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Reconstructible phylogenetic networks: do not distinguish the indistinguishable
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Abstract
Phylogenetic networks represent the evolution of organisms that have undergone reticulate events, such as recombination, hybrid speciation or lateral gene transfer. An important way to interpret a phylogenetic network is in terms of the trees it displays, which represent all the possible histories of the characters carried by the organisms in the network. Interestingly, however, different networks may display exactly the same set of trees, an observation that poses a problem for network reconstruction: from the perspective of many inference methods such networks are indistinguishable. This is true for all methods that evaluate a phylogenetic network solely on the basis of how well the displayed trees fit the available data, including all methods based on input data consisting of clades, triples, quartets, or trees with any number of taxa, and also sequence-based approaches such as popular formulations of maximum parsimony and maximum likelihood for networks. This identifiability problem is partially solved by accounting for branch lengths, although this merely reduces the frequency of the problem. Here we propose that network inference methods should only attempt to reconstruct what they can uniquely identify. To this end, we introduce a novel definition of what constitutes a uniquely reconstructible network. For any given set of indistinguishable networks, we define a canonical network that, under mild assumptions, is unique and thus representative of the entire set. Given data that underwent reticulate evolution, only the canonical form of the underlying phylogenetic network can be uniquely reconstructed. While on the methodological side this will imply a drastic reduction of the solution space in network inference, for the study of reticulate evolution this is a fundamental limitation that will require an important change of perspective when interpreting phylogenetic networks.

Author Summary
We consider here an elementary question for the inference of phylogenetic networks: what networks can be reconstructed. Indeed, whereas in theory it is always possible to reconstruct a phylogenetic tree, given sufficient data for this task, the same does not hold for phylogenetic networks: most notably, the relative order of consecutive reticulate events cannot be determined by standard network inference methods. This problem has been described before, but no solutions to deal with it have been put forward. Here we propose limiting the space of reconstructible phylogenetic networks to what we call “canonical networks”. We formally prove that each network has a (usually unique) canonical form – where a number of nodes and branches are merged – representing all that can be uniquely reconstructed about the original network. Once a canonical network $\hat{N}$ is inferred, it must be kept in mind that – even with perfect and unlimited data – the true phylogenetic network is just one of the potentially many networks having $\hat{N}$ as canonical form. This is an important difference to what biologists are used to for phylogenetic trees, where in principle it is always possible to resolve uncertainties, given enough data.
Introduction

Explicit [1] or evolutionary [2,3] phylogenetic networks are used to represent the evolution of organisms or genes that may inherit genetic material from more than one source. This may be caused by events such as hybrid speciation (e.g. in plants and animals [4,5]), horizontal gene transfer (e.g. in bacteria [6,7]), viral reassortment [8], or recombination (e.g. in viruses [9,10] or in the genomes of sexually reproducing species [11–13]). They are called “explicit” to distinguish them from “implicit” [14], “abstract” [1] or “data-display” [3] phylogenetic networks, which are used to display collections of alternative evolutionary hypotheses supported by conflicting signals in the data. In explicit networks, multiple-inheritance events are represented as reticulations, that is, nodes where two or more lineages converge to give rise to a new lineage, whose genetic material is a combination of that of its direct ancestors.

Explicit networks can be interpreted in terms of classic, tree-like evolution: if we focus on a single, indivisible and thus non-recombining inherited character (for example a single site in a DNA sequence), its history is still best described by a tree. This observation gives rise to the notion of trees displayed by a network, which are all the possible single-character histories implied by a phylogenetic network. (See, e.g., Fig. 1, where $T_1$, $T_2$ and $T_3$ are the trees displayed by networks $N_1$ and $N_2$. Formal definitions are in the Results section.)

Several works in the last few years have focused on the methodology for phylogenetic network inference, and data-display networks in particular have begun to make a real impact on the everyday practice of biologists (e.g., [15–17]). There remains, however, a strong demand for automatic reconstruction of networks that not only display conflicting signals in the data, but also seek to explain these signals with explicit inferences of past reticulation events (see, e.g., [18–20]). This is evidenced, for example, by the abundance of manually reconstructed networks in the literature [8,21–27]. As a result of this demand, the inference of explicit networks is now a rapidly growing field of research [1].

Some paradigms in the proposed methodology are beginning to emerge. Not surprisingly, the notion of trees displayed by a phylogenetic network plays a central role: the general idea is to evaluate the fit of a network $N$ with the data indirectly – on the basis of how well the trees displayed by $N$ explain the data. In the following, we describe how this applies to the two main approaches for explicit network reconstruction: consistency-based approaches (see [28] for a survey) – seeking a network consistent with a number of prior evolutionary inferences (typically trees or groupings of taxa) – and sequence-based approaches, such as...
standard formulations of maximum parsimony and maximum likelihood for networks [2,29–33].

Although evaluating a network via the trees it displays is evolutionarily meaningful, it has a problematic consequence: from the perspective of these reconstruction methods, all networks displaying the same set of trees are “indistinguishable”, as the function that these methods seek to optimize will always assign the same score to all networks displaying the same set of trees, regardless of the input data. In other words, the central parameter of phylogenetic network inference, the network itself, is in some cases not identifiable.

An Identifiability Problem

As an example, consider again networks $N_1$ and $N_2$ in Fig. 1, which display the same trees $T(N) = \{T_1, T_2, T_3\}$. (In the following, $T(N)$ denotes the set of trees displayed by $N$.) By displaying the same trees, these networks display the same clades, the same triples, the same quartets (triples and quartets are rooted subtrees with 3 leaves and unrooted subtrees with 4 leaves, respectively) and in general the same subtrees with an arbitrary number of leaves. Therefore, any method that reconstructs a network based on its consistency with collections of such data will not be able to distinguish between networks $N_1$ and $N_2$. This includes all the methods whose data consists of clusters of taxa (e.g., [34]), triples (e.g., [35]), quartets (e.g., [36]), or any trees (e.g., [37]).

The same holds for many, sequence-based, maximum parsimony and maximum likelihood approaches proposed in recent papers. For maximum parsimony, a practical approach [2,29–31] is to consider that the input is partitioned in a number of alignments $A_1, A_2, \ldots, A_m$, each from a different non-recombining genomic region (possibly consisting of just one site each), and then take, for each of these alignments, the best parsimony score $Ps(T|A_i)$ among all those of the trees displayed by a network $N$. The parsimony score of $N$ is then the sum of all the parsimony scores thus obtained. Formally, we have

$$Ps(N|A_1, A_2, \ldots, A_m) = \sum_{i=1}^{m} \min_{T \in T(N)} Ps(T|A_i).$$

It is clear that if two networks display the same set of trees (as in Fig. 1), then their parsimony score with respect to any input alignments will be the same — because they take the minimum value over the same set $T(N)$ — and thus they are indistinguishable to any method based on the maximum parsimony principle above.

As for maximum likelihood (ML), Nakhleh and collaborators [2,32,33,38] have proposed an elegant framework whereby a phylogenetic network $N$ is not only described by a network topology, but also edge lengths and inheritance probabilities associated to the reticulations of $N$. As a result, any tree $T$ displayed by $N$ has edge lengths — allowing the calculation of its likelihood $Pr(A|T)$ with respect to any alignment $A$ — and an associated probability of being observed $Pr(T|N)$. The likelihood function with respect to a set of alignments $A_1, A_2, \ldots, A_m$, each from a different non-recombining genomic region, is then given by:

$$Pr(A_1, A_2, \ldots, A_m|N) = \prod_{i=1}^{m} Pr(A_i|N) = \prod_{i=1}^{m} \sum_{T \in T(N)} Pr(A_i|T) Pr(T|N).$$

Note that an important difference with the consistency-based and parsimony methods described above is that any tree $T$ displayed by a network has now edge lengths and an associated probability $Pr(T|N)$.

Unfortunately, this ML framework is also subject to identifiability problems. For example, it does not allow us to distinguish between networks with topologies $N_1$ and $N_2$ in Fig. 1: for every assignment of edge lengths and inheritance probabilities to $N_1$, there exist corresponding assignments to $N_2$ that make the resulting networks indistinguishable, that is, displaying the same trees, with the same edge lengths and the same probabilities of being observed (see the last section in the Supporting Information, S1 Text). As a result, the likelihoods of these two networks will be identical, regardless of the data, and no method
Fig 2. **Edge lengths are informative to distinguish among different network topologies.** The only network topology, among \( N_1 \), \( N_2 \) and \( N_3 \) that can display simultaneously \( T_1 \) and \( T_2 \) with the indicated edge lengths is \( N_2 \): see for example the edge lengths assignment in the bottom right corner.

based on this definition of likelihood will be able to favour one of them over the other. We refer to S1 Text for a more detailed discussion about networks with inheritance probabilities and likelihood-based reconstruction.

In general, we believe that these identifiability problems affect all network inference methods which seek consistency with unordered collections of sequence alignments or pre-inferred attributes such as clusters, triples, quartets or trees.

**The Importance of Edge Lengths**

In this paper, as in the ML framework above, we adopt networks and trees with edge lengths as the primary objects of our study. The primary motivation for this is that this choice makes our results directly relevant to the statistical approaches for network inference, all of which need edge lengths to measure the fit of a phylogeny with the available data. In addition to ML, these approaches include distance-based and Bayesian methods [39], which are also promising for future work.

However, there is another motivation for our choice: accounting for edge lengths solves some of the identifiability problems outlined above, as in some cases it allows to distinguish between networks with different topologies, which would be otherwise impossible to tell apart. For example, consider the three network topologies in Fig. 2 (top), where taxon \( o \) is an outgroup used to identify the root of the phylogeny for \( a \), \( b \) and \( c \). These networks show three very different evolutionary histories: in \( N_1 \) taxon \( b \) is the only one issued of a reticulation event — in other words the genome of \( b \) is recombinant — whereas in \( N_2 \) and \( N_3 \), it is \( a \) and \( c \), respectively, that are recombinant. However, \( N_1 \), \( N_2 \) and \( N_3 \) display the same tree topologies — those of \( T_1 \) and \( T_2 \) — and thus would be indistinguishable to any approach that does not model edge lengths.

If instead edge lengths are accounted for (e.g. in a ML context) and the data supports \( T_1 \) and \( T_2 \) with the edge lengths in Fig. 2, then the only network fitting the data is \( N_2 \), with the edge lengths indicated at the bottom right. It is easy to check that \( N_2 \) now displays \( T_1 \) and \( T_2 \) with the shown edge lengths, whereas no edge length assignment to \( N_1 \) or \( N_3 \) can make these networks display \( T_1 \) and \( T_2 \).

We note that, throughout this paper, as in classical likelihood approaches, edge lengths measure
**Fig 3. Indistinguishable networks.** Two networks with edge lengths $N_1, N_2$ displaying the same set of trees $T(N_1) = T(N_2) = \{T_1, T_2, T_3\}$. For any choice of edge lengths $\lambda_1, \lambda_2, \ldots, \lambda_{12}$ for $N_1$, we define a family of edge length assignments for $N_2$, parameterized by $x, y$ (with $-y < x < \min\{\lambda_6, \lambda_5 + \lambda_8\}$, $0 < y < \lambda_7$).

Evolutionary divergence, for example in terms of expected number of substitutions per site. No molecular clock is assumed, meaning that we do not expect edge lengths to be proportional to time.

**Remaining Identifiability Problems, and a Proposed Solution**

Unfortunately, accounting for edge lengths only solves some of the identifiability problems for phylogenetic networks. Consider networks $N_1$ and $N_2$ in Fig. 3: for any set of edge lengths for $N_1$, there exist an infinity of edge length assignments for $N_2$ that make these two networks display exactly the same set of trees with the same edge lengths. In the following, we say that networks such as $N_1$ and $N_2$ are indistinguishable.

In fact it is not difficult to construct other examples of indistinguishable networks: each time a network has a reticulation $v$ giving birth to only one edge (i.e. with outdegree 1), then we can reduce by $\Delta \lambda$ the length of this edge and correspondingly increase by $\Delta \lambda$ the lengths of the edges ending in $v$, without altering the set of trees displayed by the network. Note that this operation, which we refer to as “unzipping” reticulation $v$, can result in $v$ coinciding with a speciation node or a leaf when $\Delta \lambda$ is taken to equal the length of the edge going out of $v$. For example in Fig. 3, one may fully unzip the two reticulation nodes in $N_1$, thus obtaining the network $N'$ of Fig. 4. As expected, $N_1$ and $N'$ display the same set of trees $\{T_1, T_2, T_3\}$ and are thus indistinguishable. What is most interesting in this example is that, if we fully unzip the two reticulations in $N_2$ (the other network in Fig. 3, also displaying $\{T_1, T_2, T_3\}$), then we eventually end up obtaining $N'$ again. As we shall see in the following, this is not a coincidence: the unzipping transformations described above lead to what we call the canonical form of a network; under mild assumptions, two networks are indistinguishable if and only if they have the same canonical form (e.g. $N_1, N_2$ in Fig. 3 have the same canonical form $N'$; formal definitions and statements in the Results section).

Here, we propose to deal with the identifiability issues for phylogenetic networks in the following
way: since no data will ever enable any of the standard inference methods described above to prefer a network over all of its indistinguishable equivalents, we propose that these methods should only attempt to reconstruct what they can uniquely identify, that is, networks in canonical form. This is a radical shift, not only for the developers of phylogenetic inference methods, who will see a drastic reduction of the solution space of their algorithms, but also for evolutionary biologists, who should abandon their hopes of seeing a network such as $N_1$ or $N_2$ in Fig. 3 being reconstructed by these inference methods.

**Previous Work and Comparison**

Limiting the scope of network reconstruction to topologically-constrained classes of networks has been a recurring theme and an important goal in the literature on phylogenetic networks. Examples of such classes include galled trees [40, 41], galled networks [42], level-k networks [43], tree-child networks [44], tree-sibling networks [45], networks with visible reticulations [1]. Although the ultimate goal should be to establish what can be inferred from biological data, most of the proposed definitions are computationally-motivated: in general the rationale behind these classes is the possibility of devising an efficient algorithm to solve some formalization of the reconstruction problem. None of these definitions claims to have biological significance.

Our goals are more basic: starting from the observation that not all phylogenetic networks are identifiable, since many of them are mutually indistinguishable with most inference approaches, we aim to define a class of networks that is (existence goal) large enough that every phylogenetic network has an equivalent (i.e. indistinguishable) network within this class and (distinguishability goal) small enough that no two networks within this class are indistinguishable. From our standpoint, the computationally-motivated definitions above are at the same time too broad and too restrictive. Too broad, because they determine a set of networks that includes many pairs of indistinguishable networks: for example the three indistinguishable networks in Fig. 2 are all galled trees — and thus belong to every single one of the classes mentioned above (which are all generalizations of galled trees). Too restrictive, because these classes of networks do not include simple networks that it should be possible to reconstruct from real data. For example, Fig. 5a shows a network $N$ with edge lengths that is not tree-sibling, nor has the visible property, and thus is not galled, nor tree-child (for definitions, see [1]), but which in practice should be reconstructible: apart from the lengths of three edges ($x, y, z$), $N$ is uniquely determined by the trees that it displays (a consequence of the formal results that we will show in the following), meaning that, given large amounts of data strongly supporting each of these (seven) trees with their correct edge lengths, any method for network inference properly accounting for edge lengths (e.g. based on ML) should be able to reconstruct $N$, or its canonical form $N'$.

To the best of our knowledge, only three classes of networks have claims of unique identifiability:
Fig 5. Examples of networks that can be uniquely recovered from the data they generate. (a) A network $N$, and its canonical form $N'$, whose topologies are not galled trees, nor galled, tree-child, tree-sibling or regular networks, nor networks with visible reticulations. $N$, however, is uniquely determined by the trees it displays, with the exception of $x$, $y$ and $z$, which can assume any value between 0 and 0.1. Because of the impossibility to determine these values, the canonical form $N'$ has the corresponding edges collapsed. As $N'$ is a network in canonical form satisfying the mild conditions of Corollary 2, $N'$ is uniquely determined by the trees it displays. Note that $N$ provides the biological interpretation for $N'$. (b) The network topology of $N''$ is such that there exists no regular network displaying the same set of (two) tree topologies as $N''$. Thus, restricting the scope of phylogenetic inference to regular networks would be very limiting. In our framework, $N''$ is a network in canonical form and thus uniquely determined by the trees it displays.

reduced networks [46,47], regular networks [48] and binary galled trees with no gall containing exactly 4 nodes [49]. These approaches bear some resemblances to ours, but do not include edge lengths in the definition of a network. Moreover, we argue that these classes of networks are still too narrow to be biologically relevant. We briefly describe and comment these previous works below.

Moret et al. [46] defined notions of reconstructible, indistinguishable and reduced networks that resemble concepts that we will introduce here. Although some of their results were flawed [47,50], some of the arguments in this introduction are inspired by their paper. Particularly relevant to the current paper is a reduction algorithm to transform a network into its reduced version. (However, the exact definition of the reduced version is unclear: as one of the authors later pointed out [47], “the reduction procedure of Moret et al. [46] is, in fact, inaccurate” and “in this paper we do not attempt to fix the procedure”). The concept of reduced version is analogous to that of canonical form here, as the authors claim that networks displaying the same tree topologies have the same reduced version (up to isomorphism; Theorem 2 in [46]). This is somehow a weaker analogue of one of our results (Corollary 1); weaker, because it does not claim that, conversely, networks with the same reduced version display the same tree topologies. To have an idea of the difference between our canonical form and the reduced version of Moret and colleagues, in Fig. 6 we compare the canonical form and the reduced version of the same network $N_1$. ($N_1$ and its reduced version are taken from Fig. 15 of [46] to avoid possible issues with the reduction algorithm.) As one can see, the canonical form retains more of the complexity of the original network.

Another reduction procedure on network topologies has been studied by Gambette and Huber [49], who proved that if two network topologies reduce to the same topology, then they must display the same tree topologies. Again, this is analogous to, but somehow weaker than our results, since it only provides a sufficient condition for networks to be indistinguishable (which in their context means to display the same tree topologies). This means that there can be irreducible networks that are indistinguishable (e.g. those in Fig. 2) thus failing to achieve the distinguishability goal. Moreover, Gambette and Huber [49] show that a particular class of network topologies (binary galled trees with no gall containing exactly 4 nodes) are uniquely identified by the tree topologies they display. It is clear that this class is too small to achieve the existence goal.

Finally, a regular network is a network topology $N$ in which, among other requirements, no two distinct nodes have the same set of descendant leaves (see [48] for a formal definition and characterizations). This requirement implies, among other things, that $N$ cannot contain any reticulation $v$ with outdegree 1 ($v$ and its direct descendant would have the same descendant leaves), which in turn implies that regular
Fig 6. Comparison between the reduced version and the canonical form of a network. $N_1$ is the network topology in Fig. 15a of [46], where edges leading to extinct taxa are shown in grey, and reticulation events are represented by horizontal lines connecting the involved edges. $N_2$ is a phylogenetic network on the same set of taxa displaying the same evolutionary history, and showing edge lengths. $R(N_1)$ is the reduced version of $N_1$ (Fig. 15b of [46]). $N'_2$ is the canonical form of $N_2$. Comparing $R(N_1)$ and $N'_2$ reveals the difference in expressive power between reduced versions and canonical forms. Collapsing the edge above $c$ and $d$ in $R(N_1)$ yields the regular network displaying the same tree topologies as $N_1$ and $N_2$. Clearly, the reduced form $R(N_1)$ (and the regular form) retain less of the complexity of the original network $N_1$ than the canonical form $N'_2$. For example in $R(N_1)$ there remains no sign of the reticulate events ancestral to taxon $e$. 
networks are special cases of our canonical networks (the latter however also specify edge lengths). In fact regular networks satisfy a property that is analogous to the one we prove here for canonical networks: a regular network $N$ is uniquely determined by the tree topologies that it displays [51], meaning that there can be no other regular network $N'$ displaying exactly the same set of tree topologies. Willson [51] shows this constructively by providing an algorithm that, given the (exponentially large) set of tree topologies displayed by a regular network $R$, reconstructs $R$ itself. However, unlike for our canonical forms, for a given network there may exist no regular network displaying the same set of trees (e.g. consider the topology of $N''$ in Fig. 5b), thus failing to meet the existence goal. Regularity is in fact a very restrictive constraint for a network. For example, none of the networks in Fig. 5 and Fig. 7 is regular, despite the fact that their topologies are uniquely determined by the trees with edge lengths that they display (a consequence of our results further below). Finally, going back to Fig. 6, collapsing the edge above taxa $c$ and $d$ in $R(N_1)$ yields the regular network displaying the same tree topologies as $N_1$ and $N_2$. Again, this shows that the canonical form retains more of the complexity of the original network than its regular counterpart.

**Fig 7. Trees displayed by a network.** A rooted network $N'_2$, and the trees it displays ($T'_1$ and $T'_2$), obtained by removing a segment of length 0.5 from the outgroup lineage of $N_2$ in Fig. 2. In our formal setting, a network such as $N_2$ in Fig. 2 can either be represented as $N'_2$ (by omitting the outgroup lineage, or part of it), or by rooting it in its outgroup (not shown).
Results

Our main result consists of formally proving that for every network \( N \) there exists a network \( N' \) in canonical form, indistinguishable from \( N \); moreover, if we restrict ourselves to networks satisfying a mild condition (the NELP property below), such canonical form \( N' \) is unique (see Theorem 1). In other words, although in general a phylogenetic network \( N \) is not uniquely recoverable from the data it generates, there always exists a canonical version \( N' \) of \( N \) that is indeed determined by the data. Informally, \( N' \) is all that can be reconstructed about \( N \).

In order to formally state this result, we here introduce a theoretical framework for explicit phylogenetic networks with branch lengths. A directed acyclic graph (DAG) is a simple directed graph that is free of directed cycles. A DAG is rooted if it contains precisely one node of indegree 0, called the root. All nodes of outdegree 0 in a DAG are called leaves. A weighted rooted phylogenetic network \( N = (V,E,\varphi,\Lambda) \) on \( \mathcal{X} \) (in this paper also called a network for simplicity) consists of a rooted DAG \( (V,E) \) whose leaves are bijectively labeled (via \( \varphi : \mathcal{X} \rightarrow V \)) with the elements of \( \mathcal{X} \) (called taxa). Moreover, each edge \( e \in E \) is associated to a set of positive weights, called lengths, \( \Lambda(e) \subset \mathbb{R}_{> 0} \). Figs. 3, 4, 5 contain examples of networks. A reticulation of a network \( N \) is a node \( v \in V \) with indegree greater than 1. A weighted phylogenetic tree on \( \mathcal{X} \) (a tree for simplicity) is a network on \( \mathcal{X} \) with no reticulations and such that each edge \( e \) has a unique length (\( |\Lambda(e)| = 1 \)), which we denote by \( \lambda(e) \). Below, we discuss the biological justification of various aspects of the definitions above.

Let \( v \) be a node with indegree 1 and outdegree 1 in a tree. Node \( v \) is said to be suppressible. Suppressing \( v \) means removing the in-edge \( e = (u,v) \) and the out-edge \( f = (v,w) \) and then creating a new edge \( g = (u,w) \) with length \( \lambda(g) = \lambda(e) + \lambda(f) \). Let \( N = (V,E,\varphi,\Lambda) \) be a network on \( \mathcal{X} \). A tree contained in \( N \) is a tree \( T = (V',E',\varphi',\lambda) \) on the same taxon set \( \mathcal{X} \) such that: (1) the roots of \( T \) and \( N \) coincide, (2) the nodes and edges of \( T \) are also nodes and edges of \( N \), that is \( V' \subseteq V \) and \( E' \subseteq E \), (3) taxon labels are unchanged, that is \( \varphi' = \varphi \), and (4) the edge lengths of \( T \) are also edge lengths of \( N \), that is, for every edge \( e \in E' \), \( \lambda(e) \in \Lambda(e) \). A tree displayed by \( N \) is a tree \( T' \) that can be obtained (up to isomorphism) by suppressing all suppressible nodes from a tree contained in \( N \). The set of trees displayed by \( N \) is denoted by \( \mathcal{T}(N) \). In Fig. 7, \( \mathcal{T}(N'_2) \) is the set of trees isomorphic to \( T'_1 \) and \( T'_2 \). Two networks \( N_1 \) and \( N_2 \) are said to be indistinguishable if they display the same set of trees, that is \( \mathcal{T}(N_1) = \mathcal{T}(N_2) \). For example, \( N_1 \) and \( N_2 \) in Fig. 3 are indistinguishable, as they display the same set of trees \( (T_1, T_2, T_3) \), up to isomorphism.

**Definition 1.** Given a network \( N \), a funnel is a node with indegree greater than 0 and outdegree 1. A funnel-free network, or canonical network, is a network that does not contain funnels. A canonical form of a network \( N \) is a network that is funnel-free and indistinguishable from \( N \).

In Fig. 3, \( N_1 \) and \( N_2 \) each contain two funnels, and thus are not funnel-free. The network \( N' \) in Fig. 4 is a canonical form of \( N_1 \) and \( N_2 \) in Fig. 3, as \( N' \) is funnel-free and indistinguishable from \( N_1 \) and \( N_2 \). Similarly, \( N'_2 \) in Fig. 6 is a canonical form of \( N_2 \). Note that nodes with indegree 1 and outdegree 1 are funnels. This implies that for trees the funnel-free condition coincides with the exclusion of suppressible nodes, which is a standard requirement in the definition of phylogenetic trees. It is thus appropriate to view the funnel-free condition as a natural extension of this requisite to networks.

**Definition 2.** A weighted path in a network \( N = (V,E,\varphi,\Lambda) \) is a pair \( (\pi,\lambda) \), where \( \pi \) is a directed path in the graph \( (V,E) \) and \( \lambda \) is a function that associates each edge \( e \) in \( \pi \) with a length \( \lambda(e) \in \Lambda(e) \). The length of a weighted path is the sum of the lengths assigned to its edges. A network satisfies the NELP (no equally long paths) property if no pair of distinct weighted paths having the same endpoints have the same length.

As we explain below, the NELP property is a mild condition to satisfy, unless edge lengths are taken to represent time. The following result states that if we restrict ourselves to networks satisfying the NELP
Theorem 1. (i) Every network $N$ has a canonical form. Moreover, (ii) if $N$ has the NELP property, then there exists a unique canonical form of $N$ among networks satisfying the NELP property (up to isomorphism).

(The notion of isomorphism between networks is only used for mathematical rigor and is defined in S1 Text.) The following result provides a necessary and sufficient condition for two networks satisfying the NELP property to be indistinguishable.

Corollary 1. Let $N_1$ and $N_2$ be networks with the NELP property and let $N'_1$ and $N'_2$ be their unique canonical forms satisfying the NELP property. Then $N_1$ and $N_2$ are indistinguishable if and only if $N'_1$ and $N'_2$ are the same network (up to isomorphism).

The following result states that a canonical network with the NELP property is uniquely determined by the trees it displays:

Corollary 2. Let $N$ be a canonical network satisfying the NELP property. Then $N$ is the unique (up to isomorphism) canonical network satisfying the NELP property that displays (all and only) the trees in $T(N)$.

We now discuss the biological significance of a number of technical aspects of our framework.

Definition of Networks and Trees Displayed by a Network

All the phylogenies considered here — trees or networks — are rooted. This is because we assume that the analysis uses an outgroup (possibly consisting of multiple taxa, and with no reticulations) for rooting. For simplicity, outgroup lineages are not included in our phylogenies (an exception to this is in Fig. 2). Note however that, because our phylogenies have edge lengths, and because omitting the outgroup is just a convention, the omitted lineages must have the same lengths for a network and all the trees it displays. For example, if we wish to omit the outgroup from $N_2$ in Fig. 2 and from the trees that it displays, then what we obtain are $N'_2, T'_1$ and $T'_2$ in Fig. 7. This has a notable consequence: the trees displayed by a rooted network with edge lengths may have a root with outdegree 1 (e.g. $T'_1$ in Fig. 7). For flexibility, we also allow a network to have a root with outdegree 1.

Moreover, we allow multiple lengths for an edge in a network, but not in a tree. For example, in Fig. 6, network $N'_3$ has an edge with two lengths ($\lambda_7 + \lambda_{12} + \lambda_{14}$ and $\lambda_7 + \lambda_{11} + \lambda_{13} + \lambda_{14}$). The motivation behind multiple lengths lies in the observation that, whereas each edge in a phylogenetic tree describing the evolution of non-reticulating organisms trivially corresponds to a unique evolutionary path in the underlying real evolutionary history, when reticulate events have occurred this is not necessarily true: Fig. 8 and Fig. 9 show that some evolutionary scenarios can either be represented using multiedges (multiple edges with the same endpoints) or edges with multiple lengths. Although these two options are mathematically equivalent, graphically the second one leads to more compact representations, and this is why we choose to allow multiple lengths rather than multiedges. For our purposes we only need to consider the case where $e$ has a finite set of lengths ($\Lambda(e) = \{\lambda_1(e), \ldots, \lambda_k(e)\}$).

Another unconventional aspect of our networks is the possibility of having nodes with in-degree and out-degree both greater than one. (See, e.g., the last common ancestor of $c$ and $d$ in $N'_2$ in Fig. 6.) Traditionally, the internal nodes in a phylogenetic network are constrained to belong to one of two different categories: reticulate nodes, with more than one incoming edge and just one outgoing edge, and speciation (or coalescence) nodes, with one incoming edge and multiple outgoing edges. Because reticulate and speciation events are clearly distinct, it is reasonable to constrain internal nodes to only fall in the two
Fig 8. A non-reticulating evolutionary history (left) and a reticulating evolutionary history (right). The black lineages are those leading to a sampled set of taxa $\mathcal{X}$. The horizontal jagged lines represent reticulation events. Whereas representing the scenario on the left with a phylogenetic tree on $\mathcal{X}$ is straightforward, for the one on the right several options are possible. We show three alternative representations in Fig. 9.

Fig 9. Alternative network representations for the evolutionary scenario in Fig. 8 (right). In our framework only $N_2$ and $N_3$ are networks.

categories above. In our framework, this requirement is dropped, and some networks, notably those in canonical form, may have nodes that both represent reticulate and speciation events. In this case, it is important to understand that these nodes represent a potentially complex (and unrecoverable) reticulate scenario, followed by one or more speciation events. Compare, for example, network $N$ and its canonical form $N'$ in Fig. 5, or $N_2$ and $N_3'$ in Fig. 6. (In the latter, it is especially instructive to consider the reticulate history above the direct ancestor of taxon $e$.)

The NELP Property

We use network $N_1$ of Fig. 3 to illustrate the NELP property. In $N_1$ there are three distinct weighted paths having as endpoints the root of $N_1$ and the direct ancestor of $b$. The lengths of these paths are
Fig 10. Different (non-isomorphic) but indistinguishable funnel-free networks. All edges are assumed to have the (unique) length 1 unless otherwise displayed. These networks do not satisfy the NELP property, showing that this is a necessary condition for the uniqueness of canonical forms (Theorem 1(ii)). The ellipsis at the end represents the fact that an infinite number of such networks can be obtained by adding any number of copies of the subgraph in grey in the last network.

\[ \ell_1 = \lambda_1 + \lambda_6, \quad \ell_2 = \lambda_2 + \lambda_3 + \lambda_5 + \lambda_8 \] and \[ \ell_3 = \lambda_2 + \lambda_{10} + \lambda_9 + \lambda_8. \] Moreover, there is another pair of paths having the same endpoints: those of lengths \[ \ell_4 = \lambda_3 + \lambda_5 \] and \[ \ell_5 = \lambda_{10} + \lambda_9. \] Thus \( N_1 \) has the NELP property if and only if the three numbers \( \ell_1, \ell_2 \) and \( \ell_3 \) are all different (note that this implies that also \( \ell_4 \) and \( \ell_5 \) are different). If edge lengths are taken to represent evolutionary change, rather than time, this is a very mild requirement: when edge lengths are drawn at random from a continuous distribution, the probability that two paths get exactly the same length is zero.

On the other hand, the NELP property does not hold for phylogenetic networks where edge lengths are taken to represent time. For these networks, canonical forms may not be unique (see Fig. 10 for an example of this). Even in this case, we believe that inference methods should only consider phylogenetic networks in their canonical form, as this allows to reduce the solution space without any loss in “expressive power”: since every network \( N \) has (at least one) canonical form that displays exactly the same set of trees — and therefore has the same fit with the data as \( N \) — restricting the solution space to canonical forms always leaves at least one optimal network within this space. The real weakness of using canonical forms in a molecular clock context is that if a canonical form is not unique, then it cannot be considered representative of all the networks indistinguishable from it. As an example of this, consider the indistinguishable networks in Fig. 10: none of these is representative of all the others.

**Discussion**

Our results are both negative and positive. The bad news is that any method that scores the fit between a network \( N \) and the available data — which may be sequences, distances, splits, trees (with or without edge lengths) — based on the set of trees displayed by \( N \) must face an important theoretical limitation: regardless of the amount of available data from the taxa under consideration, some parts of the network representing their evolutionary history may be impossible to recover — most notably the relative order of consecutive reticulate events (see, e.g., Fig. 3). The good news is that, when edge lengths are taken into account, we can set precise limits to what is recoverable: the canonical form of a network \( N \) is
a simplified version of $N$ that excludes all the unrecoverable aspects of $N$. In a canonical form, reticulate events are brought as forward in time as possible, causing the collapse of multiple consecutive nodes. (Compare network $N_2$ and its canonical form $N'_2$ in Fig. 6.) The importance of the canonical form $N'$ of a network $N$ lies in the fact that, if we restrict our consideration to networks with the NELP property, $N'$ is the unique canonical network consistent with perfect and unlimited data from the taxa in $N$.

There is an interesting analogy between soft polytomies in classical phylogenetics and collapsed nodes in a canonical network. Both represent lack of knowledge about the order of evolutionary events: speciations or more generally lineage splits in the first case, and reticulate events in the second. However, there is also an important difference between them: whereas in principle polytomies can be resolved by collecting further data from the taxa in the tree (for example, by extensive sequencing of their genomes [52]), the standard network inference methods considered here cannot resolve collapsed nodes in a canonical network, irrespective of the amount of data from the taxa under consideration. This difference is mitigated by the observation that increased taxon sampling may indeed permit to resolve the collapsed nodes, when the new lineages break adjacencies between reticulate nodes. However, such lineages may not always exist or they may be difficult to sample.

The present work has several consequences that should be of interest both to the biologists concerned by the use of methods for phylogenetic network inference, and to the researchers interested in the development of these methods. We illustrate these consequences starting from a well-known problem of network inference methods, that of multiple optima. It has been noted before that many of the inference methods that have been recently proposed — especially those solely based on topological features — often return multiple optimal networks: Huson and Scornavacca show a striking example of this (Fig. 2 in [53]), where the problem of finding the simplest network displaying two given tree topologies admits at least 486 optimal solutions.

The existence of multiple optimal networks for a given data set is essentially due to two reasons: insufficient data and non-identifiability. For the example of 486 optimal solutions, this large number may be partly due to the fact that the goal was to achieve consistency with only two tree topologies. More data may enable to discriminate among the 486 returned networks. Non-identifiability, which occurs when none of the allowed data can discriminate between two or more networks, is a more serious problem than insufficient data, as it cannot be solved by simply increasing the size of the input sample. Another interesting example appears in a paper by Albrecht et al. [54], which we reproduce here in Fig. 11. Here, there are only three optimal networks, essentially differing for which of the three clades $\{A.bicornis, A.longissima, A.sharonensis\}$, $\{A.uniariistata, A.comosa\}$ and $\{A.tauschii\}$ is considered as a hybrid (in this example reticulations represent hybridizations). This pattern is entirely analogous to that of the three networks in Fig. 2 (with a,b and c replaced by the three clades above), meaning that these three networks are indistinguishable to methods not accounting for edge lengths. Therefore, in this example, the existence of multiple optimal solutions is entirely due to non-identifiability.

All this motivates three recommendations:

1. It is important to use data in a way that causes non-identifiability to be as limited as possible. For example, as we have seen, accounting for edge lengths solves some cases of non-identifiability (e.g., in Fig. 2) although it does not eliminate this problem altogether (e.g., in Fig. 3).

2. Given an inferred network $\hat{N}$, it is important to know the set of networks that are theoretically impossible to distinguish from $\hat{N}$: no matter the amount of data, they will all receive the same support as $\hat{N}$. We may call this set the indistinguishable class of $\hat{N}$. The biologist using an inference method must be aware that $\hat{N}$ is not the only network supported by the data.

3. It would be highly useful to devise inference methods that instead of searching for (or directly constructing) solutions in the space of all possible networks, only considers one element per indistinguishable class. This has the potential to significantly speed up the inference.
Correspondingly, we recommend that edge lengths should be accounted for in the analyses (point 1) and, for each of the indistinguishable classes resulting from this choice, we identify a canonical network that, for all practical purposes, can be considered to be unique. Most important to the end users, we propose that a canonical network $\hat{N}$ is what should be given as the result of the inference, with the caveat that $\hat{N}$ is a way to represent a class of networks that are all equally supported (point 2). In a canonical form $\hat{N}$, the aspects that are not common to all networks in this class are collapsed, as described above. This will help the evolutionary biologist to locate the uncertainties in the phylogeny, and possibly to choose further taxa to resolve them. Finally, we propose that inference methods only attempt to search among — or construct — phylogenetic networks in their canonical form (point 3).

We note that accounting for yet more characteristics of the data may reduce (or eliminate altogether) the identifiability issues for phylogenetic networks. In the case of sequence-based methods, one may take into account the natural order of sites within a sequence [11–13, 55, 56]. Similarly, for reconstruction methods based on collections of subtrees, one could observe and use the relative position of the different genomic regions supporting the input trees. However, these relative positions must be conserved across the genomes being analyzed, a condition which may hold for recombining organisms (e.g. individuals within a population or different viral strains), but which is not obvious when studying a group of taxa that have undergone reticulate events (e.g., hybridization) at some point in a distant past.

The main conclusion of the present study is the following: unless one abandons any optimization criterion that scores a network solely based on the trees it displays, the reconstruction should be carried out in a reduced space of networks: that of the canonical forms defined here. The motivation for this lies in the fact that canonical networks are guaranteed to be uniquely determined, if sufficient data are available. Once a canonical form $\hat{N}$ is inferred, it must be kept in mind that even assuming that the inference is free of statistical error, the true phylogenetic network is just one of the many networks having $\hat{N}$ as canonical form. Compared to what biologists are used to for phylogenetic trees — where in principle it is always possible to resolve uncertainties — it is clear that this requires an important change of perspective.
Methods

The following three subsections describe the proofs of Theorem 1 part (i), of Theorem 1 part (ii), and of their corollaries, respectively. In the case of Theorem 1 part (ii), only the gist of the proof is provided here. The proof in full detail is deferred to S1 Text.

Reduction Algorithm

In order to prove that any network \( N \) has a canonical form, we describe an algorithm to transform \( N \) into a canonical network indistinguishable from \( N \). The algorithm simply consists of repeatedly applying to \( N = (V,E,\varphi,\Lambda) \) one of the following two reduction rules, until neither can be executed (see Fig. 12):

**Funnel suppression (R1)**

Given a funnel \( v \) with \( k \geq 1 \) in-edges \((u_1,v), (u_2,v), \ldots, (u_k,v)\) and out-edge \((v,w)\), remove \( v \) and all these edges from \( N \) and introduce \( k \) new edges \((u_1,w), (u_2,w), \ldots, (u_k,w)\).

For all \( i \in \{1,2,\ldots,k\} \) assign to \((u_i,w)\) the lengths \( \Lambda((u_i,w)) := \Lambda((u_i,v)) + \Lambda((v,w)) \), where the sum of two sets of numbers \( A \) and \( B \) is defined as \( A + B = \{a + b : a \in A, b \in B\} \).

**Multiedge merging (R2)**

Given a collection of multi-edges \((u,w)\) with multiplicity \( k \) and lengths \( \Lambda'_1, \Lambda'_2, \ldots, \Lambda'_k \), replace these edges with a single edge with lengths \( \bigcup_{i=1}^{k} \Lambda'_i \).

An example of the reduction of a network to its canonical form is shown in Fig. 13. Note that, even if the algorithm may temporarily produce multi-edges, the network produced in the end obviously does not have any multi-edge (otherwise we could still apply rule R2).

**Proof of part (i) of Theorem 1.** We must prove that any network \( N = (V,E,\varphi,\Lambda) \) has a canonical form. For this, we apply the reduction algorithm described above, thus obtaining a sequence \( N_0 = N, N_1, \ldots, N_m \), where each \( N_{i+1} \) is obtained from \( N_i \) by applying either R1 or R2. Neither R1 nor R2 can be applied to \( N_m \). We prove that \( N_m \) is a canonical form of \( N \). Although, strictly speaking, \( N_i \) may not be a network (as it potentially contains multi-edges), the notion of trees displayed by \( N_i \), and thus that of indistinguishability, trivially extends to these multigraphs.

First, note that the algorithm terminates after a finite number of iterations (\( m \)). This is true because at each iteration the size of \( E \) is reduced by at least one. Moreover, the resulting network \( N_m \) is funnel-free, since no reduction of type R1 can be applied to it.
Fig 13. Reduction of a network to its canonical form. Gray edges are those to which the next reduction rule is applied. All edges are assumed to have the (unique) length 1 unless otherwise displayed.

What is left to prove is that $N_m$ is indistinguishable from $N = N_0$. To this end we prove that, at each iteration, $N_i$ and $N_{i+1}$ are indistinguishable, i.e. $T(N_i) = T(N_{i+1})$. In other words any tree $T$ is displayed by $N_i$ if and only if $T$ is displayed by $N_{i+1}$.

Let $T$ be displayed by $N_i$. Then $T$ can be obtained by suppressing all suppressible nodes from a tree $T_i$ contained in $N_i$. We consider three cases. (1) If none of the edges in $T_i$ is involved in the reduction transforming $N_i$ into $N_{i+1}$, then clearly $T_i$ is still contained in $N_{i+1}$ and thus $T$ is still displayed by $N_{i+1}$.

(2) If $T_i$ is involved in a R1 reduction, then it contains a funnel $v$ and it contains one of the in-edges of the funnel, say $(u_j, v)$, with length $\lambda_j \in \Lambda_j = \Lambda((u_j, v))$, along with the out-edge $(v, w)$, with length $\lambda_0 \in \Lambda_0 = \Lambda((v, w))$. Now, let $T_{i+1}$ be the tree obtained from $T_i$ by suppressing the suppressible node $v$ and thus creating a new edge $(u_j, w)$ with length $\lambda_j + \lambda_0$. Because the R1 reduction creates a new edge $(u_j, w)$ with length set $\Lambda_j + \Lambda_0$, containing the value $\lambda_j + \lambda_0$, then $T_{i+1}$ is contained in $N_{i+1}$. Moreover, it easy to see that $T$ can still be obtained by suppressing all suppressible nodes from $T_{i+1}$. Thus $T$ is still displayed by $N_{i+1}$. (3) If $T_i$ is involved in a R2 reduction, then it contains one of the edges of a multi-edge $(u, w)$, with a length $\lambda$ belonging to one of the length sets $\Lambda'_1, \Lambda'_2, \ldots, \Lambda'_k$ associated to the $k$ copies of $(u, w)$. Thus we have that $\lambda \in \bigcup_{i=1}^{k} \Lambda'_i$, which implies that $T_i$ is still contained in $N_{i+1}$ and thus $T$ is still displayed by $N_{i+1}$. This concludes the proof of $T(N_i) \subseteq T(N_{i+1})$.

In order to prove that, conversely, $T(N_{i+1}) \subseteq T(N_i)$, one can proceed in a similar way as above: if $T$ is displayed by $N_{i+1}$, then $T$ can be obtained by suppressing all suppressible nodes from a tree $T_{i+1}$ contained in $N_i$. By considering three cases analogous to the ones above regarding the involvement of $T_{i+1}$ in the reduction transforming $N_i$ into $N_{i+1}$, we can prove that in all these cases $T$ is already displayed by $N_i$. Thus $N_i$ and $N_{i+1}$ are indistinguishable, which concludes our proof. □

We note informally that the order of application of the possible reductions in the algorithm above is irrelevant to the end result. To see this, it suffices to show that if two different reductions are applicable
to a network, then the result of applying them is the same irrespective of the order of application. As we do not need this remark for the other results in this paper, we do not give a formal proof of it.

Lemma 1. Let \( N \) be a network and \( N' \) a canonical form of \( N \) obtained by applying the reduction algorithm. If \( N \) satisfies the NELP property, then \( N' \) satisfies the NELP property.

Proof. We prove that for each basic step of the reduction algorithm — transforming \( N_i \) into \( N_{i+1} \) via a reduction rule R1/R2 — if \( N_i \) satisfies the NELP property, then \( N_{i+1} \) also satisfies it. Suppose the contrary; then, \( N_{i+1} \) contains two distinct weighted paths \( \rho_1,\rho_2 \) with the same endpoints \( u \) and \( v \) and same lengths. Because R1/R2 cannot create new nodes, \( u \) and \( v \) are also nodes in \( N_i \). Moreover, it is easy to see that each weighted path \( \rho \) in \( N_i \) from \( u \) to \( v \) gives rise to exactly one weighted path \( f(\rho) \) in \( N_{i+1} \) from \( u \) to \( v \), with exactly the same length as \( \rho \). Now take two weighted paths in \( N_i \), one in the preimage \( f^{-1}(\rho_1) \) and the other in the preimage \( f^{-1}(\rho_2) \). These two weighted paths in \( N_i \) are distinct (as \( \rho_1 \neq \rho_2 \)), have the same endpoints (\( u \) and \( v \)) and the same length. But then \( N_i \) violates the NELP property, leading to a contradiction. We thus have that if \( N_2 \) satisfies the NELP property, then \( N_{i+1} \) also satisfies it. By iterating the argument above for each step in the reduction algorithm, the lemma follows.

Uniqueness of the Canonical Form for Networks Satisfying the NELP

The proof of Theorem 1, part (ii), is rather technical. In this section, we introduce a number of new concepts and state the main intermediate results that are necessary to obtain this result. We leave their detailed proofs to S1 Text, together with the obvious definitions of basic concepts such as that of isomorphic networks, sub-network and union of two networks.

Definition 3. (Root-leaf path, prefix, postfix, wishbone, crack.) Let \( N \) be a network on \( X \) and \((\pi,\lambda)\) be a weighted path in \( N \) from the root of \( N \) to a leaf labelled by \( x \in X \). Now consider the sub-network \( P = (V(\pi),E(\pi),\varphi|_{x},\lambda) \) on \( \{x\} \) consisting of all the nodes and edges in \( \pi \) and associated labels. Any sub-network of \( N \) such as \( P \) is called a root-leaf path of \( N \). Given a root-leaf path \( P \) and a node \( v \) belonging to it, any weighted path formed by all the ancestors [descendants] of \( v \) in \( P \) is a prefix [suffix] of \( P \). Note that a prefix [suffix] only consists of one node when \( v \) is the root [leaf] of \( P \). A wishbone of \( N \) is any sub-network of \( N \) formed by taking the union of two root-leaf paths that have in common only a prefix. A crack of \( N \) is any sub-network of \( N \) formed by taking the union of two root-leaf paths that have in common only a prefix and a suffix.

Fig. 14 illustrates the definitions above. Note that any root-leaf path \( P \) is both a wishbone and a crack, as \( P \) is the result of the union of \( P \) with itself, and \( P \) has a common prefix and a common suffix with \( P \). Moreover, any sub-network \( R \) that can be obtained from a root-leaf path by attributing two lengths to one of its edges \( e \) is a crack. Finally, note that wishbones and cracks are networks, and thus the notion of isomorphism (Definition 5 in S1 Text) can be applied to them.

The proof of part (ii) in Theorem 1 depends on two important results (Propositions 1 and 2 below), whose proofs can be found in S1 Text. The first states that a network with the NELP property is uniquely determined by the wishbones and cracks it contains.

Proposition 1. Two networks \( N_1 \) and \( N_2 \) with the NELP property are isomorphic if and only if they contain the same wishbones and cracks (up to isomorphism).

Proposition 1 is interesting on its own as it suggests an enumerative algorithm to verify whether two networks with the NELP property are isomorphic. Unfortunately this algorithm would be impractical, as the number of wishbones (or cracks) in a network is not polynomial in the size of the network. Also note that we require \( N_1 \) and \( N_2 \) to satisfy the NELP property because there exist non-isomorphic networks containing the same wishbones and cracks: for example the networks in the bottom line of Fig. 10. The second result that we need is the following:
Fig 14. Illustration of Definition 3. P (edges in black) is a root-leaf path of $N$ and thus both a wishbone and a crack of $N$. R and S (black) are cracks of $N$. Q (black) is a wishbone of $N$. All edges are assumed to have the (unique) length 1 unless otherwise displayed.

**Proposition 2.** Let $N_1$ and $N_2$ be two indistinguishable funnel-free networks, satisfying the NELP property. Then they contain the same wishbones and cracks (up to isomorphism).

Proof of part (ii) of Theorem 1. Let $N$ be a network with the NELP property and $N'$ a canonical form of $N$ obtained by applying the reduction algorithm. By Lemma 1, $N'$ satisfies the NELP property. Now suppose that there exists another canonical form of $N$, called $N''$, satisfying the NELP property. By transitivity, $N'$ and $N''$ are indistinguishable. Because $N'$ and $N''$ are indistinguishable, funnel-free and with the NELP property, $N'$ and $N''$ must contain the same wishbones and cracks (because of Proposition 2). But then, because of Proposition 1, $N'$ and $N''$ are isomorphic. □

We note that some of our arguments in S1 Text lead us to conjecture that a funnel-free network satisfying the NELP property cannot be indistinguishable from a funnel-free network violating the NELP property. This claim would allow us to simplify the statement of Theorem 1: networks with the NELP property would be guaranteed to have a unique canonical form (not just among networks with the NELP property, but among all networks). Unfortunately, to this date, we were unable to prove this conjecture. Nonetheless, note that the reduction algorithm returns, for any network with the NELP property, its unique canonical form with the NELP property (by Lemma 1).

**Corollaries**

It remains to prove the two corollaries at the end of the Results section. The first one states that two networks $N_1$ and $N_2$ satisfying the NELP property are indistinguishable if and only if their unique canonical forms with the NELP property, $N'_1$ and $N'_2$ respectively, are isomorphic. By Lemma 1, $N'_1$ and $N'_2$ can be obtained by applying the reduction algorithm to $N_1$ and $N_2$.

Proof of Corollary 1. The if part trivially follows from the transitivity of indistinguishability. As for the only if part, note that (again by transitivity) $N'_1$ is indistinguishable from $N_2$. As it is also funnel-free, $N'_1$ is a canonical form of $N_2$. Because $N_2$ can only have one canonical form satisfying the NELP property (by Theorem 1(ii)), $N'_1$ and $N'_2$ must be the same network (up to isomorphism). □

As for Corollary 2, we recall that it states that a canonical network $N$ with the NELP property is uniquely determined by the trees it displays.
Proof of Corollary 2. Let $N$ and $N'$ be indistinguishable canonical networks satisfying the NELP property. Then, $N$ and $N'$ are both canonical forms of $N$ satisfying the NELP. But then, by Theorem 1(ii), $N$ and $N'$ must be the same network (up to isomorphism).

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References


**Supporting Information Legends**

**S1 Text.** Supporting Information: a mathematical theory of explicit phylogenetic networks with edge lengths. This document provides an introduction to the mathematical theory of explicit phylogenetic networks with edge lengths, leading in particular to the proofs of Propositions 1 and 2, which are necessary for the proof of Theorem 1, part (ii). In the last section, we consider networks with inheritance probabilities and their relevance for likelihood-based reconstruction.
A mathematical theory of explicit phylogenetic networks with edge lengths.

This document is structured in six sections, in which we develop a theory of phylogenetic networks with edge lengths. The first section introduces the notion of isomorphism between such networks and states some obvious propositions; the second section looks in more detail at the process whereby a network displays a tree; the third shows characterizations for both the NELP and the funnel-free property; the fourth and fifth sections derive the proofs of the two propositions that are necessary for the proof of Theorem 1, part (ii). The last section shows an example relevant to likelihood frameworks modelling inheritance probabilities, in addition to edge lengths. The notation and definitions introduced in the Results section and in the Methods section within the paper will be used here. We recall in particular that, throughout this paper, networks are rooted DAGs whose leaves are bijectively labeled by taxa and whose edges have a finite set of strictly positive lengths.

**Isomorphisms and sub-networks**

**Definition 4.** Given a network $N = (V, E, \varphi, \Lambda)$, a sub-network $N' = (V', E', \varphi', \Lambda')$ of $N$ is any 4-tuple such that: (a) $N'$ is a network, (b) $V' \subseteq V$, (c) $E' \subseteq V'^2 \cap E$, (d) $\varphi'$ is the restriction of $\varphi$ to the taxa associated to the leaves of $(V', E')$ and, (e) for every edge $e \in E'$, $\Lambda'(e) \subseteq \Lambda(e)$. The union $N_1 \cup N_2$ of two sub-networks of $N$, $N_1 = (V_1, E_1, \varphi_1, \Lambda_1)$, $N_2 = (V_2, E_2, \varphi_2, \Lambda_2)$, having the same root, is the sub-network $N' = (V_1 \cup V_2, E_1 \cup E_2, \varphi', \Lambda')$, where $\varphi'$ is the restriction of $\varphi$ to the taxa associated to the leaves of $(V_1 \cup V_2, E_1 \cup E_2)$ and for every edge $e \in E_1 \cup E_2$, $\Lambda'(e) = \Lambda_1(e) \cup \Lambda_2(e)$ (where we take the liberty to let $\Lambda_i(e) = \emptyset$ whenever $e \notin E_i$).

Note that we define the union for sub-networks of $N$ sharing the same root, because this ensures that such union is still a network. Although these requirements could be relaxed, the definition above is sufficient for the purposes of the current paper.

**Definition 5.** Let $N_1 = (V_1, E_1, \varphi_1, \Lambda_1)$ and $N_2 = (V_2, E_2, \varphi_2, \Lambda_2)$ be two networks on $\mathcal{X}$. $N_1$ and $N_2$ are isomorphic if there exists a bijection $f : V_1 \to V_2$ (called an isomorphism) such that:

(i) for every $x \in \mathcal{X}$, $f(\varphi_1(x)) = \varphi_2(x)$;

(ii) for every pair of nodes $(u, v) \in V_1^2$, $(u, v) \in E_1 \iff (f(u), f(v)) \in E_2$ and whenever both these edges exist, $\Lambda_1((u, v)) = \Lambda_2((f(u), f(v)))$.

The following four lemmas are trivially true, and for brevity we do not include their proofs here.

**Lemma 2.** Let $N_1$ and $N_2$ be isomorphic networks. Then every sub-network of $N_1$ is isomorphic to some sub-network of $N_2$.

**Definition 6.** Let $P$ be a root-leaf path of a network $N$. The depth of a node $v$ in $P$ is the length of the weighted path in $P$ from the root of $P$ to $v$. We say that $P$ is to $x$, if $P$ is a network on the set $\{x\}$, that is, if its only leaf is labelled by taxon $x$.

**Lemma 3.** A root-leaf path $P_1$ is isomorphic to $P_2$ if and only if (a) $P_2$ is a root-leaf path to the same taxon as $P_1$, and (b) for every node $v_i$ in any of the two root-leaf paths, say $P_i$, there exist a node $v_j$ in the other root-leaf path, $P_j$, such that the depth of $v_i$ in $P_1$ is equal to the depth of $v_j$ in $P_j$.

**Lemma 4.** A wishbone $W = P_1 \cup P_2$ is isomorphic to $W'$ if and only if $W'$ is a wishbone and can be written as $W' = P'_1 \cup P'_2$, so that (a) the longest common prefix of $P_1$ and $P_2$ has the same length as the longest common prefix of $P'_1$ and $P'_2$, and (b) $P'_1$ and $P'_2$ are isomorphic to $P_1$ and $P_2$, respectively.

**Lemma 5.** A crack $K = P_1 \cup P_2$ is isomorphic to $K'$ if and only if $K'$ is a crack and can be written as $K' = P'_1 \cup P'_2$, so that (a) the longest common prefix of $P_1$ and $P_2$ has the same length as the longest common prefix of $P'_1$ and $P'_2$, (b) the longest common suffix of $P_1$ and $P_2$ has the same length as the longest common suffix of $P'_1$ and $P'_2$, and (c) $P'_1$ and $P'_2$ are isomorphic to $P_1$ and $P_2$, respectively.
Figure S1. Illustration of the definitions of switching of a network, tree contained in a network, tree displayed by a network and embeddings. $N$ is a network, $S_1$, $S_2$ and $S_3$ are three of its switchings, $T_1$ and $T_2$ are two trees contained in $N$, $T$ is a tree displayed by $N$, and $T_1$ and $T_2$ are two embeddings of $T$ in $N$. Nodes are explicitly shown in $S_1$, $S_2$, $S_3$, $T_1$ and $T_2$ for clarity. Unless otherwise shown, all edges are assumed to have length 1.

Switchings, trees weakly displayed and embeddings

Definition 7. Let $N = (V, E, \varphi, \Lambda)$ be a network. A switching $S = (V, E', \varphi, \lambda)$ of $N$ is obtained from $N$ by doing the following:

(i) for each node $v \in V$, delete all incoming edges of $v$ except one; let $E'$ be the resulting set of edges;
(ii) for each edge $e \in E'$, assign to $e$ a single length $\lambda(e) \in \Lambda(e)$.

Note that technically a switching is not a network, as action (i) above may create leaves in $S$ that are not labelled by any taxon. Biologically, a switching corresponds to the evolutionary tree describing the history of a single (indivisible and thus non-recombining) character carried by the root of $N$; some lineages in this tree may never reach any leaf of $N$. Moreover, since each edge in this tree corresponds to a unique path in the underlying real evolutionary history, it is clear why a switching only allows one length per edge (see Fig. 8 in the main text). Note that the trees contained by a network (defined in the Results section within the paper) can also be seen as the trees that can be obtained from a switching by removing all nodes and edges that have no descendant labelled by a taxon.

Definition 8. Let $N$ be a network. A tree weakly contained in $N$ is a tree $T$ that is a sub-network of $N$ and has the same root as $N$. A tree weakly displayed by $N$ is any tree $T$ can be obtained (up to isomorphism) by suppressing all suppressible nodes from a tree $T'$ weakly contained in $N$. Tree $T'$ is called an embedding of $T$ in $N$. The set of trees weakly displayed by $N$ is denoted by $\tilde{T}(N)$.

The difference with the definition of trees displayed by a network is that a network on $\mathcal{X}$ can only display trees on $\mathcal{X}$, whereas it can weakly display any tree on $\mathcal{X}'$, with $\emptyset \subset \mathcal{X}' \subseteq \mathcal{X}$. Note that if a tree $T$ is displayed by a network $N$, then $T$ is also weakly displayed by $N$. The definitions of switching of a network, tree contained in a network, tree displayed by a network and embeddings are illustrated in Fig. S1. We note that every switching of a network $N$ gives rise to a unique tree contained in $N$ and...
every tree contained in \( N \) gives rise to a unique tree displayed by \( N \). However, the converses of these two propositions are not true: several switchings of \( N \) can give rise to the same tree displayed by \( N \). The latter means that \( T \in \mathcal{T}(N) \) can have several embeddings in \( N \).

**Lemma 6.** Let \( T \) be a tree on \( \mathcal{X} = \{x_1, x_2, \ldots, x_n\} \) with no suppressible nodes. Let \( P_1, P_2, \ldots, P_n \) be the root-leaf paths in \( T \) to \( x_1, x_2, \ldots, x_n \), respectively. For all \( i, j \in \{1, 2, \ldots, n\} \), let \( \lambda_i \) be the length of \( P_i \) and let \( \lambda_{ij} \) be the length of the longest common prefix of \( P_i \) and \( P_j \). Any embedding of \( T \) is the union of \( n \) root-leaf paths \( P'_1, P'_2, \ldots, P'_n \) to \( x_1, x_2, \ldots, x_n \), such that for all \( i, j \in \{1, 2, \ldots, n\} \), \( P'_i \) has length \( \lambda_i \), \( P'_i \cup P'_j \) is a wishbone and the length of the longest common prefix of \( P'_i \) and \( P'_j \) is \( \lambda_{ij} \).

**Proof.** Let \( T_e \) be an embedding of \( T \). By definition of embedding, suppressing all suppressible nodes in \( T_e \) gives rise to a tree \( T' \) isomorphic to \( T \). Because suppressing nodes does not change the taxon set of a tree, \( T_e \) is a tree on the same taxa as \( T' \), and thus on the same taxa as \( T \), that is \( \mathcal{X} = \{x_1, x_2, \ldots, x_n\} \). Because \( T_e \) is a tree on \( \mathcal{X} \), \( T_e \) equals the union of its \( n \) root-leaf paths to \( x_1, x_2, \ldots, x_n \) — which we call \( P'_1, P'_2, \ldots, P'_n \) respectively — and each pair of these paths have nothing in common other than a prefix, meaning that \( P'_i \cup P'_j \) is a wishbone. Now let \( \lambda'_i \) denote the length of \( P'_i \) and \( \lambda'_{ij} \) denote the length of the longest common prefix of \( P'_i \) and \( P'_j \). It remains to prove that \( \lambda'_i = \lambda_i \) and \( \lambda'_{ij} = \lambda_{ij} \), for all \( i, j \in \{1, 2, \ldots, n\} \).

Because suppressing nodes does not change the length of a root-leaf path nor the taxon labelling its leaf, then the root-leaf path to \( x_i \) in \( T' \), which is derived from \( P'_i \), must have length \( \lambda'_i \). But because \( T \) and \( T' \) are isomorphic, then they contain the same root-leaf paths, up to isomorphism (by Lemma 2 and Lemma 3). The (unique) root-leaf paths to \( x_i \) in \( T \) and \( T' \) are thus isomorphic and have therefore the same length, that is, \( \lambda_i = \lambda'_i \).

Similarly, because suppressing nodes transforms a wishbone in another wishbone, without changing the length of the longest prefix common to its two root-leaf paths, nor the taxon labelling their leaves, then the root-leaf paths to \( x_i \) and \( x_j \) in \( T' \), which are derived from \( P'_i \) and \( P'_j \), respectively, must form a wishbone \( W' \), with a longest common prefix of length \( \lambda'_{ij} \). Because \( T \) and \( T' \) are isomorphic, then they contain the same wishbones, up to isomorphism (by Lemma 2 and Lemma 4), meaning that \( W' \) and \( P_i \cup P_j \) must be isomorphic. By Lemma 4, the longest common prefix of \( P_i \) and \( P_j \) must then have length \( \lambda_{ij} = \lambda'_{ij} \). \( \square \)

**Lemma 7.** Let \( T, T' \) denote trees and \( N, N' \) denote networks.

(a) If \( N \) and \( N' \) are isomorphic, then \( \mathcal{T}(N) = \mathcal{T}(N') \).

(b) If \( T' \) is obtained by suppressing all suppressible nodes from \( T \), then \( \mathcal{T}(T) = \mathcal{T}(T') \).

(c) If \( T' \) is an embedding of \( T \), then \( \mathcal{T}(T) = \mathcal{T}(T') \).

(d) If \( T \) is weakly contained in \( N \), then \( \mathcal{T}(T) \subseteq \mathcal{T}(N) \).

(e) If \( T \) is weakly displayed by \( N \), then \( \mathcal{T}(T) \subseteq \mathcal{T}(N) \).

**Proof.** (a) Let \( T \) be a tree weakly contained in \( N \). Let \( f \) be an isomorphism between \( N \) and \( N' \). It is easy to see that the restriction of \( f \) to the nodes of \( T \) defines an isomorphism between \( T \) and a tree \( T' \) that is weakly contained in \( N' \), meaning that \( N \) and \( N' \) must weakly contain the same trees (up to isomorphism). Therefore \( N \) and \( N' \) must display the same trees. (b) Let \( T \) and thus \( T' \), be trees on \( \mathcal{Y} \). Given a nonempty subset \( \mathcal{X} \subseteq \mathcal{Y} \), let \( T_X \) be the (unique) tree on \( \mathcal{X} \) weakly contained in \( T \), and let \( T'_X \) be the (unique) tree on \( \mathcal{X} \) weakly contained in \( T' \). It is easy to see that \( T'_X \) can be obtained by suppressing some suppressible nodes from \( T_X \). It follows that the tree obtained by suppressing all suppressible nodes from \( T_X \) is the same as that obtained by suppressing all suppressible nodes from \( T_X \). This means that a tree on \( \mathcal{X} \) is weakly displayed by \( T \) if and only if it is weakly displayed by \( T' \). Since this is true for any nonempty \( \mathcal{X} \subseteq \mathcal{Y} \), point (b) follows. (c) By definition, if \( T' \) is an embedding of \( T \), then there exist a tree \( T_i \), isomorphic to \( T \), that can be obtained by suppressing all suppressible nodes from \( T' \). But then, by point (a), \( \mathcal{T}(T) = \mathcal{T}(T_i) \) and, by point (b), \( \mathcal{T}(T_i) = \mathcal{T}(T') \). By transitivity, \( \mathcal{T}(T) = \mathcal{T}(T') \). (d) If \( T \) is weakly contained in \( N \), then every tree weakly contained in \( T \) is also weakly contained in \( N \). Therefore
every tree weakly displayed by $T$ is also weakly displayed by $N$. (e) If $T$ is weakly displayed by $N$, then by definition there exist an embedding $T'$ of $T$ in $N$. But then, by point (c), $\tilde{T}(T) = \tilde{T}(T')$, and, because $T'$ is weakly contained in $N$, $\tilde{T}(T') \subseteq \tilde{T}(N)$ (by point(d)). By transitivity, $\tilde{T}(T) \subseteq \tilde{T}(N)$.

Lemma 8. For every tree $T_{\text{wc}}$ weakly contained in a network $N$, there exists a tree $T_e$ contained in $N$ that weakly contains $T_{\text{wc}}$.

Proof. Let $N$ be a network on $\mathcal{X}$ and $T_{\text{wc}}$ be a tree on $\mathcal{X}' \subseteq \mathcal{X}$. Number the taxa in $\mathcal{X} \setminus \mathcal{X}'$ so that we have $\mathcal{X} \setminus \mathcal{X}' = \{x_1, x_2, \ldots, x_{|\mathcal{X}| - |\mathcal{X}'|}\}$. For convenience of notation, let $T_0 = T_{\text{wc}}$. For every $i \in \{1, 2, \ldots, |\mathcal{X}| - |\mathcal{X}'|\}$, we now show how to define $T_i$ on $\mathcal{X}' \cup \{x_1, \ldots, x_i\}$ that weakly contains $T_{i-1}$ and is weakly contained in $N$. Let $P_i$ be a root-leaf path in $N$ composed by the edges traversed by the following walk from the leaf labelled by $x_i$ to the root in $N$: from the leaf labelled by $x_i$ always take an edge $e$ of $N$ in its inverse direction (from its head to its tail), with any of $e$'s associated lengths, until you end up in a node $v$ belonging to $T_{i-1}$: from then on, follow the (inverse) path from $v$ to the root of $T_{i-1}$ (which coincides with that of $N$). By construction, $P_i$ and $T_{i-1}$ share a common prefix and nothing else. If we now define $T_i$ as the union of $T_{i-1}$ and $P_i$, it is clear that $T_i$ is a tree on $\mathcal{X}' \cup \{x_1, \ldots, x_i\}$ that weakly contains $T_{i-1}$ and is weakly contained in $N$. Now consider the sequence of trees $T_{\text{wc}} = T_0, T_1, \ldots, T_{|\mathcal{X}| - |\mathcal{X}'|}$. Each of these trees is weakly contained in $N$, and (because the relation of weak containment is transitive) weakly containing all its predecessors. Because $T_{|\mathcal{X}| - |\mathcal{X}'|}$ is a tree on $\mathcal{X}$ weakly contained in $N$, $T_{|\mathcal{X}| - |\mathcal{X}'|}$ is contained in $N$ and weakly contains $T_{\text{wc}}$, thus concluding the proof.

Proposition 3. Let $N$ and $N'$ be networks. Then they are indistinguishable if and only if they weakly display the same trees.

Proof. The if part is trivial. Suppose $N$ and $N'$ weakly display the same trees. Then they must be networks on the same taxon set $\mathcal{X}$, otherwise the network on the larger set of taxa, say, $N$, would weakly display trees with taxa that can never be present in trees weakly displayed by $N'$. Because $N$ and $N'$ weakly display the same trees on all taxon subsets $\mathcal{X}' \subseteq \mathcal{X}$, then they also display the same trees on $\mathcal{X}$, that is, they are indistinguishable.

As for the only if part, we prove that, assuming $N$ and $N'$ are indistinguishable, $\tilde{T}(N) = \tilde{T}(N')$. We just prove $\tilde{T}(N) \subseteq \tilde{T}(N')$, as the proof of $\tilde{T}(N') \subseteq \tilde{T}(N)$ is symmetric. Let $T \in \tilde{T}(N)$ be a tree weakly displayed by $N$, and let $T_e$ be an embedding of $T$ in $N$. Because $T_e$ is weakly contained in $N$, by Lemma 8 there exists a tree $T_{\text{wc}}$ contained in $N$ that weakly contains $T_e$. But then, by points (c) and (d) in Lemma 7,

$$\tilde{T}(T) = \tilde{T}(T_e) \subseteq \tilde{T}(T_e). \quad (1)$$

Now let $T_d$ be the tree obtained by suppressing all suppressible nodes in $T_e$. Because $T_e$ is contained in $N$, then $T_d$ is displayed by $N$ and thus by $N'$, given that $N$ and $N'$ are indistinguishable. Therefore, by points (b) and (c) in Lemma 7,

$$\tilde{T}(T_e) = \tilde{T}(T_d) \subseteq \tilde{T}(N'). \quad (2)$$

By putting together relations (1) and (2), we have that $T \in \tilde{T}(T) \subseteq \tilde{T}(N')$. Because this holds for any $T \in \tilde{T}(N)$, we conclude $\tilde{T}(N) \subseteq \tilde{T}(N')$.

Corollary 3. Let $N$ and $N'$ be indistinguishable networks. Then, for every root-leaf path $P$ in $N$, there exists in $N'$ an equally long root-leaf path to the same taxon as $P$.

Proof. Suppose there exists in $N$ a root-leaf path to $x$ of length $\lambda$. By suppressing all suppressible nodes from this path, one obtains a tree $T$ consisting of one edge of length $\lambda$, whose head is labelled by $x$. Because $N$ and $N'$ are indistinguishable, they both weakly display $T$ (by Proposition 3). But since any embedding of $T$ must consist of a single root-leaf path to $x$ of length $\lambda$ (by Lemma 6), such a root-leaf path must exist in $N'$.
Characterizing the funnel-free and the NELP properties

In this section we investigate what it means for a network to be canonical and to satisfy the NELP property. We start by characterizing the topological constraint (absence of funnels) that defines canonical networks.

**Proposition 4.** A network $N$ is funnel-free if and only if, for any two distinct non-root nodes $u$ and $w$ in $N$, there exist two node-disjoint directed paths from $u$ and $w$ to two distinct leaves in $N$.

**Proof.** First, note that if $N$ is not funnel-free, then the stated property does not hold: it suffices to take a funnel as $u$ and its only direct descendant as $w$ and then it is clear that any two paths from $u$ and $w$ cannot be node-disjoint.

Second, we prove that in a funnel-free network, for any two distinct nodes $u$ and $w$, neither of which coincides with the root, there always exist two node-disjoint paths $\pi_{ux}$ and $\pi_{wy}$, from $u$ to $x$ and from $w$ to $y$ respectively, where $x$ and $y$ are leaves of $N$. Let $d_N(v)$ denote the number of proper descendants (that is, not including $v$) of a node $v$ in a network $N$. We prove our claim by induction on $d_N(u) + d_N(w)$.

First, suppose $d_N(u) + d_N(w) = 0$. In this case $u$ and $w$ are leaves and the thesis trivially holds.

Then, suppose $d_N(u) + d_N(w) = n > 0$. Assume, without loss of generality, that $d_N(w) \geq d_N(u)$. Then $d_N(w) > 0$, which means that $w$ is an internal node. As a consequence, $w$ must have at least two children nodes, because otherwise $w$ would be a funnel or a root with outdegree 1 (which we have excluded). Of these children nodes, at least one must be different from $u$. Call this node $w'$. Because $w'$ is a descendant of $w$, then $d_N(w') < d_N(w)$ and so $d_N(u) + d_N(w') < n$. By the inductive hypothesis, we can then assume that the thesis holds for the pair of (different, non-root) nodes $(u, w')$. That is, there exist in $N$ two disjoint directed paths $\pi_{ux}$ and $\pi_{w'y}$, from $u$ to $x$ and from $w'$ to $y$ respectively, such that $x$ and $y$ are leaves of $N$. Now form a new path $\pi_{wy}$ by appending the edge $(w, w')$ at the beginning of $\pi_{w'y}$. Note that $w$ cannot be part of $\pi_{ux}$ because otherwise $w$ would be a descendant of $u$ and that would contradict $d_N(w) \geq d_N(u)$. Because of this, and because $\pi_{w'y}$ and $\pi_{ux}$ are disjoint, then also $\pi_{wy}$ and $\pi_{ux}$ are disjoint. The thesis then holds also in the inductive step and the theorem follows. \qed

We now examine the NELP property. It turns out that this property is equivalent to requiring the uniqueness of embeddings for all trees weakly displayed by the network. In order to prove this, we show the following trivial result that will be useful throughout this document.

**Lemma 9.** A network $N$ satisfies the NELP property if and only if, for every taxon $x$ in $N$, all root-leaf paths in $N$ to $x$ have different lengths.

**Proof.** For the if part, note that if $N$ does not satisfy the NELP property, then there are two distinct weighted paths $(\pi_1, \lambda_1)$, $(\pi_2, \lambda_2)$ with the same lengths and endpoints. In this case it is easy to extend these paths to two distinct root-leaf paths $P_1$ and $P_2$ leading to the same taxon $x$ and having the same length. As for the only if part, the existence of two root-leaf paths in $N$ of equal lengths to the same taxon $x$ clearly implies that $N$ violates the NELP property. \qed

**Proposition 5.** A network $N$ satisfies the NELP property if and only if every tree weakly displayed by $N$ has a unique embedding in $N$.

**Proof.** For the only if part, let $T$ be a tree on $\{x_1, x_2, \ldots, x_n\}$, weakly displayed by $N$, with $\lambda_1, \lambda_2, \ldots, \lambda_n$ being the lengths of the root-leaf paths to $x_1, x_2, \ldots, x_n$ in $T$. Suppose $T_1$ and $T_2$ are two embeddings of $T$ in $N$. Then $T_1$ and $T_2$ must each consist of the union of $n$ root-leaf paths to $x_1, x_2, \ldots, x_n$ of lengths $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively (Lemma 6). But because $N$ satisfies the NELP property, then for any $i \in \{1, 2, \ldots, n\}$ there can be only one path to $x_i$ of length $\lambda_i$ (Lemma 9), meaning that $T_1 = T_2$.

As for the if part, suppose every tree displayed by a network $N$ has a unique embedding in $N$, but that $N$ does not satisfy the NELP property. Then (by Lemma 9) there exist two distinct root-leaf paths $P_1$ and $P_2$ to the same taxon (say, $x$) that have the same length $\lambda$. But then $P_1$ and $P_2$ are distinct
embeddings in $N$ of the tree consisting of a single edge whose head is labelled by $x$, thus contradicting the uniqueness of embeddings.

Note that the network in Fig. S1, which has several embeddings for $T$ (namely $T_1$ and $T_2$), violates the NELP property: for example there are three different paths of length 5 from the root to the leaf labelled by $a$. Also note that the uniqueness of embeddings for the trees in $T(N)$ does not imply the NELP property: consider for example the top-left network in Fig. 10 in the main text, where the two trees displayed by this network have unique embeddings, but the network does not satisfy the NELP property.

Proving Proposition 1

Lemma 10. Let $P_1, Q_1$ be two root leaf paths in a network $N_1$ and let $P_2$ and $Q_2$ be two root-leaf paths respectively isomorphic to $P_1$ and $Q_1$, in another network $N_2$. Suppose that $N_2$ satisfies the NELP property and that, for every wishbone or crack $K_1$ contained in $P_1 \cup Q_1$ and containing $P_1$, there exists in $N_2$ a wishbone or crack $K_2$ isomorphic to $K_1$. Then $P_2$ and $Q_2$ intersect each other at the same depths as $P_1$ and $Q_1$, that is:

$P_1$ and $Q_1$ have a node $u$ in common that is at depth $\delta$ in $P_1$ and depth $\delta'$ in $Q_1$ if and only if $P_2$ and $Q_2$ have a node $v$ in common that is at depth $\delta$ in $P_2$ and depth $\delta'$ in $Q_2$.

Proof. In the following, we prove that if $P_1$ and $Q_1$ have in common the nodes $u_0, u_1, \ldots, u_k$ and only these nodes (where $u_0$ is the root of $N_1$), with respective depths $\delta_0 = 0 < \delta_1 < \ldots < \delta_k$ in $P_1$ and depths $\delta'_0 = 0 < \delta'_1 < \ldots < \delta'_k$ in $Q_1$, then in $N_2$ there must be exactly $k$ nodes in common between $P_2$ and $Q_2$, and these nodes must be at depths $\delta_0, \delta_1, \ldots, \delta_k$ in $P_2$ and depths $\delta'_0, \delta'_1, \ldots, \delta'_k$ in $Q_2$. This statement is clearly equivalent to the statement that we wish to prove.

First, let $v_0, v_1, \ldots, v_k$ be the nodes that have respective depths $\delta_0, \delta_1, \ldots, \delta_k$ in $P_2$. The existence of these nodes is guaranteed by the fact that $P_2$ is isomorphic to $P_1$, and $P_1$ has nodes $u_0, u_1, \ldots, u_k$ at exactly those depths (Lemma 3). Furthermore let $\Delta_P$ be the length of $P_1$ and $P_2$, let $\Delta_Q$ be the length of $Q_1$ and $Q_2$, let $x_P$ be the taxon labelling the leaves of $P_1$ and $P_2$ and let $x_Q$ be the taxon labelling the leaves of $Q_1$ and $Q_2$. Below, we prove the following two claims:

(C1) For every $i \in \{1, 2, \ldots, k\}$, there exists in $N_2$ a weighted path from $v_{i-1}$ to $v_i$ of length $\delta_i - \delta_{i-1}$ that has no node in common with $P_2$ other than $v_{i-1}$ and $v_i$. 

Figure S2. Two root-leaf paths $P_1$ and $Q_1$ intersecting in 5 nodes, their corresponding cracks $P_1 \cup R^0_1$, $P_1 \cup R^3_1$, $P_1 \cup R^1_1$ and wishbone $P_1 \cup R^2_1$ (see the proof of Lemma 10).
(C2) If \( v_k \) is a leaf, then it is labelled by \( x_Q \); otherwise there exists in \( N_2 \) a weighted path from \( v_k \) to the leaf labelled by \( x_Q \) that has no node in common with \( P_2 \) other than \( v_k \) and whose length is equal to \( \Delta_Q - \delta'_k \).

We start with the proof of C1. See Fig. S2 in order to follow the reasoning below. First consider the trivial case where \((u_{i-1}, u_i)\) is an edge in both \( P_1 \) and \( Q_1 \), and that it is assigned the same length \( \lambda \) in both of them. In this case, it is clear that \( \delta_i - \delta_{i-1} = \delta'_i - \delta'_{i-1} = \lambda \). Because \( P_2 \) is isomorphic to \( P_1 \) and because \( u_{i-1} \) and \( u_i \) are consecutive nodes in \( P_1 \), the nodes at the same depths as \( u_{i-1} \) and \( u_i \) in \( P_2 \), that is \( v_{i-1} \) and \( v_i \), must also be consecutive in \( P_2 \) (by Lemma 3). That is, \((v_{i-1}, v_i)\) is an edge of \( P_2 \). Its length in \( P_2 \) must be equal to the difference between the depths of \( v_{i-1} \) and \( v_i \) in \( P_2 \), that is \( \delta_i - \delta_{i-1} = \delta'_i - \delta'_{i-1} \). Edge \((v_{i-1}, v_i)\) along with its length in \( P_2 \) constitutes a weighted path from \( v_{i-1} \) to \( v_i \) of length \( \delta'_i - \delta'_{i-1} \) that has no node in common with \( P_2 \) other than \( v_{i-1} \) and \( v_i \), thus proving C1 in this trivial case. In all other cases, define \( R_1 \) as the root-leaf path that shares with \( P_1 \) its prefix down to \( u_{i-1} \) and its suffix from \( u_i \) onwards, and shares with \( Q_1 \) its portion between \( u_{i-1} \) and \( u_i \). By construction, \( Q_1 \) has no node in common with \( P_1 \) between \( u_{i-1} \) and \( u_i \), meaning that \( R_1 \) and \( P_1 \) in common only a prefix and a suffix, and thus that \( P_1 \cup R_1 \) is a crack. Because \( P_1 \cup R_1 \) is a crack containing \( P_1 \) and contained in \( P_1 \cup Q_1 \), then there exists in \( N_2 \) a crack isomorphic to \( P_1 \cup R_1 \). Because isomorphic cracks are the union of isomorphic root-leaf paths (Lemma 5) and because in \( N_2 \) there can be no root-leaf path isomorphic to \( P_1 \) other than \( P_2 \) (as \( N_2 \) satisfies the NELP property), then the crack isomorphic to \( P_1 \cup R_1 \) in \( N_2 \) can be written as \( P_2 \cup R_2 \), where \( R_2 \) is a root-leaf path isomorphic to \( R_1 \). Note now that \( P_1 \) and \( R_1 \) have lengths \( \Delta_P \) and \( \Delta_P - (\delta_i - \delta_{i-1}) + (\delta'_i - \delta'_{i-1}) \), respectively, and longest common prefix and suffix of lengths \( \delta_{i-1} \) and \( \Delta_P - \delta_i \), respectively (as \( \delta_{i-1} \) and \( \delta_i \) are the respective depths of \( u_{i-1} \) and \( u_i \) in \( P_1 \)). By Lemma 5, also \( P_2 \) and \( R_2 \) and their longest common prefix and suffix must have these lengths. This implies that \( R_2 \) separates from \( P_2 \) at the node at depth \( \delta_{i-1} \) in \( P_2 \) — which by construction is \( v_{i-1} \) — then follows a weighted path of length \( \delta'_i - \delta_{i-1} \) that has no node in common with \( P_2 \) other than its extremes, and finally joins up with \( P_2 \) at the node at depth \( \delta_i \) in \( P_2 \) — which by construction is \( v_i \). The portion of \( R_2 \) between \( v_{i-1} \) and \( v_i \) has length \( \delta'_i - \delta'_{i-1} \) and no node in common with \( P_2 \) other than \( v_{i-1} \) and \( v_i \), thus proving claim C1.

As for C2, if \( v_k \) is a leaf, since \( v_k \) belongs to both \( P_2 \) and \( Q_2 \), then it is the leaf of both \( P_2 \) and \( Q_2 \), and thus it must be labelled by \( x_P = x_Q \). If instead \( v_k \) is not a leaf, then there must have strict descendants in \( P_2 \). Then, because \( P_1 \) and \( P_2 \) are isomorphic, also \( u_k \) (the node at the same depth in \( P_1 \) as \( v_k \) in \( P_2 \)) must have strict descendants and is thus not a leaf. Define then \( R_1^{k+1} \) as the root-leaf path that shares with \( P_1 \) its prefix down to \( u_k \), and shares with \( Q_1 \) its suffix from \( u_k \) to the leaf labelled by \( x_Q \) (as \( u_k \) is not a leaf, this suffix contains at least one edge). By construction, \( Q_1 \) has no node in common with \( P_1 \) after separating from it in \( u_k \), meaning that \( R_1^{k+1} \) and \( P_1 \) have in common only a prefix, and thus that \( P_1 \cup R_1^{k+1} \) is a wishbone. Because \( P_1 \cup R_1^{k+1} \) is a wishbone containing \( P_1 \) and contained in \( P_1 \cup Q_1 \), then there exists in \( N_2 \) a wishbone isomorphic to \( P_1 \cup R_1^{k+1} \). Because isomorphic wishbones are the union of isomorphic root-leaf paths (Lemma 4) and because in \( N_2 \) there can be no root-leaf path isomorphic to \( P_1 \) other than \( P_2 \) (as \( N_2 \) satisfies the NELP property), then the crack isomorphic to \( P_1 \cup R_1^{k+1} \) in \( N_2 \) can be written as \( P_2 \cup R_2^{k+1} \), where \( R_2^{k+1} \) is a root-leaf path isomorphic to \( R_1^{k+1} \). Now note that \( P_1 \) and \( R_1^{k+1} \) have lengths \( \Delta_P \) and \( \Delta_Q - \delta_k' + \delta_k \), respectively, and longest common prefix of length \( \delta_k \). By Lemma 4, also \( P_2 \) and \( R_2^{k+1} \) and their longest common prefix must have these lengths. This implies that \( R_2^{k+1} \) separates from \( P_2 \) at the node at depth \( \delta_k \) in \( P_2 \) — which by construction is \( v_k \) — and then follows a weighted path of length \( \Delta_Q - \delta'_k \), that has no node in common with \( P_2 \) other than \( v_k \), and that ends up in a leaf labelled by \( x_Q \) (as \( R_2^{k+1} \) is isomorphic to \( R_1^{k+1} \)). Thus C2 is also proved.

As a consequence of C1 and C2, one can construct a root-leaf path \( R \) in \( N_2 \) by concatenating all the weighted paths whose existence has been proven in C1 and C2. Clearly, \( R \) is a root-leaf path to \( x_Q \) and it has a total length of \( \Delta_Q \), as:

\[
\begin{align*}
\sum_{i=1}^{k} (\delta'_i - \delta'_{i-1}) &= \delta'_k = \Delta_Q & \text{if } v_k \text{ is a leaf}, \\
\Delta_Q - \delta'_k + \sum_{i=1}^{k} (\delta'_i - \delta'_{i-1}) &= \Delta_Q & \text{otherwise}.
\end{align*}
\]
Because in $N_2$ there can be no root-leaf path to $x_Q$ of length $\Delta_Q$ other than $Q_2$ (as $N_2$ satisfies the NELP property), then $R = Q_2$. Now note that C1 and C2 imply that the only nodes that $R = Q_2$ has in common with $P_2$ are $v_0, v_1, \ldots, v_k$. Moreover, for every $i \in \{0, 1, \ldots, k\}$, the depth of $v_i$ in $P_2$ equals $\delta_i$ (by definition), and the depth of $v_i$ in $R = Q_2$ equals $\sum_{j=1}^{i} (\delta'_{j} - \delta'_{j-1}) = \delta'_i$, which is what we set out to prove.

\[\square\]

**Proposition 1.** Two networks $N_1$ and $N_2$ with the NELP property are isomorphic if and only if they contain the same wishbones and cracks (up to isomorphism).

**Proof.** The only if part is trivial: if $N_1$ and $N_2$ are isomorphic, then each wishbone or crack $W$ contained in one of the two networks must have an isomorphic sub-network $W'$ in the other (by Lemma 2), and $W'$ must be either be a wishbone or a crack (by Lemmas 4 and 5).

As for the if part, let us now suppose networks $N_1 = (V_1, E_1, \varphi_1, \Lambda_1)$ and $N_2 = (V_2, E_2, \varphi_2, \Lambda_2)$ satisfy the NELP property and contain isomorphic wishbones and cracks. We prove that $N_1$ and $N_2$ are isomorphic.

Because $N_1$ and $N_2$ contain the same wishbones and cracks (up to isomorphism), and because a root-leaf path (which is both a wishbone and a crack) can only be isomorphic to another root-leaf path (Lemma 3), then $N_1$ and $N_2$ must contain the same root-leaf paths (up to isomorphism). Note that $N_1$ and $N_2$ must be networks on the same taxon set $\mathcal{X}$, because otherwise any root-leaf path that leads to a leaf corresponding to a taxon present in only one of the networks would have no isomorphic root-leaf path in the other network. Then, define a relation $\sim$ between $V_1$ and $V_2$ as follows: for any $v_1 \in V_1$ and $v_2 \in V_2$, we write $v_1 \sim v_2$ if there exist two isomorphic root-leaf paths $P_1$ and $P_2$, in $N_1$ and $N_2$, respectively, that contain $v_1$ and $v_2$, respectively, such that $v_1$ has the same depth in $P_1$ as $v_2$ in $P_2$. Note that, because $N_1$ and $N_2$ contain isomorphic root-leaf paths, for every $v_1 \in V_1$ there exists a $v_2 \in V_2$ such that $v_1 \sim v_2$ (by Lemma 3). Moreover, we now prove that such $v_2$ is unique. Suppose there exist $v_2$ and $v_2'$ such that $v_1 \sim v_2$ and $v_1 \sim v_2'$. This would mean (see Fig. S3) that there exist two isomorphic root-leaf paths $P_1$ and $P_2$, in $N_1$ and $N_2$, respectively, that have $v_1$ and $v_2$ at the same depth $\delta$ and that there exist two isomorphic root-leaf paths $Q_1$ and $Q_2$, in $N_1$ and $N_2$, respectively, that have $v_1$ and $v_2'$ at the same depth $\delta'$. Because $N_1$ and $N_2$ contain the same wishbones and cracks, in particular for every wishbone or crack contained in $P_1 \cup Q_1$ and containing $P_1$ there exists an isomorphic wishbone or crack in $N_2$. Thus the assumptions of Lemma 10 are verified, meaning that $P_2$ and $Q_2$ must intersect each other at the same depths as $P_1$ and $Q_1$. As a result, because $P_1$ and $Q_1$ have a node $v_1$ in common that is at depth $\delta$ in $P_1$ and depth $\delta'$ in $Q_1$, then $P_2$ and $Q_2$ must have a node in common that is at depth $\delta$ in $P_2$ and depth $\delta'$ in $Q_2$. But $v_2$ and $v_2'$ are precisely the nodes in $P_2$ and $Q_2$ at depths $\delta$ and $\delta'$, respectively, meaning that we must have $v_2 = v_2'$. We thus conclude that there is a unique $v_2 \in V_2$ such that $v_1 \sim v_2$. Similarly, for every $v_2 \in V_2$ there exists a unique $v_1 \in V_1$ such that $v_1 \sim v_2$. Thus, relation $\sim$ identifies a bijection $f : V_1 \to V_2$, defined by $f(v_1) = v_2 \iff v_1 \sim v_2$.

We now prove that bijection $f$ is an isomorphism between $N_1$ and $N_2$, by showing that it verifies the two requirements in Definition 5. First, note that for every $x \in \mathcal{X}$ we can consider a root-leaf path $P_1$ in $N_1$ ending in $\varphi_1(x)$ and its isomorphic equivalent $P_2$ in $N_2$ ending in $\varphi_2(x)$. Because the two paths are isomorphic, $\varphi_1(x)$ must lie at the same depth in $P_1$ as $\varphi_2(x)$ in $P_2$. Thus $\varphi_1(x) = \varphi_2(x)$, that is $f(\varphi_1(x)) = \varphi_2(x)$.

Second, we show that if $c_1 = (u, v) \in E_1$ is an edge of $N_1$ having a length $\lambda \in \Lambda_1(c_1)$ then $(f(u), f(v))$ is an edge of $N_2$ having also length $\lambda$. Let $P_1$ be any root-leaf path that passes via $c_1$ and assigns length $\lambda$ to $c_1$. Let $\delta$ and $\delta + \lambda$ be the depths of $u$ and $v$, respectively, in $P_1$. Because $N_1$ and $N_2$ contain isomorphic root-leaf paths, $N_2$ must contain a root-leaf path $P_2$, isomorphic to $P_1$. As a consequence of Lemma 3, $P_2$ must have two nodes at depths $\delta$ and $\delta + \lambda$ — which by construction must be $f(u)$ and $f(v)$, respectively — connected by an edge $c_2 = (f(u), f(v)) \in E_2$ having length $\lambda$ in $P_2$, that is, $\lambda \in \Lambda_2(c_2)$. This allows to conclude that $\Lambda_1((u, v)) \subseteq \Lambda_2((f(u), f(v)))$. Similarly, we can prove that $\Lambda_2((f(u), f(v))) \subseteq \Lambda_1((u, v))$, which allows us to conclude that $f$ satisfies point (ii) in Definition 5. Bijection $f$ is thus an isomorphism.
between $N_1$ and $N_2$. \hfill \square

**Proving Proposition 2**

In order to prove that two indistinguishable funnel-free networks with the NELP property have isomorphic wishbones and cracks, we need some more accessory results and notation.

**Definition 9.** Let $P$ and $Q$ be two root-leaf paths in a network $N$. We denote the length of the longest common prefix of $P$ and $Q$ by $\mu(P, Q)$. We say that $P$ and $Q$ separate at node $v$, if $P \neq Q$, and $v$ is the last node in the longest common prefix of $P$ and $Q$. Finally, $P$ and $Q$ separate openly at $v$ if they separate at $v$ and the direct descendants of $v$ in $P$ and $Q$ are distinct nodes.

Note that if the longest common prefix of $P$ and $Q$ consists of only one node, then $\mu(P, Q) = 0$. In general, if $P$ and $Q$ separate at $v$, then the depth of $v$ in $P$ and $Q$ is precisely $\mu(P, Q)$. Moreover, note that two distinct root-leaf paths $P$ and $Q$ do not separate openly at $v$ when they contain the same edge $(v, v')$, but with different lengths in $P$ and $Q$.

**Definition 10.** Let $P$ and $Q$ denote root-leaf paths. An open crack is a crack $P \cup Q$ where $P$ and $Q$ separate openly. A closed crack is a crack $P \cup Q$ where $P$ and $Q$ differ for the length of exactly one edge.

Note that all cracks are either open, closed or root-leaf paths. In Fig. 14 in the main text, $S$ and $R$ are an open crack and a closed crack, respectively.

**Lemma 11.** (Three-prefix condition). Let $P_1$, $P_2$, $P_3$ be root-leaf paths in a network $N$. Then the two smallest of $\mu(P_1, P_2)$, $\mu(P_1, P_3)$, $\mu(P_2, P_3)$ are equal, or, equivalently, for $\{i, j, k\} = \{1, 2, 3\}$:

$$\mu(P_i, P_j) \geq \min\{\mu(P_i, P_k), \mu(P_j, P_k)\}.$$ 

*Proof.* First note that the two smallest of three numbers $x_1, x_2, x_3$ are equal if and only if $x_i \geq \min\{x_j, x_k\}$, for $\{i, j, k\} = \{1, 2, 3\}$: without loss of generality suppose $x_1 \leq x_2 \leq x_3$ and note that — while $x_2 \geq \min\{x_1, x_3\} = x_1$ and $x_3 \geq \min\{x_1, x_2\} = x_1$ are trivially true — $x_1 \geq \min\{x_2, x_3\}$ holds if and only if $x_1 = x_2$. In order to prove that the two smallest of $\mu(P_1, P_2)$, $\mu(P_1, P_3)$, $\mu(P_2, P_3)$ are equal, imagine following $P_1$, $P_2$ and $P_3$ from the root until at least one of them separates from the others: let $v$ be a node at depth $\delta$ in $P_1$, $P_2$, $P_3$, where either only $P_i$ separates from $P_j$ and $P_k$, or all three $P_1, P_2, P_3$ separate. In both cases, $\delta = \mu(P_i, P_j) = \mu(P_i, P_k) \leq \mu(P_j, P_k)$, thus concluding the proof of this lemma. \hfill \square

**Lemma 12.** Let $N$ and $N'$ be two indistinguishable funnel-free networks, and let $N'$ satisfy the NELP property. Let $P_1$ and $P_2$ be root-leaf paths in $N$, whose union $P_1 \cup P_2$ is a wishbone. Let $P'_1$ and $P'_2$
be the root-leaf paths in $N'$ to the same taxa as $P_1$ and $P_2$, and having the same lengths as $P_1$ and $P_2$, respectively (whose existence and uniqueness are guaranteed by Corollary 3 and Lemma 9, respectively). Then, $P'_1 \cup P'_2$ is a wishbone of $N'$, and

$$\mu(P'_1, P'_2) = \mu(P_1, P_2).$$

Proof. Let $x_1$ and $x_2$ be the taxa labelling the leaves of $P_1$ and $P_2$ (and thus $P'_1$ and $P'_2$), respectively, and let $\lambda_1$ and $\lambda_2$ be the lengths of $P_1$ and $P_2$ (and thus $P'_1$ and $P'_2$). Note that we may have $x_1 = x_2$, and $\lambda_1 = \lambda_2$, if $P_1 = P_2$. Because $N$ and $N'$ are indistinguishable, they must both weakly display $T_W$, the tree obtained by suppressing all suppressible nodes in the wishbone $W = P_1 \cup P_2$ (by Proposition 3). $T_W$ must be the union of two root-leaf paths to $x_1$ and $x_2$, of lengths $\lambda_1$ and $\lambda_2$, respectively, and having a longest common prefix of length $\mu(P_1, P_2)$ (because these properties are not lost by suppressing suppressible nodes). Then, the embedding of $T_W$ in $N'$ must be a wishbone $W'$ consisting of the union of two root-leaf paths to $x_1$ and $x_2$, of lengths $\lambda_1$ and $\lambda_2$, respectively, whose longest common prefix has length $\mu(P'_1, P'_2)$ (by Lemma 6). Because $P'_1$ and $P'_2$ are the unique root-leaf paths in $N'$ to $x_1$ and $x_2$, and of lengths $\lambda_1$ and $\lambda_2$, respectively, this implies $W' = P'_1 \cup P'_2$ and the lemma follows. \qed

Note that the lemma that we just proved includes the trivial case where $P_1 = P_2$.

Lemma 13. Let $N$ and $N'$ be two indistinguishable funnel-free networks, and let $N'$ satisfy the NELP property. Let $P$ and $Q$ be distinct root-leaf paths in $N$, whose union $P \cup Q$ is a crack. Let $P'$ and $Q'$ be the root-leaf paths in $N'$ to the same taxa, and having the same lengths as $P$ and $Q$, respectively (whose existence and uniqueness are guaranteed by Corollary 3 and Lemma 9, respectively). Then

$$\mu(P', Q') \begin{cases} = \mu(P, Q) & \text{if } P \cup Q \text{ is an open crack,} \\
\geq \mu(P, Q) & \text{if } P \cup Q \text{ is a closed crack.} \end{cases}$$

Proof. The proof is by induction on the number of edges in the longest suffix common to $P$ and $Q$. Throughout this proof, for any root-leaf path $X$ in $N$, let $X'$ denote the root-leaf path in $N'$ to the same taxon and having the same length as $X$. Moreover, let $u$ be the last node in the longest prefix common to $P$ and $Q$ and let $v$ be the first node in the longest suffix common to $P$ and $Q$. Because $P \neq Q$, $u$ must be a strict ancestor of $v$. Moreover, let $u_P$ and $u_Q$ be the direct ancestors of $v$ in $P$ and $Q$, respectively. Of these two nodes, at least one is not a strict ancestor of the other, otherwise $N$ would contain a cycle. Thus, without loss of generality, we assume throughout that $u_Q$ is not a strict ancestor of $u_P$. Note that if $P \cup Q$ is an open crack, at least one between $u_P$ and $u_Q$ must be different from $u$. Because we require that $u_Q$ is not a strict ancestor of $u_P$, it follows that $u_Q \neq u$, when $P \cup Q$ is an open crack. Finally, in the particular case where $P \cup Q$ is a closed crack and $u = u_Q$ is the root of $N$, the statement trivially holds, as $\mu(P, Q) = 0 \leq \mu(P', Q')$. Thus, we assume throughout that $u_Q$ is not the root of $N$. Because it cannot be a leaf or a funnel, either, we can assume that $u_Q$ has outdegree 2 or more.

Base case. (See Fig. S4, left, to follow the argument below.) Suppose that the longest suffix common to $P$ and $Q$ contains no edge, or, equivalently, that it consists of just a leaf $v$. Because $u_Q$ has outdegree 2 or more, we can define a root-leaf path $R$ in $N$ that separates openly from $Q$ at $u_Q$: let $R$ have a common prefix with $Q$ consisting of the portion of $Q$ from the root down to $u_Q$; then, let $R$ take an edge $(u_Q, w)$ with $w \neq v$ (with any of this edge's lengths in $N$) and finally let $R$ take any weighted path from $w$ that does not end up in $v$ (which is possible because $N$ is funnel-free). Clearly,

$$\mu(P, Q) = \mu(P, R) \leq \mu(Q, R). \quad (3)$$

Note that $R$ and $Q$ have no node in common below $u_Q$ — as otherwise $R$ would contain $v$, which we excluded by construction — meaning that $Q \cup R$ is a wishbone. Moreover, $P \cup R$ is also a wishbone: assuming otherwise would mean that $R$ has a node in common with $P$ below $u$, either contradicting the
fact that $P$ and $Q$ have no nodes in common between $u$ and $v$, or contradicting the requirement that $u_Q$ is not a strict ancestor of $u_P$, or contradicting the requirement that $v$ does not belong to $R$. Because $Q \cup R$ and $P \cup R$ are wishbones, then (by Lemma 12)

$$
\mu(Q', R') = \mu(Q, R), \quad \mu(P', R') = \mu(P, R)
$$

(4)

Now consider two cases: either $P \cup Q$ is an open crack or it is closed. If it is open, then $u \neq u_Q$ (see above) and thus we have $\mu(P, R) < \mu(Q, R)$ in (3). Then, because of the equalities in (4), we have $\mu(P', R') < \mu(Q', R')$. But then, because of the three-prefix condition (Lemma 11), we must have $\mu(P', Q') = \mu(P', R')$. Combine this with Equations (3) and (4), to show that $\mu(P', Q') = \mu(P, Q)$. If instead $P \cup Q$ is a closed crack, then $u = u_Q$ and thus we have $\mu(P, R) = \mu(Q, R)$. Then, because of the equalities in (4), we have $\mu(P', R') = \mu(Q', R')$. In this case, the three-prefix condition (Lemma 11) implies $\mu(P', Q') \geq \mu(P', R') = \mu(Q', R')$ and thus $\mu(P', Q') \geq \mu(P, Q)$.

Inductive step. (See the two drawings on the right in Fig. S4 to follow the argument below.) Now suppose that the longest suffix common to $P$ and $Q$ contains $k > 0$ edges, and assume that the lemma’s statements hold for every pair of root-leaf paths whose union is a crack and whose longest common suffix contains fewer than $k$ edges. Because $u_Q$ has outdegree 2 or more, we can define a root-leaf path $R$ in $N$ that separates openly from $Q$ at $u_Q$: let $R$ have a common prefix with $Q$ consisting of the prefix of $Q$ from the root down to $u_Q$; then, let $R$ take an edge $(u_Q, w)$ with $w \neq v$ (with any of this edge’s lengths in $N$) and then let $R$ continue taking edges always avoiding ending up in $v$ (recall that $N$ is funnel-free), until it either arrives in a leaf or in a node $t$ belonging to $P \cup Q$. Note that such $t$ must be a strict descendant of $v$ in the suffix common to $P$ and $Q$. (By construction $t \neq v$; moreover, if $t$ belonged to $Q$ and not to $P$, then $t$ would lie between $u_Q$ and $v$ in $Q$, which contradicts the assumption that $u_Q$ is a direct ancestor of $v$ in $Q$; finally, if $t$ belonged to $P$ and not to $Q$, then $t$ would either coincide with $u_P$ or be a strict descendant of $u_P$, implying that $u_Q$ is an ancestor of $u_P$, which is not possible by construction.) Finally, from $t$ onwards, let $R$ coincide with the suffix common to $P$ and $Q$. Clearly, as in the proof of the base case, the following holds:

$$
\mu(P, Q) = \mu(P, R) \leq \mu(Q, R).
$$

(5)

If $R$ arrives in a leaf without hitting a node in $P \cup Q$ (i.e., without crossing neither $P$ or $Q$), then $Q \cup R$ and $P \cup R$ are wishbones. If instead $R$ hits $P \cup Q$ in $t$, then $Q \cup R$ and $P \cup R$ are open cracks: they are cracks, because, by construction, $t$ is the first node that $R$ has in common with $P$ and $Q$ after separating
from them (at u and \( u_Q \), respectively), and because R has its suffix following \( t \) in common with \( P \) and \( Q \); they are open because the edge \( (u_Q, w) \) in R does not belong to neither \( Q \) or \( P \). Moreover, these open cracks are such that the two root-leaf paths that compose them have a longest common suffix of fewer than \( k \) edges, as \( t \) is a strict descendant of \( v \). Irrespective of \( Q \cup R \) and \( P \cup R \) being wishbones or open cracks, we then have (by Lemma 12 or by inductive hypothesis) that

\[
\mu(Q', R') = \mu(Q, R), \quad \mu(P', R') = \mu(P, R). \tag{6}
\]

Now consider two cases: either \( P \cup Q \) is an open crack or it is closed. If it is open, then \( u \neq u_Q \) (see above) and thus we have \( \mu(P, R) < \mu(Q, R) \) in (5). Then, because of the equalities in (6), we have \( \mu(P', Q') = \mu(P', R') \). If instead \( P \cup Q \) is a closed crack, then \( u = u_Q \) and thus we have \( \mu(P, R) = \mu(Q, R) \). Then, because of the equalities in (6), we have \( \mu(P', Q') = \mu(Q', R') \). In this case, the three-prefix condition (Lemma 11) implies \( \mu(P', Q') \geq \mu(P', R') = \mu(Q', R') \) and thus \( \mu(P', Q') \geq \mu(P, Q) \).

**Lemma 14.** Let \( N \) and \( N' \) be two indistinguishable funnel-free networks, and let \( N' \) satisfy the NELP property. Let \( P \) and \( Q \) be any root-leaf paths in \( N \), and let \( P' \) and \( Q' \) be the root-leaf paths in \( N' \) to the same taxa, and having the same lengths as \( P \) and \( Q \), respectively (whose existence and uniqueness are guaranteed by Corollary 3 and Lemma 9, respectively). Then, (i)

\[
\mu(P', Q') = \begin{cases} 
= \mu(P, Q) & \text{if } P = Q \text{ or if they separate openly}, \\
\geq \mu(P, Q) & \text{otherwise}.
\end{cases}
\]

Moreover, (ii) if \( P \) and \( Q \) separate openly, then \( P' \) and \( Q' \) also separate openly.

**Proof.** Once again, throughout this proof, for any root-leaf path \( X \) in \( N \), we let \( X' \) denote the root-leaf path in \( N' \) to the same taxon and having the same length as \( X \). Let \( \Delta(P, Q) \) denote the number of edges in \( P \cup Q \) that are either present in only one of \( P \) and \( Q \), or that are present in both, but with different lengths. We prove part (i) by induction on \( \Delta(P, Q) \).

**Base case.** If \( \Delta(P, Q) = 0 \), then \( P = Q \) and \( P' = Q' \), trivially implying that both \( \mu(P, Q) \) and \( \mu(P', Q') \) equal the length of these paths, and thus \( \mu(P', Q') = \mu(P, Q) \).

**Inductive step.** If \( \Delta(P, Q) > 0 \), then \( P \) and \( Q \) must separate at a node \( u \). If they separate openly and do not have any node in common after \( u \), then they form a wishbone, meaning that \( \mu(P', Q') = \mu(P, Q) \) (by Lemma 12), thus verifying the statement. In all other cases, let \( v \) be the first node that \( P \) and \( Q \) have

![Figure S5. Illustration of the inductive case in the proof of Lemma 14.](image-url)
in common after $u$. Define a new root-leaf path $R$ that coincides with $P$ and $Q$ along all their common prefix down to $u$, then coincides with $Q$ along the weighted path in $Q$ from $u$ to $v$, and finally coincides with $P$ along all its suffix from $v$ to the leaf in $P$ (see Fig. S5). Clearly,

$$\mu(P, Q) = \mu(P, R) < \mu(Q, R). \quad (7)$$

Note that $\Delta(Q, R) = \Delta(P, Q) - \Delta(P, R)$. Because $P \neq R$, then $\Delta(P, R) > 0$. Therefore, we have $\Delta(Q, R) < \Delta(P, Q)$ and we can assume, by inductive hypothesis:

$$\mu(Q', R') \geq \mu(Q, R). \quad (8)$$

Because $P$ and $R$ are node-disjoint between $u$ and $v$, and coincide everywhere else, clearly they form a crack. This crack is open or closed, depending on whether $P$ and $Q$ separate openly or by taking the same edge $(u, v)$ with different lengths. Consider these two cases separately.

If $P$ and $Q$ separate openly, then $P \cup R$ is an open crack and Lemma 13 implies that

$$\mu(P', R') = \mu(P, R). \quad (9)$$

Now combine Equations (7), (8), (9) to show that $\mu(P', R') < \mu(Q', R')$. But then, because of the three-prefix condition (Lemma 11), we must have $\mu(P', Q') = \mu(P, Q)$, which together with Equations (7) and (9) implies $\mu(P', Q') \leq \mu(P, Q)$, when $P$ and $Q$ separate openly.

If instead $P$ and $Q$ separate by taking the same edge $(u, v)$ with different lengths, then $P \cup R$ is a closed crack. Then, by Lemma 13,

$$\mu(P', R') \geq \mu(P, R). \quad (10)$$

Now combine Equations (7), (8), (10) to show that $\mu(P, Q) \leq \mu(Q', R')$ and $\mu(P, Q) \leq \mu(P', R')$. But then, because of the three-prefix condition (Lemma 11), we must have

$$\mu(P', Q') \geq \min\{\mu(Q', R'), \mu(P', R')\} \geq \mu(P, Q),$$

which concludes our proof by induction of part (i).

As for part (ii), suppose that $P$ and $Q$ separate openly at a node $u$, at depth $\alpha$ in their common prefix, by taking two distinct edges $(u, w_P)$ and $(u, w_Q)$. Because $N$ is funnel-free, because $w_P$ and $w_Q$ are distinct, and because neither of them is the root of $N$, then there exist two node-disjoint directed paths $\pi_P$ and $\pi_Q$ from $w_P$ and $w_Q$, respectively, to two leaves of $N$ (by Proposition 4). Let $R_P$ be any root-leaf path that coincides with $P$ along its prefix down to $w_P$ and then follows $\pi_P$ by taking its edges with any of their lengths in $N$. Similarly, let $R_Q$ be any root-leaf path that coincides with $Q$ along its prefix down to $w_Q$ and then follows $\pi_Q$ by taking its edges with any of their lengths in $N$. Because $R_P$ and $R_Q$ coincide with $P$ and $Q$ down to node $u$, and all their nodes that are strict descendants of $u$ belong to (the node-disjoint paths) $\pi_P$ and $\pi_Q$, respectively, then $R_P \cup R_Q$ is a wishbone in $N$, with $\mu(R_P, R_Q) = \alpha$. By Lemma 12, also $R_P' \cup R_Q'$ is a wishbone in $N'$ and $\mu(R_P', R_Q') = \alpha$.

Moreover, because, by construction, $\mu(P, R_P) > \alpha$ and $\mu(Q, R_Q) > \alpha$, and because, by part (i) of the present lemma, $\mu(P', R_P') \geq \mu(P, R_P)$ and $\mu(Q', R_Q') \geq \mu(Q, R_Q)$, then we have

$$\mu(P', R_P') > \alpha \quad \text{and} \quad \mu(Q', R_Q') > \alpha.$$

These two relationships, together with the fact that $R_P'$ and $R_Q'$ separate openly at depth $\alpha$, imply that $P'$ and $Q'$ also separate openly at depth $\alpha$: if we let $u'$ be the node at depth $\alpha$ in $R_P'$ and $R_Q'$, the successors of $u'$ in $R_P'$ and $R_Q'$ must be distinct, and belonging to $P'$ and $Q'$, respectively, thus implying that $P'$ and $Q'$ separate openly.

The lemmas above only require one of the two networks to verify the NELP property. In the rest of this section, we concentrate on the case where both networks satisfy the NELP property, as this is among the hypotheses of Proposition 2.
Lemma 15. Let \( N \) and \( N' \) be two indistinguishable funnel-free networks satisfying the NELP property. Let \( P \) and \( Q \) be any root-leaf paths in \( N \), and let \( P' \) and \( Q' \) be the root-leaf paths in \( N' \) to the same taxa, and having the same lengths as \( P \) and \( Q \), respectively (whose existence and uniqueness are guaranteed by Corollary 3 and Lemma 9, respectively). Then, \( P' \) and \( Q' \) separate openly if and only if \( P \) and \( Q \) separate openly, and

\[
\mu(P', Q') = \mu(P, Q).
\]

Proof. Apply Lemma 14 both to \( P \) and \( Q \) in \( N \) and to \( P' \) and \( Q' \) in \( N' \), showing that \( \mu(P', Q') \geq \mu(P, Q) \) and \( \mu(P', Q') \leq \mu(P, Q) \), respectively (and thus \( \mu(P', Q') = \mu(P, Q) \)) and that if one pair of root-leaf paths separates openly, also the other separates openly.

Lemma 16. Let \( N \) and \( N' \) be two indistinguishable funnel-free networks satisfying the NELP property. Then \( N \) and \( N' \) have the same root-leaf paths (up to isomorphism). Moreover, for each root-leaf path \( P \) in one of the two networks, the root-leaf path \( P' \) isomorphic to \( P \) in the other network is unique.

Proof. Let \( P \) be a root-leaf path in one of the two networks, say (without loss of generality), \( N \), and let \( P' \) be the unique root-leaf path in \( N' \) to the same taxon and having the same length as \( P \) (whose existence and uniqueness are guaranteed by Corollary 3 and Lemma 9, respectively). For any node \( v \) in \( P \), there exists a node \( v' \) at the same depth in \( P' \); this is trivial when \( v \) is at depth 0; otherwise, if we let \( Q \) be a root-leaf path that separates openly from \( P \) at \( v \) (which exists because \( N \) is funnel-free), then \( Q' \) — the root-leaf path in \( N' \) to the same taxon and having the same length as \( Q \) — must separate from \( P' \) at a node at the same depth \( \mu(P', Q') = \mu(P, Q) \) as \( v \) (by Lemma 14). Symmetrically, for any node in \( P' \) there exists a node at the same depth in \( P \). Thus, because \( P \) and \( P' \) are root-leaf paths to the same taxon and have nodes at the same depths, then \( P \) and \( P' \) are isomorphic (by Lemma 3). In conclusion, for any root-leaf path \( P \) in one of the two networks, there exists a unique isomorphic root-leaf path \( P' \) in the other, which is what we wanted to prove.

Lemma 17. Let \( N \) and \( N' \) be two indistinguishable funnel-free networks satisfying the NELP property. Let \( P_1 \) and \( P_2 \) be root-leaf paths in \( N \), whose union \( W = P_1 \cup P_2 \) is a wishbone. Let \( P'_1 \) and \( P'_2 \) be the unique root-leaf paths in \( N' \) isomorphic to \( P_1 \) and \( P_2 \), respectively (Lemma 16). Then \( W' = P'_1 \cup P'_2 \) is a wishbone in \( N' \) isomorphic to \( W \).

Proof. By Lemma 12, \( W' = P'_1 \cup P'_2 \) is a wishbone with \( \mu(P'_1, P'_2) = \mu(P_1, P_2) \). Because \( W \) and \( W' \) are wishbones, and equal to the union of pairs of isomorphic root-leaf paths, with longest common prefixes of the same length, then \( W \) and \( W' \) are isomorphic (by Lemma 4).

Corollary 4. Let \( N \) and \( N' \) be two indistinguishable funnel-free networks satisfying the NELP property. Then \( N \) and \( N' \) have the same wishbones (up to isomorphism).

Proof. Apply Lemma 17 to all wishbones in \( N \) and \( N' \).

Lemma 18. Let \( N \) and \( N' \) be two indistinguishable funnel-free networks with the NELP property. Then they have the same cracks (up to isomorphism).

Proof. We prove that for any crack \( K = P_1 \cup P_2 \) contained in one of the two networks, there exists a crack \( K' \), isomorphic to \( K \), in the other network. Without loss of generality, we assume here that \( K \) is a crack in \( N \), but all arguments hold symmetrically for the case where \( K \) is a crack in \( N' \).

We introduce some notation that is useful throughout the proof. For any root-leaf path \( X \) in \( N \), let \( X' \) denote the unique root-leaf path in \( N' \) that is isomorphic to \( X \) (Lemma 16). The thesis is trivial when \( P_1 = P_2 \): in this case \( K \) is a root-leaf path, and \( K' \) is isomorphic to it. Thus we assume throughout that \( P_1 \neq P_2 \). Now let \( u \) be the last node in the longest prefix common to \( P_1 \) and \( P_2 \) and let \( v \) be the first node in the longest suffix common to \( P_1 \) and \( P_2 \). Because \( P_1 \neq P_2 \), \( u \) must be a strict ancestor of \( v \). Also, let \( \alpha = \mu(P_1, P_2) = \mu(P'_1, P'_2) \) (the equality is guaranteed by Lemma 15), meaning that \( P'_1 \) and \( P'_2 \)
separate at a node $u'$ that has the same depth $\alpha$ in $P'_1$ and $P'_2$ as $u$ in $P_1$ and $P_2$. Moreover let $\beta_1$ and $\beta_2$ be the (strictly positive) lengths of the weighted paths from $u$ to $v$ within $P_1$ and $P_2$, respectively, and let $\gamma \geq 0$ be the length of the longest suffix common to $P_1$ and $P_2$. Because $v$ has depth $\alpha + \beta_1$ and $\alpha + \beta_2$ in $P_1$ and $P_2$, respectively, there must exist in $N$ two nodes $v'_1$ and $v'_2$ at depths $\alpha + \beta_1$ and $\alpha + \beta_2$ in $P'_1$ and $P'_2$, respectively. Finally, note that because $P_1$ and $P_2$ have a common suffix, they are root-leaf paths to the same taxon. Therefore also $P'_1$ and $P'_2$ are root-leaf paths to the same taxon, and thus end up in the same leaf. Thus $P'_1$ and $P'_2$ must share at least one node after separating at $u'$. Let $v'$ be the first node that $P'_1$ and $P'_2$ have in common after separating at $u'$ (thus a strict descendant of $u'$). All these notations are shown in Fig. S6.

We now prove that $v' = v'_1 = v'_2$ implies that $K' = P'_1 \cup P'_2$ is a crack isomorphic to $K$. First, note that the suffixes of $P'_1$ and $P'_2$ following $v'_1$ and $v'_2$, respectively, are weighted paths of length $\gamma$ (because $P'_1$ and $P'_2$ have lengths $\alpha + \beta_1 + \gamma$ and $\alpha + \beta_2 + \gamma$, respectively), ending up in the same leaf. Thus, when $v'_1 = v'_2$, these two weighted paths have equal lengths and the same endpoints. Because $N$ has the NELP property, these two weighted paths must then coincide, meaning that $P'_1$ and $P'_2$ have a common suffix of length $\gamma$. Moreover, by the definition of $v'$, $P'_1$ and $P'_2$ have no node in common between $u'$ and $v'$. Therefore $v' = v'_1 = v'_2$ implies that $K' = P'_1 \cup P'_2$ is a crack with a longest common suffix of length $\gamma$, which is the same as the length of the longest common suffix of $P_1$ and $P_2$. Because of this, because $P'_1$ and $P'_2$ have a longest common prefix with the same length as the longest common prefix of $P_1$ and $P_2$ (a), and finally because $P'_1$ and $P'_2$ are isomorphic to $P_1$ and $P_2$, respectively, then $K$ and $K'$ are isomorphic (by Lemma 5).

Now consider the case where $K$ is a closed crack: in this case, $P_1$ and $P_2$ only differ for the length assigned to edge $(u, v)$, which must be $\beta_1$ in $P_1$ and $\beta_2$ in $P_2$. Note that $P'_1$ and $P'_2$ cannot separate openly, because otherwise also $P_1$ and $P_2$ would separate openly (by Lemma 15). Therefore, $P'_1$ and $P'_2$ must both contain the edge $(u', v')$, but assign different lengths to it. (We have called the head of this edge $v'$, as it is clearly the first node that $P'_1$ and $P'_2$ have in common after separating at $u'$.) Because $P'_1$ is isomorphic to $P_1$, the length assigned to $(u', v')$ in $P'_1$ must be $\beta_1$; if this edge were assigned a length $\beta'_1 < \beta_1$ in $P'_1$, then a node at depth $\alpha + \beta'_1$ would exist in $P'_1$ and therefore in $P_1$, but this contradicts the fact that there is no node in $P_1$ between $u$ (at depth $\alpha$) and $v$ (at depth $\alpha + \beta_1$); if instead $(u', v')$ were assigned a length $\beta'_1 > \beta_1$ in $P'_1$, then no node would exist at depth $\alpha + \beta_1$ in $P'_1$, which contradicts the fact that $v$ has depth $\alpha + \beta_1$ in $P_1$. Symmetrically, because $P'_2$ is isomorphic to $P_2$, the length assigned to $(u', v')$ in $P'_2$ must be $\beta_2$. Thus $v'$ has depth $\alpha + \beta_1$ in $P'_1$ and $\alpha + \beta_2$ in $P'_2$, meaning that $v' = v'_1 = v'_2$. As we showed above, this implies that $K' = P'_1 \cup P'_2$ is a crack isomorphic to $K$, whenever $K$ is a closed crack.

It remains to prove that for any open crack $K = P_1 \cup P_2$ contained in one of the two networks, there exists a crack $K'$ in the other network that is isomorphic to $K$. We prove this by induction on the number of edges in the longest suffix common to $P_1$ and $P_2$. In both the base case and the inductive step, we let $u_1$ and $u_2$ be the direct ancestors of $v$ in $P_1$ and $P_2$, respectively. Of these two nodes, at least one is not an ancestor of the other, otherwise either $N$ would contain a cycle or $u_1 = u_2$ (the latter would imply that $K$ is a closed crack). Thus, without loss of generality, we assume throughout that $u_2$ is not an ancestor of $u_1$. Note that this implies that $u_2 \neq u$.

Base case. (See Fig. S7 to follow the argument below.) If the longest suffix common to $P_1$ and $P_2$ contains no edge, then $v$ coincides with the leaf in $P_1$ and $P_2$, and $\gamma = 0$. Thus $P'_1$ and $P'_2$ have lengths $\alpha + \beta_1$ and $\alpha + \beta_2$, respectively, meaning that $v'_1$ and $v'_2$ are their leaves. But $P'_1$ and $P'_2$ have the same leaf, thus implying $v'_1 = v'_2$. It remains to prove that $v' = v'_1 = v'_2$, or in other words that $P'_1$ and $P'_2$ have no node in common between $u'$ and $v'_1 = v'_2$. Because $u_2$ has outdegree 2 or more, we can define a root-leaf path $R$ in $N$ that separates openly from $P_2$ at $u_2$: let $R$ have a common prefix with $P_2$ consisting of the portion of $P_2$ from the root down to $u_2$; then, let $R$ take an edge $(u_2, w)$ with $w \neq v$ (with any of this edge’s lengths in $N$) and finally let $R$ take any weighted path from $w$ that does not end up in $v$ (which is possible because $N$ is funnel-free). Note that $R$ and $P_2$ have no node in common below $u_2$ — as otherwise $R$ would contain $v$, which we excluded by construction — meaning that $P_2 \cup R$ is a wishbone.
Figure S6. Illustration of the notation for Lemma 18 and of the argument against claim C1 within it. $P_1$ and $P_1'$ are in black and the portions of $P_2$ and $P_2'$ not overlapping with $P_1$ and $P_1'$ are in grey. Note that the position of $v_2'$ along $P_2'$ is not shown as it may be either above or below $v'$.

Figure S7. Illustration of the base case of the proof of Lemma 18. $P_1$ and $P_1'$ are in black and the portions of $P_2$ and $P_2'$ not overlapping with $P_1$ and $P_1'$ are in grey. Finally, the portions of $R$ and $R'$ not overlapping with $P_1$ and $P_1'$ are represented by the dashed lines. Moreover, $P_1 \cup R$ is also a wishbone: assuming otherwise would mean that $R$ has a node in common with $P_1$ below $u$, either contradicting the fact that $P_1$ and $P_2$ have no nodes in common between $u$ and $v$, or contradicting the requirement that $u_2$ is not an ancestor of $u_1$, or contradicting the requirement that $v$ does not belong to $R$. Now let $W_1 = P_1 \cup R$ and $W_2 = P_2 \cup R$. Because these are wishbones, by Lemma 17, $W_1' = P_1' \cup R'$ and $W_2' = P_2' \cup R'$ are also wishbones isomorphic to $W_1$ and $W_2$, respectively. Moreover, because the same holds for $P_1, P_2$ and $R$ in $N$, the following holds:

$$\mu(P_1', P_2') = \mu(P_1', R') = \alpha \leq \mu(P_2', R'),$$

meaning that $P_1'$ and $R'$ separate at the same node, $u'$, where $P_1'$ and $P_2'$ separate.

Now recall that $v'$ is the first node that $P_1'$ and $P_2'$ have in common after separating at $u'$. Note that $v'$ cannot be in the prefix common to $P_2'$ and $R'$, because otherwise $v'$ would be a node common to $P_1'$ and $R'$, which (together with the fact that $v'$ is a strict descendant of $u'$) contradicts the fact that $P_1'$ and $R'$ form a wishbone and separate at $u'$. Thus $v'$ must belong to the suffix of $P_2'$ after separation from $R'$, or, in other words, $v'$ must have a depth in $P_2'$ strictly greater than $\mu(P_2', R')$. Now note that the only node in $P_2$ at a depth strictly greater than $\mu(P_2, R)$ is $v$, meaning (as $P_2'$ is isomorphic to $P_2$) that there can only be one node in $P_2'$ at a depth strictly greater than $\mu(P_2', R') = \mu(P_2, R)$. This node is $v_1' = v_2'$, thus allowing to conclude that $v' = v_1' = v_2'$.

Inductive case. Now suppose that the longest suffix common to $P_1$ and $P_2$ contains $k > 0$ edges, and
We show that for every wishbone or crack \( W \), there exist in \( N \) a wishbone or crack \( W \) not overlapping with \( P_1 \) and \( P'_1 \) are represented by the dashed lines. In \( (a) \), \( P_1 \) and \( R \) — and thus \( P'_1 \) and \( R' \) — form a wishbone, whereas in \( (b) \), \( P_1 \) and \( R \) — and thus \( P'_1 \) and \( R' \) — form an open crack.

Figure S8. Illustration of the argument against claim C3 in the proof of Lemma 18. \( P_1 \) and \( P'_1 \) are in black and the portions of \( P_2 \) and \( P'_2 \) not overlapping with \( P_1 \) and \( P'_1 \) are in grey. Finally, the portions of \( R \) and \( R' \) not overlapping with \( P_1 \) and \( P'_1 \) are represented by the dashed lines.

To understand the proof, assume that for every open crack with up to \( k - 1 \) edges in its longest common suffix in one of the two networks, there exists an isomorphic crack in the other network. We show that each of the following claims leads to contradiction: (C1) \( v' \) is a strict descendant of \( v'_1 \) along \( P'_1 \); (C2) \( v' \) is a strict descendant of \( v'_2 \) along \( P'_2 \); (C3) \( v' \) is a strict ancestor of \( v'_2 \) along \( P'_2 \); (C4) \( v' = v'_2 \) and \( v' \) is a strict ancestor of \( v'_1 \) along \( P'_1 \).

First, let us deal with claim C1: suppose that \( v' \) is a strict descendant of \( v'_1 \) along \( P'_1 \) (see Fig. S6). We show that for every wishbone or crack \( W' \) contained in \( P'_1 \cup P'_2 \) and containing \( P'_1 \), there exists in \( N \) a wishbone or crack \( W \) isomorphic to \( W' \). This is trivial when \( W' \) is a wishbone: because \( P'_1 \) and \( P'_2 \) are root-leaf paths to the same taxon, the only wishbone \( W' \) contained in \( P'_1 \cup P'_2 \) and containing \( P'_1 \) is \( P'_1 \) itself, for which \( W = P_1 \). This is also trivial when \( W' \) is a closed crack, as we have already proved that \( N \) and \( N' \) must have the same closed cracks (up to isomorphism). It remains the case where \( W' \) is an open crack. In this case, \( W' \) can be obtained by combining \( P'_1 \) with a weighted path contained in \( P'_2 \) that has no node or edge in common with \( P'_1 \), other than its endpoints \( w \) and \( z \). Because the first such weighted path in \( P'_2 \) is the one between \( w = u' \) and \( z = v' \), it follows that \( z \) must be a descendant of \( v' \) and thus a strict descendant of \( v'_1 \) along \( P'_1 \). Now note that, because \( P'_1 \) is isomorphic to \( P_1 \), and because the suffix of \( P_1 \) starting in \( v \) contains \( k \) edges, then the suffix of \( P'_1 \) starting in \( v'_1 \) (the node at the same depth in \( P'_1 \) as \( v \) in \( P_1 \)) also contains exactly \( k \) edges. But then, because \( z \) is a strict descendant of \( v'_1 \) along \( P'_1 \), then the suffix of \( P'_1 \) starting in \( z \) contains strictly less than \( k \) edges. That is, the longest suffix common to the two root-leaf paths composing the open crack \( W' \) contains strictly less than \( k \) edges. Then, by inductive hypothesis, there must be a crack \( W \) in \( N \) that is isomorphic to \( W' \). We have thus proved that for every wishbone or crack \( W' \) contained in \( P'_1 \cup P'_2 \) and containing \( P'_1 \), there exists in \( N \) a wishbone or crack \( W \) isomorphic to \( W' \). This, together with the fact that \( N \) satisfies the NELP property, allows us to apply Lemma 10 and conclude that the unique root-leaf paths isomorphic to \( P'_1 \) and \( P'_2 \) in \( N \), that is \( P_1 \) and \( P_2 \), must intersect each other at the same depths as \( P'_1 \) and \( P'_2 \). But this leads to a contradiction, as the node at depth \( \alpha + \beta_1 \) in \( P_1 \), that is \( v \), belongs to both \( P_1 \) and \( P_2 \), whereas the node at the same depth in \( P'_1 \), that is \( v'_1 \) only belongs to \( P'_1 \). Similarly, one can prove that claim C2 leads to contradiction.

Now assume (claim C3) that \( v' \) is a strict ancestor of \( v'_2 \) along \( P'_2 \). Recall that \( u_1 \) and \( u_2 \) are the two direct ancestors of \( v \) in \( P_1 \) and \( P_2 \), respectively, with \( u_2 \) assumed to not be an ancestor of \( u_1 \). Because \( u_2 \) has outdegree 2 or more, we can define a root-leaf path \( R \) in \( N \) that separates openly from \( P_2 \) at \( u_2 \); let \( R \) have a common prefix with \( P_2 \) consisting of the prefix of \( P_2 \) from the root down to \( u_2 \); then, let \( R \) take an edge \((u_2, w)\) with \( w \neq v \) (with any of this edge’s lengths in \( N \)) and then let \( R \) continue taking
edges always avoiding ending up in \( v \) (recall that \( N \) is funnel-free), until it either arrives in a leaf or in a node \( t \) belonging to \( P_1 \cup P_2 \). Note that, if such \( t \) exists, then it must be a strict descendant of \( u \) in the suffix common to \( P_1 \) and \( P_2 \). (By construction \( t \neq v \); moreover, if \( t \) belonged to \( P_2 \) and not to \( P_1 \), then \( t \) would lie between \( u_2 \) and \( v \) in \( P_2 \), which contradicts the assumption that \( u_2 \) is a direct ancestor of \( v \) in \( P_2 \); finally if \( t \) belonged to \( P_1 \) and not to \( P_2 \), then \( t \) would be an ancestor of \( u_1 \), implying that \( u_2 \) is an ancestor of \( u_1 \), which is not possible by construction.) Finally, from \( t \) onwards, let \( R \) coincide with the suffix common to \( P_1 \) and \( P_2 \). (See Fig. S9 to follow the argument below.)

Now consider \( P_1 \cup R \). Because \( P_1 \cup P_2 \) is a crack and \( u_2 \) is strictly between \( u \) and \( v \), \( R \) cannot have any node in common with \( P_1 \) between \( u \) and \( u_2 \) (\( u_2 \) included). Thus, if \( R \) arrives in a leaf without hitting a node in \( P_1 \cup P_2 \), then \( P_1 \cup R \) is a wishbone. Then, by Lemma 17, \( P_1' \cup R' \) is a wishbone isomorphic to \( P_1 \cup R \) (see Fig. S8(b)). If instead \( R \) hits \( P_1 \cup P_2 \) in \( t \), then \( P_1 \cup R \) is an open crack: it is a crack because, by construction, \( t \) is the first node that \( R \) has in common with \( P_1 \) after separating from it at \( u \), and because \( R \) has its suffix following \( t \) in common with \( P_1 \): it is open, because \( u_2 \neq u \) implies that the path from \( u \) to \( t \) in \( R \) is composed by more than one edge. Moreover the suffix of \( P_1 \) and \( R \) following \( t \) contains fewer than \( k \) edges, as \( t \) is a strict descendant of \( v \). Thus, by inductive hypothesis, in \( N' \) there exists a crack isomorphic to \( P_1 \cup R \). Because isomorphic cracks are the union of isomorphic root-leaf paths (Lemma 5) and because in \( N' \) there can be no root-leaf path isomorphic to \( P_1 \) and \( R \) other than \( P_1' \) and \( R' \), respectively (as \( N' \) satisfies the NELP property, by Lemma 9), then this crack can be written as \( P_1' \cup R' \) (see Fig. S8(b)). Thus, we have proved that \( P_1' \cup R' \) is either a wishbone or a crack isomorphic to \( P_1 \cup R \). Moreover, because the same holds for \( P_1, P_2 \) and \( R \) in \( N \), the following holds:

\[
\nu(P_1', P_2') = \nu(P_1', R') = \alpha \leq \nu(P_2', R') = \alpha + \beta_2 - \varepsilon,
\]

where \( \varepsilon \) denotes the length of edge \((u_2, v)\) in \( P_2 \). Thus \( P_2' \) and \( R' \) separate at a node that we denote by \( u_2' \), which has depth \( \alpha + \beta_2 - \varepsilon \) in \( P_2' \) (the same as \( u_2 \) in \( P_2 \)). When \( P_1' \cup R' \) is a crack, because it is isomorphic to \( P_1 \cup R \), the first node in the suffix common to \( P_1' \) and \( R' \) must be at the same depths in \( P_1' \) and \( R' \) as \( t \) in \( P_1 \) and \( R \). We call this node \( t' \). Because the depth of \( t \) in \( R \) is strictly larger than \( \alpha + \beta_2 - \varepsilon \), the same holds for the depth of \( t' \) in \( R' \), implying that \( t' \) must be a strict descendant of \( u_2' \).

Now recall that \( v' \) is the first node that \( P_1' \) and \( P_2' \) have in common after separating at \( u' \). There are two possibilities regarding its position in \( P_2' \) relative to \( u_2' \). First consider the case where \( v' \) is an ancestor of \( u_2' \) along \( P_2' \) (including \( v' = u_2' \)). In this case \( v' \) is a strict descendant of \( u' \) in the prefix common to \( P_2 \) and \( R' \), and at the same time \( v' \) is a node in \( P_1' \). But this contradicts both possible relations between \( P_1' \) and \( R' \): \( P_1' \) and \( R' \) forming a wishbone at \( u_2' \) and \( v_2' \) forming a crack by separating at \( u' \) and rejoining in \( t' \) (which, as we showed, must be a strict descendant of \( u_2' \), implying \( t' \neq v' \)). The other case to consider is that of \( v' \) being a strict descendant of \( u_2' \) along \( P_2' \), while being a strict ancestor of \( v_2' \) along \( P_2' \) (claim C3). That is, \( v' \) is strictly between \( u_2' \) and \( v_2' \) in \( P_2' \). But this is impossible, as together with the fact that \( P_2' \) and \( P_2' \) are isomorphic, it would imply the existence of a node strictly between \( u_2 \) and \( v \) in \( P_2 \), which is excluded by construction. Since all these possibilities lead to a contradiction, we conclude that C3 is also impossible.

Finally, let us deal with claim C4: suppose that \( v' = v_2' \), and that \( v' \) is a strict ancestor of \( v_1' \) along \( P_1' \). (See Fig. S9 to follow the argument below.) Recall that, because \( P_1' \) and \( P_2' \) are isomorphic to \( P_1 \) and \( P_2 \), they have lengths \( \alpha + \beta_1 + \gamma \) and \( \alpha + \beta_2 + \gamma \), respectively. Also recall that \( P_1' \) and \( P_2' \) separate at \( u' \) (with depth \( \alpha \) in \( P_1' \) and \( P_2' \)) and that the respective depths of \( v_1' \) and \( v_2' \) in \( P_1' \) and \( P_2' \) are \( \alpha + \beta_1 \) and \( \alpha + \beta_2 \). Now, let \( \delta > 0 \) denote the length of the weighted path in \( P_1' \) from \( v' \) to \( v_1' \), implying that the weighted path in \( P_1' \) from \( u' \) to \( v' \) has length \( \beta_1 - \delta > 0 \) (strictly positive because by construction \( u' \neq v' \)).

Now define in \( N' \) a new root-leaf path \( S' \) that coincides with \( P_1' \) and \( P_2' \) along all their common prefix down to \( u' \) (of length \( \alpha \)), then coincides with \( P_1' \) along the weighted path in \( P_1' \) from \( u' \) to \( v' \) (of length \( \beta_1 - \delta \)), and finally coincides with \( P_2' \) along all its suffix from \( v' \) onwards (of length \( \gamma \)). As a result, \( S' \) and \( P_1' \) have a common prefix down to \( v' \) and therefore \( \nu(P_1', S') \geq \alpha + \beta_1 - \delta \).

Let us now focus on the consequences on \( N \) of these definitions. Let \( S \) denote the unique root-leaf path in \( N \) isomorphic to \( S' \) (Lemma 16) and let \( s \) be the node at depth \( \alpha + \beta_1 - \delta \) in \( P_1 \) (whose existence
is guaranteed by the existence of node $v'$ at the same depth in $P_1'$. Because the same holds for $P_1'$, $P_2'$ and $S'$ in $N'$, the following holds (Lemma 15):

$$\mu(P_1, P_2) = \mu(P_2, S) = \alpha < \alpha + \beta_1 - \delta \leq \mu(P_1, S),$$

meaning that the prefix of $P_1$ down to $s$ is entirely in common with $S$ (and therefore $S$ and $P_2$ separate in $u$). But this implies that all the nodes that $S$ has in common with $P_2$ after separating from it at $u$ must be strict descendants of $s$, as otherwise $P_1$ and $P_2$ would have nodes in common between $u$ and $v$, which is excluded by construction.

Now note that, because $P_2'$ is isomorphic to $P_2$, and because the suffix of $P_2$ starting in $v$ contains $k$ edges, then the suffix of $P_2'$ starting in $v' = v_2'$ (the node at the same depth in $P_2'$ as $v$ in $P_2$) also contains exactly $k$ edges. But this suffix coincides with that of $S'$ starting in $v' = v_2'$. Then, also the suffix of $S$ starting in $s$ (the node at the same depth in $S$ as $v'$ in $S'$) contains exactly $k$ edges. This observation, together with the fact that all the nodes that $S$ has in common with $P_2$ after separating from it must be strict descendants of $s$, allows us to show that for every wishbone or crack $W$ contained in $P_2 \cup S$ and containing $S$, there exists in $N'$ a wishbone or crack $W'$ isomorphic to $W$. This is trivial when $W$ is a wishbone: because $P_2$ and $S$ are root-leaf paths to the same taxon, the only wishbone $W$ contained in $P_2 \cup S$ and containing $S$ is $S$ itself, for which $W' = S'$. This is also trivial when $W$ is a closed crack, as we have already proved that $N$ and $N'$ must have the same closed cracks (up to isomorphism). It remains the case where $W$ is an open crack. In this case, it can be written as $W = S \cup Q$, where $Q$ is a root-leaf path contained in $P_2 \cup S$, separating openly from $S$ at some internal node in $S$. Because all the nodes that $S$ and $P_2$ have in common after separating at $u$ must be strict descendants of $s$, also the first node in the longest suffix common to $S$ and $Q$ must be a strict descendant of $s$, meaning that this suffix must contain strictly less than $k$ edges. Then, by inductive hypothesis, there must be a crack $W'$ in $N'$ that is isomorphic to $W = S \cup Q$. Because for every wishbone or crack $W$ contained in $P_2 \cup S$ and containing $S$, there exists in $N'$ a wishbone or crack $W'$ isomorphic to $W$, then $P_2'$ and $S'$ must intersect each other at the same depths as $P_2$ and $S$ (by Lemma 10). But this leads to a contradiction, as the node at depth $\alpha + \beta_1 - \delta$ in $S'$, that is $v' = v_2'$, belongs to both $S'$ and $P_2'$, whereas the node at the same depth in $S$, that is $s$, only belongs to $S$.

We have thus proved that each of C1-C4 leads to a contradiction. The fact that neither C2 nor C3 can hold implies that $v' = v_2'$. This, together with the fact that neither C1 nor C4 can hold, implies $v' = v_1'$. Thus, $v' = v_1' = v_2'$, which, as explained in the introduction of this proof, implies that $P_1' \cup P_2'$ is a crack isomorphic to $P_1 \cup P_2$, and thus the lemma follows. \end{proof}
Proposition 2 follows from Corollary 4 and Lemma 18.

Networks with inheritance probabilities and likelihood-based reconstruction

As described in the main text, the ML framework we consider [2, 32, 33, 38] not only models edge lengths, but also inheritance probabilities. The latter provide, for each reticulate edge in a network \( N \), the probability that a random tree \( T \) displayed by \( N \) includes that edge (i.e., that inheritance follows that edge). The inheritance probabilities determine, for each tree \( T \in \mathcal{T}(N) \), an associated probability \( \text{Pr}(T|N) \).

Unfortunately, including inheritance probabilities does not solve identifiability problems: in Fig. S10 we show an example of two phylogenetic networks \( N_1, N_2 \) with edge lengths and inheritance probabilities that cannot be distinguished on the basis of the trees they display and associated probabilities. By setting the edge lengths as shown, and the inheritance probabilities so that \( p_2 = 1 - (1 - p_1)(1 - q_1) \) and \( q_2 = p_1/p_2 \), it is easy to check that \( \text{Pr}(T_1|N_1) = \text{Pr}(T_1|N_2) \), \( \text{Pr}(T_2|N_1) = \text{Pr}(T_2|N_2) \) and \( \text{Pr}(T_3|N_1) = \text{Pr}(T_3|N_2) \). Thus, for every assignment of edge lengths and inheritance probabilities to \( N_1 \), there exist corresponding assignments to \( N_2 \) that make the resulting networks display the same trees, with the same edge lengths and the same probabilities of being observed. Because \( \mathcal{T}(N_1) = \mathcal{T}(N_2) \), and \( \text{Pr}(T|N_1) = \text{Pr}(T|N_2) \) for any \( T \) displayed by these two networks, it is easy to see that

\[
\prod_{i=1}^{m} \sum_{T \in \mathcal{T}(N_k)} \text{Pr}(A_i|T)\text{Pr}(T|N_k).
\]

is the same for \( k = 1 \) or \( k = 2 \), that is, the likelihoods of \( N_1 \) and \( N_2 \) are identical regardless of the data. In other words, no method based on this definition of likelihood will be able to discriminate between them.

The example above is not an exception: for any two distinct indistinguishable networks (i.e., with \( \mathcal{T}(N_1) = \mathcal{T}(N_2) \)), it is possible to provide assignments of inheritance probabilities to their reticulations, so that not only these networks display the same trees, but that also the probabilities associated to the trees they display are identical.

Moreover, we can extend the notion of indistinguishability, as well as that of canonical form, to networks with inheritance probabilities. Results entirely analogous to those we presented in this paper will then hold. We will not prove any of this here, as it lies beyond the scope of the present study.
Figure S10. An example showing that the ML framework considered in the main text is also subject to identifiability problems: the two networks $N_1$ and $N_2$ (top) with edge lengths (gray) and inheritance probabilities (black), display the same trees $T_1, T_2, T_3$ (middle), with the same associated probabilities (bottom), when $p_2 = 1 - (1 - p_1)(1 - q_1)$ and $q_2 = p_1/p_2$. 

$$
\begin{align*}
\Pr(T_1|N_1) &= p_1 & \Pr(T_2|N_1) &= (1 - p_1)(1 - q_1) & \Pr(T_3|N_1) &= (1 - p_1)q_1 \\
\Pr(T_1|N_2) &= p_2q_2 & \Pr(T_2|N_2) &= 1 - p_2 & \Pr(T_3|N_2) &= p_2(1 - q_2)
\end{align*}
$$