B. $\square$

5. Conclusion

Coverability is a local rule. Our aim was to determine whether this notion enforces some global properties on covered words. Although this is not the case in one dimension, Theorem 2 shows that, under some natural hypotheses on the cover (natural in the sense they take into account the two dimensions), this enforcement is possible in two dimensions. However, many other questions have to be considered to understand better the power of coverability, especially when considering that covers are rectangular blocks.

One could also ask what happens with non-rectangular covers. In the multi-scale case, it would be natural to require that the covers "grow" in all directions. As a consequence, they would eventually contain larger and larger squares. Thus we expect all the results about multi-scale coverability to remain unchanged if the "rectangle" constraint is relaxed.

Our approach could be linked to considerations from dynamical systems and tilings. For instance, a natural question is: are self-similarity and multi-scale coverability linked? Our study already states that, as in the one-dimensional case, multi-scale coverability implies other properties, such as existence of frequencies and zero topological entropy. A difference with the one-dimensional case is that multiscale coverability does not impy uniform recurrence. Proposition 20 and discussion before explain that this difference does not come directly from the change of dimension but much more from the fact that we consider the full plane $\mathbb{Z}^2$ instead of the quarter of plane $\mathbb{N}^2$.

One could ask: why did we consider $\mathbb{Z}^2$-words as a two-dimensional generalization of $\mathbb{N}$-words? This stems from the fact that $\mathbb{Z}^2$-words are much more relevant in the area of tilings. Let us observe that most results, such as Proposition 1, Proposition 12 and Theorem 13, can be directly adapted to $\mathbb{N}^2$-words. However, it is much more difficult for Theorem 2 and Theorem 8, as the condition on border is not adequate for $\mathbb{N}^2$-words.

Acknowledgements

We acknowledge Anaël Grandjean for his help with the proof of Theorem 13 and many interesting discussions during the preparation of this paper. We thank the referees for their corrections and numerous interesting suggestions.

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