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2-distance coloring of sparse graphs

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Abstract

A 2-distance coloring of a graph is a coloring of the vertices such that two vertices at distance at most 2 receive distinct colors. We prove that every graph with maximum degree $\Delta$ at least 4 and maximum average degree less than $\frac{7}{3}$ admits a 2-distance $(\Delta + 1)$-coloring. This result is tight. This improves previous known results of Dolama and Sopena.

Keywords: 2-distance coloring; square coloring; maximum average degree.

1 Introduction

All the graphs we consider here are simple, finite and undirected. Let $G = (V,E)$ be a graph. For any subgraph $H$ of $G$, we denote $V(H)$ and $E(H)$ the vertices and edges of $H$. For any vertex $v \in V$, the degree of $v$ in $G$, denoted $d(v)$, is the number of neighbors of $v$ in $G$. The maximum degree of $G$, denoted $\Delta(G)$, is $\max_{v \in V} d(v)$. The maximum average degree of $G$, denoted $\text{mad}(G)$, is the maximum for every subgraph $H$ of $G$ of $\frac{2|E(H)|}{|V(H)|}$. A 2-distance coloring

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of a graph $G$ is a coloring of the vertices of $G$ such that two vertices that are adjacent or have a common neighbor receive distinct colors. This is equivalent to a proper vertex-coloring of the square of $G$. We define $\chi^2(G)$ as the smallest $k$ such that $G$ admits a 2-distance $k$-coloring. Note that any graph $G$ satisfies $\chi^2(G) \geq \Delta(G) + 1$. The girth $g(G)$ is the length of a shortest cycle in $G$. Two vertices $x$ and $y$ are $p$-linked if there exists a path $x-v_1\cdots v_p-y$ such that vertices $v_1,\ldots,v_p$ have degree 2, and $v_1\cdots v_p$ is called a branch of $x$ (or $y$).

Borodin, Ivanova and Neustroeva [1] studied sparse planar graphs, and prove the following result:

**Theorem 1.1 ([1])** Every planar graph $G$ with $g(G) \geq 15$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

Note that this result was later extended to list-coloring [2].

Dolama and Sopena [3] proved a more general result than Theorem 1.1, which is not restricted to planar graphs anymore. Theorem 1.2 however presents a slight loss in quality compared to Theorem 1.1: since for any planar graph $G$, $(\text{mad}(G) - 2)(g(G) - 2) < 4$, Theorem 1.2 implies only that Theorem 1.1 holds for $g(G) \geq 16$.

**Theorem 1.2 ([3])** Every graph $G$ with $\text{mad}(G) < \frac{16}{7}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

We aim at making the upper bound on the maximum average degree optimal, and prove the following.

**Theorem 1.3** Every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

The bound we obtain is optimal. Indeed, as pointed out by Montassier [6], there is a graph $G$ with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$ (see Figure 1).

![Graph](https://example.com/graph.png)

**Fig. 1.** A graph $G$ with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$.

When restricted to planar graphs, Theorem 1.3 is an improvement of Theorem 1.1 as it implies that Theorem 1.1 holds with $g(G) \geq 14$. It is not comparable to the more general result in [2], since we are not considering list-coloring.
We are going to use a discharging method to prove Theorem 1.3. We will prove that there are some configurations a minimal counter-example cannot contain, and, then use discharging rules to show that this graph does not exist.

2 Proof

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label 'i' means "exactly i neighbors", the label 'i+' (resp. 'i-') means that it has at least (resp. at most) i neighbors. Note that the white vertices may coincide with other vertices. The label 'T(v, a)' inside a vertex v means that T(v, a) exists, as defined below.

A configuration T(v, a4) (see Figure 2), is inductively defined as a vertex v of degree 4 with neighbors a1, a2, a3, a4, where for i ∈ {1, 2, 3}, vertex v is 2-linked by a path v-a_i-b_i-w_i either to a vertex w_i of degree at most 3 or to a configuration T(w_i, b_i).

Now we define configurations (C1) to (C5) (see Figure 3).

• (C1) is a vertex of degree 0 or 1.
• (C2) is a vertex 3-linked to a vertex not of maximal degree.
• (C3) is a vertex of degree 3 that is 2-linked to two vertices of degree 3, and 1-linked to a vertex of degree at most 3.
• (C4) is a vertex u of degree at most 3 that is 2-linked by a path u-y-x-v to a vertex v such that T(v, x) exists.
• (C5) is a vertex u of degree 3 that is 2-linked to two vertices, and 1-linked by a path u-x-v to a vertex v such that T(v, x) exists.
In the following lemma, we actually use $k$ instead of $\Delta(G)$ in order to ensure that any subgraph of $G$ admits a $(k + 1)$-coloring even though $\Delta$ can decrease.

A graph is minimal for a property if it satisfies this property but none of its subgraphs does.

**Lemma 2.1** Let $k \geq 4$ and $G$ such that $\Delta(G) \leq k$ and $G$ admits no 2-distance $(k + 1)$-coloring, and $G$ is minimal for this property. Then $G$ does not contain any of Configurations $(C_1)$ to $(C_5)$.

The following lemma will ensure that the discharging rules we introduce later are well-defined.

**Lemma 2.2** In a graph $G$ where $(C_4)$ is forbidden, and $x$ and $y$ are two vertices of degree 4 that are 2-linked by a path $x-a-b-y$, at most one of $T(x,a)$ and $T(y,b)$ exists.

We design discharging rules $R_1$, $R_2$, $R_3$ (see Figure 4). We use them in the proof of Lemma 2.3, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least $\frac{7}{3}$. For any two vertices $x$ and $y$ of degree at least 3, with $d(x) \geq d(y)$,

- Rule $R_1$ is when $x$ and $y$ are 1-linked by a path $x - a - y$.
  - $(R_{1,1})$ If $d(x) = d(y)$, then both $x$ and $y$ give $\frac{1}{6}$ to $a$.
  - $(R_{1,2})$ If $d(x) > d(y)$ and $T(x,a)$ exists, then both $x$ and $y$ give $\frac{1}{6}$ to $a$.
  - $(R_{1,3})$ If $d(x) > d(y)$ and $T(x,a)$ does not exist, then $x$ gives $\frac{1}{3}$ to $a$.
- Rule $R_2$ is when $x$ and $y$ are 2-linked by a path $x - a - b - y$.
  - $(R_{2,1})$ If $d(x) = d(y)$ and neither $T(x,a)$ nor $T(y,b)$ exist, then $x$ (resp. $y$) gives $\frac{1}{3}$ to $a$ (resp. $b$).
- \((R_{2.2})\) If \(d(x) = d(y)\) and \(T(y, b)\) exists, then \(x\) gives \(\frac{1}{3}\) to \(a\) and both \(x\) and \(y\) give \(\frac{1}{6}\) to \(b\).
- \((R_{2.3})\) If \(d(x) > d(y)\), then \(x\) gives \(\frac{1}{3}\) to \(a\) and both \(x\) and \(y\) give \(\frac{1}{6}\) to \(b\).

- Rule \(R_3\) is when \(x\) and \(y\), both of degree at least 4, are 3-linked by a path \(x - a - b - c - y\). Then \(x\) gives \(\frac{1}{3}\) to \(a\) and \(\frac{1}{6}\) to \(b\), and symmetrically for \(y\).

**Rule 1:** \(x\) and \(y\) are 1-linked

**Rule 2:** \(x\) and \(y\) are 2-linked

**Rule 3:** \(x\) and \(y\) are 3-linked.

![Fig. 4. Discharging rules \(R_1, R_2, R_3\).](image)

We use these discharging rules to prove the following lemma:

**Lemma 2.3** A graph \(G\) that does not contain Configurations \((C_1)\) to \((C_5)\) verifies \(\text{mad}(G) \geq \frac{7}{3}\).

**Proof of Theorem 1.3**

We prove a stronger version of Theorem 1.3 by contradiction. For \(k \geq 4\), let \(G\) be a minimal graph such that \(\Delta(G) \leq k\), \(\text{mad}(G) < \frac{7}{3}\) and \(G\) does not admit a \((k + 1)\)-coloring. Graph \(G\) is also a minimal graph such that \(\Delta(G) \leq k\) and \(G\) does not admit a \((k + 1)\)-coloring (all its proper subgraphs verify \(\Delta \leq k\) and \(\text{mad} < \frac{7}{3}\), so they admit a \((k + 1)\)-coloring). By Lemma 2.1, graph \(G\) cannot contain \((C_1)\) to \((C_5)\). Lemma 2.3 implies that \(\text{mad}(G) \geq \frac{7}{3}\). Contradiction. \(\square\)
3 Conclusion

We actually proved a slightly stronger result than Theorem 1.3. However, the addition, namely that every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) \leq 3$ admits a 2-distance 5-coloring, can be derived from a result of Dvořák, Škrekovski and Tancer [4].

Note that the proof of Theorem 1.3 also provides an $O(|V|^3)$ algorithm to find a 2-distance coloring of a graph $G$ with $\Delta(G) + 1$ colors if $G$ verifies the hypothesis of Theorem 1.3; indeed Lemma 2.3 proves that every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ contains $(C_1)$, $(C_2)$, ... or $(C_5)$. Consequently, we can find a $(C_i)$ in $G$, remove the corresponding vertices, and extend the coloring to the initial graph using the proof of Lemma 2.1.

As it was conjectured by Kostochka and Woodall [5] that 2-distance list-coloring requires exactly as many colors as 2-distance coloring, future work could aim at extending Theorem 1.3 to list-coloring.

References


