2-Distance Coloring of Sparse Graphs

Marthe Bonamy, Benjamin Lévêque, Alexandre Pinlou

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Marthe Bonamy [a,1] Benjamin Lévêque [a] Alexandre Pinlou [a,2]

[a] *LIRMM, Université Montpellier 2, CNRS*
{marthe.bonamy,benjamin.leveque,alexandre.pinlou}@lirmm.fr

Abstract

A 2-distance coloring of a graph is a coloring of the vertices such that two vertices at distance at most 2 receive distinct colors. We prove that every graph with maximum degree Δ at least 4 and maximum average degree less that $\frac{7}{3}$ admits a 2-distance $(\Delta + 1)$-coloring. This result is tight. This improves previous known results of Dolama and Sopena.

Keywords: 2-distance coloring; square coloring; maximum average degree.

1 Introduction

All the graphs we consider here are simple, finite and undirected. Let $G = (V, E)$ be a graph. For any subgraph H of G, we denote $V(H)$ and $E(H)$ the vertices and edges of H. For any vertex $v \in V$, the *degree* of v in G, denoted $d(v)$, is the number of neighbors of v in G. The *maximum degree* of G, denoted $\Delta(G)$, is $\max_{v \in V} d(v)$. The *maximum average degree* of G, denoted $\mathrm{mad}(G)$, is the maximum for every subgraph H of G of $\frac{2|E(H)|}{|V(H)|}$. A 2-*distance coloring*

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[1] École Normale Supérieure de Lyon

[2] Second affiliation: Département MIAp, Université Paul-Valéry, Montpellier 3

of a graph G is a coloring of the vertices of G such that two vertices that are adjacent or have a common neighbor receive distinct colors. This is equivalent to a proper vertex-coloring of the square of G. We define $\chi^2(G)$ as the smallest k such that G admits a 2-distance k-coloring. Note that any graph G satisfies $\chi^2(G) \geq \Delta(G) + 1$. The *girth* $g(G)$ is the length of a shortest cycle in G. Two vertices x and y are *p-linked* if there exists a path x-v_1-$\cdots$-v_p-y such that vertices $v_1, \ldots, v_p$ have degree 2, and v_1-$\cdots$-v_p is called a *branch* of x (or y).

Borodin, Ivanova and Neustroeva [1] studied sparse planar graphs, and prove the following result:

Theorem 1.1 ([1]) *Every planar graph G with $g(G) \geq 15$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.*

Note that this result was later extended to list-coloring [2].

Dolama and Sopena [3] proved a more general result than Theorem 1.1, which is not restricted to planar graphs anymore. Theorem 1.2 however presents a slight loss in quality compared to Theorem 1.1: since for any planar graph G, $(\text{mad}(G) - 2)(g(G) - 2) < 4$, Theorem 1.2 implies only that Theorem 1.1 holds for $g(G) \geq 16$.

Theorem 1.2 ([3]) *Every graph G with $\text{mad}(G) < \frac{16}{7}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.*

We aim at making the upper bound on the maximum average degree optimal, and prove the following.

Theorem 1.3 *Every graph G with $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.*

The bound we obtain is optimal. Indeed, as pointed out by Montassier [6], there is a graph G with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$ (see Figure 1).

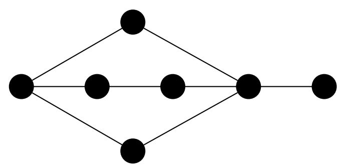

Fig. 1. A graph G with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$.

When restricted to planar graphs, Theorem 1.3 is an improvement of Theorem 1.1 as it implies that Theorem 1.1 holds with $g(G) \geq 14$. It is not comparable to the more general result in [2], since we are not considering list-coloring.

We are going to use a discharging method to prove Theorem 1.3. We will prove that there are some configurations a minimal counter-example cannot contain, and, then use discharging rules to show that this graph does not exist.

2 Proof

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label 'i' means "exactly i neighbors", the label 'i^+' (resp. 'i^-') means that it has at least (resp. at most) i neighbors. Note that the white vertices may coincide with other vertices. The label '$T(v,a)$' inside a vertex v means that $T(v,a)$ exists, as defined below.

A configuration $T(v,a_4)$ (see Figure 2), is inductively defined as a vertex v of degree 4 with neighbors a_1, a_2, a_3, a_4, where for $i \in \{1,2,3\}$, vertex v is 2-linked by a path v-a_i-b_i-w_i either to a vertex w_i of degree at most 3 or to a configuration $T(w_i,b_i)$.

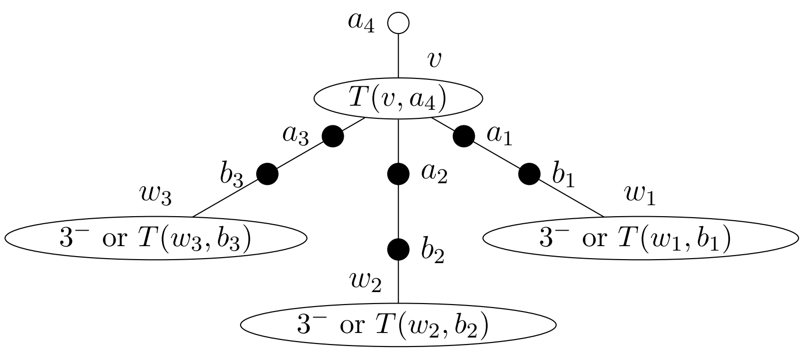

Fig. 2. A $T(v,u_4)$.

Now we define configurations (C_1) to (C_5) (see Figure 3).

- (C_1) is a vertex of degree 0 or 1.
- (C_2) is a vertex 3-linked to a vertex not of maximal degree.
- (C_3) is a vertex of degree 3 that is 2-linked to two vertices of degree 3, and 1-linked to a vertex of degree at most 3.
- (C_4) is a vertex u of degree at most 3 that is 2-linked by a path u-y-x-v to a vertex v such that $T(v,x)$ exists.
- (C_5) is a vertex u of degree 3 that is 2-linked to two vertices, and 1-linked by a path u-x-v to a vertex v such that $T(v,x)$ exists.

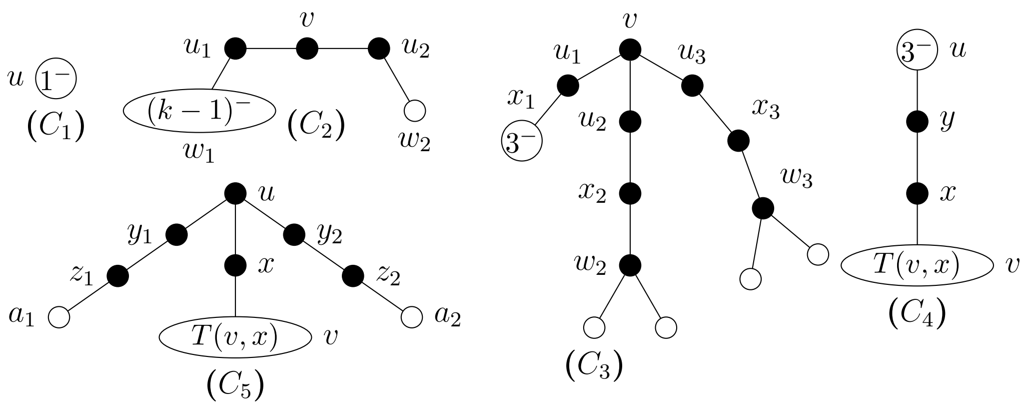

Fig. 3. Forbidden configurations.

In the following lemma, we actually use k instead of $\Delta(G)$ in order to ensure that any subgraph of G admits a $(k+1)$-coloring even though Δ can decrease.

A graph is *minimal* for a property if it satisfies this property but none of its subgraphs does.

Lemma 2.1 *Let $k \geq 4$ and G such that $\Delta(G) \leq k$ and G admits no 2-distance $(k+1)$-coloring, and G is minimal for this property. Then G does not contain any of Configurations (C_1) to (C_5).*

The following lemma will ensure that the discharging rules we introduce later are well-defined.

Lemma 2.2 *In a graph G where (C_4) is forbidden, and x and y are two vertices of degree 4 that are 2-linked by a path x-a-b-y, at most one of $T(x,a)$ and $T(y,b)$ exists.*

We design discharging rules R_1, R_2, R_3 (see Figure 4). We use them in the proof of Lemma 2.3, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least $\frac{7}{3}$. For any two vertices x and y of degree at least 3, with $d(x) \geq d(y)$,

- Rule R_1 is when x and y are 1-linked by a path $x - a - y$.
  - · $(R_{1.1})$ If $d(x) = d(y)$, then both x and y give $\frac{1}{6}$ to a.
  - · $(R_{1.2})$ If $d(x) > d(y)$ and $T(x,a)$ exists, then both x and y give $\frac{1}{6}$ to a.
  - · $(R_{1.3})$ If $d(x) > d(y)$ and $T(x,a)$ does not exist, then x gives $\frac{1}{3}$ to a.
- Rule R_2 is when x and y are 2-linked by a path $x - a - b - y$.
  - · $(R_{2.1})$ If $d(x) = d(y)$ and neither $T(x,a)$ nor $T(y,b)$ exist, then x (resp. y) gives $\frac{1}{3}$ to a (resp. b).

- · ($R_{2.2}$) If $d(x) = d(y)$ and $T(y, b)$ exists, then x gives $\frac{1}{3}$ to a and both x and y give $\frac{1}{6}$ to b.
- · ($R_{2.3}$) If $d(x) > d(y)$, then x gives $\frac{1}{3}$ to a and both x and y give $\frac{1}{6}$ to b.

- Rule R_3 is when x and y, both of degree at least 4, are 3-linked by a path $x - a - b - c - y$. Then x gives $\frac{1}{3}$ to a and $\frac{1}{6}$ to b, and symmetrically for y.

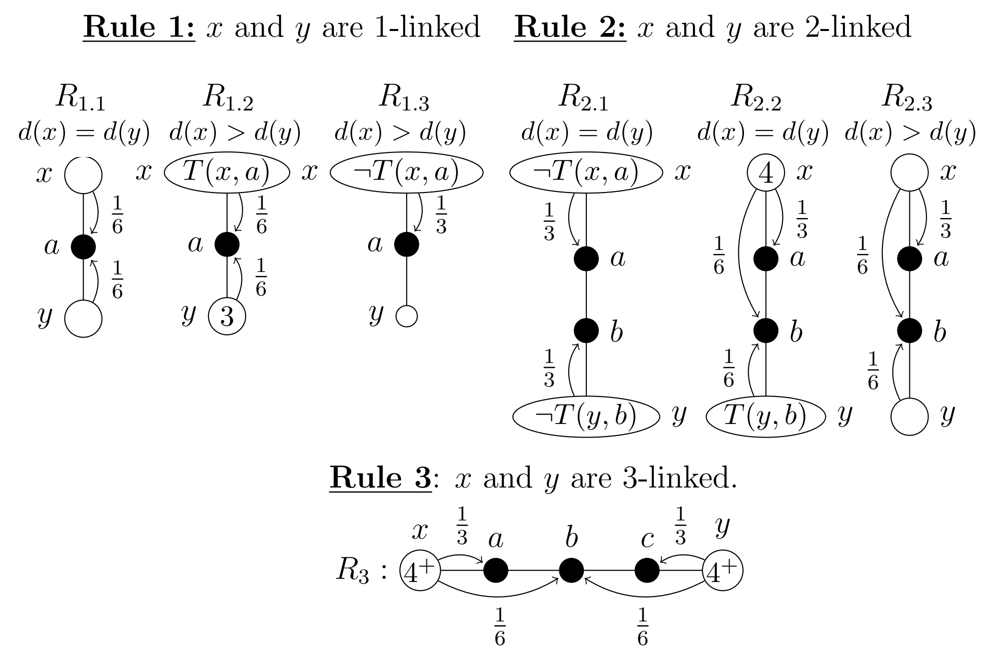

Fig. 4. Discharging rules R_1, R_2, R_3.

We use these discharging rules to prove the following lemma:

Lemma 2.3 *A graph G that does not contain Configurations (C_1) to (C_5) verifies* $\text{mad}(G) \geq \frac{7}{3}$.

Proof of Theorem 1.3

We prove a stronger version of Theorem 1.3 by contradiction. For $k \geq 4$, let G be a minimal graph such that $\Delta(G) \leq k$, $\text{mad}(G) < \frac{7}{3}$ and G does not admit a $(k+1)$-coloring. Graph G is also a minimal graph such that $\Delta(G) \leq k$ and G does not admit a $(k+1)$-coloring (all its proper subgraphs verify $\Delta \leq k$ and mad $< \frac{7}{3}$, so they admit a $(k+1)$-coloring). By Lemma 2.1, graph G cannot contain (C_1) to (C_5). Lemma 2.3 implies that $\text{mad}(G) \geq \frac{7}{3}$. Contradiction. $\square$

3 Conclusion

We actually proved a slightly stronger result than Theorem 1.3. However, the addition, namely that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ and $\Delta(G) \leq 3$ admits a 2-distance 5-coloring, can be derived from a result of Dvořák, Škrekovski and Tancer [4].

Note that the proof of Theorem 1.3 also provides an $O(|V|^3)$ algorithm to find a 2-distance coloring of a graph G with $\Delta(G) + 1$ colors if G verifies the hypothesis of Theorem 1.3: indeed Lemma 2.3 proves that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ contains (C_1), (C_2), ... or (C_5). Consequently, we can find a (C_i) in G, remove the corresponding vertices, and extend the coloring to the initial graph using the proof of Lemma 2.1.

As it was conjectured by Kostochka and Woodall [5] that 2-distance list-coloring requires exactly as many colors as 2-distance coloring, future work could aim at extending Theorem 1.3 to list-coloring.

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