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2-distance coloring of sparse graphs

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Abstract
A 2-distance coloring of a graph is a coloring of the vertices such that two vertices at distance at most 2 receive distinct colors. We prove that every graph with maximum degree $\Delta$ at least 4 and maximum average degree less than $\frac{7}{3}$ admits a 2-distance $(\Delta + 1)$-coloring. This result is tight. This improves previous known results of Dolama and Sopena.

Keywords: 2-distance coloring; square coloring; maximum average degree.

1 Introduction

All the graphs we consider here are simple, finite and undirected. Let $G = (V, E)$ be a graph. For any subgraph $H$ of $G$, we denote $V(H)$ and $E(H)$ the vertices and edges of $H$. For any vertex $v \in V$, the degree of $v$ in $G$, denoted $d(v)$, is the number of neighbors of $v$ in $G$. The maximum degree of $G$, denoted $\Delta(G)$, is $\max_{v \in V} d(v)$. The maximum average degree of $G$, denoted $\text{mad}(G)$, is the maximum for every subgraph $H$ of $G$ of $\frac{|E(H)|}{|V(H)|}$. A 2-distance coloring

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of a graph $G$ is a coloring of the vertices of $G$ such that two vertices that are adjacent or have a common neighbor receive distinct colors. This is equivalent to a proper vertex-coloring of the square of $G$. We define $\chi^2(G)$ as the smallest $k$ such that $G$ admits a 2-distance $k$-coloring. Note that any graph $G$ satisfies $\chi^2(G) \geq \Delta(G) + 1$. The girth $g(G)$ is the length of a shortest cycle in $G$.

Two vertices $x$ and $y$ are $p$-linked if there exists a path $x-v_1\cdots v_p-y$ such that vertices $v_1,\ldots,v_p$ have degree 2, and $v_1\cdots v_p$ is called a branch of $x$ (or $y$).

Borodin, Ivanova and Neustroeva [1] studied sparse planar graphs, and prove the following result:

**Theorem 1.1 ([1])** Every planar graph $G$ with $g(G) \geq 15$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

Note that this result was later extended to list-coloring [2].

Dolama and Sopena [3] proved a more general result than Theorem 1.1, which is not restricted to planar graphs anymore. Theorem 1.2 however presents a slight loss in quality compared to Theorem 1.1: since for any planar graph $G$, $(\text{mad}(G) - 2)(g(G) - 2) < 4$, Theorem 1.2 implies only that Theorem 1.1 holds for $g(G) \geq 16$.

**Theorem 1.2 ([3])** Every graph $G$ with $\text{mad}(G) < \frac{16}{7}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

We aim at making the upper bound on the maximum average degree optimal, and prove the following.

**Theorem 1.3** Every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

The bound we obtain is optimal. Indeed, as pointed out by Montassier [6], there is a graph $G$ with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$ (see Figure 1).

![Fig. 1. A graph $G$ with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$.](image)

When restricted to planar graphs, Theorem 1.3 is an improvement of Theorem 1.1 as it implies that Theorem 1.1 holds with $g(G) \geq 14$. It is not comparable to the more general result in [2], since we are not considering list-coloring.
We are going to use a discharging method to prove Theorem 1.3. We will prove that there are some configurations a minimal counter-example cannot contain, and, then use discharging rules to show that this graph does not exist.

2 Proof

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label \(i\) means "exactly \(i\) neighbors", the label \(i^+\) (resp. \(i^-\)) means that it has at least (resp. at most) \(i\) neighbors. Note that the white vertices may coincide with other vertices. The label \(T(v,a)\) inside a vertex \(v\) means that \(T(v,a)\) exists, as defined below.

A configuration \(T(v,a_4)\) (see Figure 2), is inductively defined as a vertex \(v\) of degree 4 with neighbors \(a_1, a_2, a_3, a_4\,\text{, where for } i \in \{1, 2, 3\}, \text{ vertex } v \text{ is } 2\text{-linked by a path } v-a_i-b_i-w_i \text{ either to a vertex } w_i \text{ of degree at most 3 or to a configuration } T(w_i, b_i)\).

![Fig. 2. A \(T(v, u_4)\).](image)

Now we define configurations \((C_1)\) to \((C_5)\) (see Figure 3).

- \((C_1)\) is a vertex of degree 0 or 1.
- \((C_2)\) is a vertex 3-linked to a vertex not of maximal degree.
- \((C_3)\) is a vertex of degree 3 that is 2-linked to two vertices of degree 3, and 1-linked to a vertex of degree at most 3.
- \((C_4)\) is a vertex \(u\) of degree at most 3 that is 2-linked by a path \(u-y-x-v\) to a vertex \(v\) such that \(T(v, x)\) exists.
- \((C_5)\) is a vertex \(u\) of degree 3 that is 2-linked to two vertices, and 1-linked by a path \(u-x-v\) to a vertex \(v\) such that \(T(v, x)\) exists.
In the following lemma, we actually use \( k \) instead of \( \Delta(G) \) in order to ensure that any subgraph of \( G \) admits a \((k + 1)\)-coloring even though \( \Delta \) can decrease.

A graph is \emph{minimal} for a property if it satisfies this property but none of its subgraphs does.

**Lemma 2.1** Let \( k \geq 4 \) and \( G \) such that \( \Delta(G) \leq k \) and \( G \) admits no 2-distance \((k + 1)\)-coloring, and \( G \) is minimal for this property. Then \( G \) does not contain any of Configurations \((C_1)\) to \((C_5)\).

The following lemma will ensure that the discharging rules we introduce later are well-defined.

**Lemma 2.2** In a graph \( G \) where \((C_4)\) is forbidden, and \( x \) and \( y \) are two vertices of degree 4 that are 2-linked by a path \( x-a-b-y \), at most one of \( T(x, a) \) and \( T(y, b) \) exists.

We design discharging rules \( R_1 \), \( R_2 \), \( R_3 \) (see Figure 4). We use them in the proof of Lemma 2.3, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least \( \frac{7}{3} \). For any two vertices \( x \) and \( y \) of degree at least 3, with \( d(x) \geq d(y) \),

- Rule \( R_1 \) is when \( x \) and \( y \) are 1-linked by a path \( x-a-y \).
  - \((R_{1,1})\) If \( d(x) = d(y) \), then both \( x \) and \( y \) give \( \frac{1}{6} \) to \( a \).
  - \((R_{1,2})\) If \( d(x) > d(y) \) and \( T(x, a) \) exists, then both \( x \) and \( y \) give \( \frac{1}{6} \) to \( a \).
  - \((R_{1,3})\) If \( d(x) > d(y) \) and \( T(x, a) \) does not exist, then \( x \) gives \( \frac{1}{3} \) to \( a \).

- Rule \( R_2 \) is when \( x \) and \( y \) are 2-linked by a path \( x-a-b-y \).
  - \((R_{2,1})\) If \( d(x) = d(y) \) and neither \( T(x, a) \) nor \( T(y, b) \) exist, then \( x \) (resp. \( y \)) gives \( \frac{1}{3} \) to \( a \) (resp. \( b \)).
• \((R_{2.2})\) If \(d(x) = d(y)\) and \(T(y, b)\) exists, then \(x\) gives \(\frac{1}{3}\) to \(a\) and both \(x\) and \(y\) give \(\frac{1}{6}\) to \(b\).

• \((R_{2.3})\) If \(d(x) > d(y)\), then \(x\) gives \(\frac{1}{3}\) to \(a\) and both \(x\) and \(y\) give \(\frac{1}{6}\) to \(b\).

- Rule \(R_3\) is when \(x\) and \(y\), both of degree at least 4, are 3-linked by a path \(x - a - b - c - y\). Then \(x\) gives \(\frac{1}{3}\) to \(a\) and \(\frac{1}{6}\) to \(b\), and symmetrically for \(y\).

**Rule 1:** \(x\) and \(y\) are 1-linked

**Rule 2:** \(x\) and \(y\) are 2-linked

**Rule 3:** \(x\) and \(y\) are 3-linked.

We use these discharging rules to prove the following lemma:

**Lemma 2.3** A graph \(G\) that does not contain Configurations \((C_1)\) to \((C_5)\) verifies \(\text{mad}(G) \geq \frac{7}{3}\).

**Proof of Theorem 1.3**

We prove a stronger version of Theorem 1.3 by contradiction. For \(k \geq 4\), let \(G\) be a minimal graph such that \(\Delta(G) \leq k\), \(\text{mad}(G) < \frac{7}{3}\) and \(G\) does not admit a \((k + 1)\)-coloring. Graph \(G\) is also a minimal graph such that \(\Delta(G) \leq k\) and \(G\) does not admit a \((k + 1)\)-coloring (all its proper subgraphs verify \(\Delta \leq k\) and \(\text{mad} < \frac{7}{3}\), so they admit a \((k + 1)\)-coloring). By Lemma 2.1, graph \(G\) cannot contain \((C_1)\) to \((C_5)\). Lemma 2.3 implies that \(\text{mad}(G) \geq \frac{7}{3}\). Contradiction. \(\square\)
3 Conclusion

We actually proved a slightly stronger result than Theorem 1.3. However, the
addition, namely that every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) \leq 3$ admits
a 2-distance 5-coloring, can be derived from a result of Dvořák, Škrekovski
and Tancer [4].

Note that the proof of Theorem 1.3 also provides an $O(|V|^3)$ algorithm to
find a 2-distance coloring of a graph $G$ with $\Delta(G) + 1$ colors if $G$ verifies the
hypothesis of Theorem 1.3; indeed Lemma 2.3 proves that every graph $G$ with
$\text{mad}(G) < \frac{7}{3}$ contains $(C_1), (C_2), \ldots$ or $(C_5)$. Consequently, we can find a
$(C_i)$ in $G$, remove the corresponding vertices, and extend the coloring to the
initial graph using the proof of Lemma 2.1.

As it was conjectured by Kostochka and Woodall [5] that 2-distance list-
coloring requires exactly as many colors as 2-distance coloring, future work
could aim at extending Theorem 1.3 to list-coloring.

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