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2-distance coloring of sparse graphs

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\textbf{Abstract}

A 2-distance coloring of a graph is a coloring of the vertices such that two vertices at distance at most 2 receive distinct colors. We prove that every graph with maximum degree $\Delta$ at least 4 and maximum average degree less that $\frac{7}{3}$ admits a 2-distance ($\Delta + 1$)-coloring. This result is tight. This improves previous known results of Dolama and Sopena.

\textit{Keywords:} 2-distance coloring; square coloring; maximum average degree.

\section{Introduction}

All the graphs we consider here are simple, finite and undirected. Let $G = (V,E)$ be a graph. For any subgraph $H$ of $G$, we denote $V(H)$ and $E(H)$ the vertices and edges of $H$. For any vertex $v \in V$, the degree of $v$ in $G$, denoted $d(v)$, is the number of neighbors of $v$ in $G$. The maximum degree of $G$, denoted $\Delta(G)$, is $\max_{v \in V} d(v)$. The maximum average degree of $G$, denoted $\text{mad}(G)$, is the maximum for every subgraph $H$ of $G$ of $\frac{2|E(H)|}{|V(H)|}$. A 2-distance coloring

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of a graph $G$ is a coloring of the vertices of $G$ such that two vertices that are adjacent or have a common neighbor receive distinct colors. This is equivalent to a proper vertex-coloring of the square of $G$. We define $\chi^2(G)$ as the smallest $k$ such that $G$ admits a 2-distance $k$-coloring. Note that any graph $G$ satisfies $\chi^2(G) \geq \Delta(G) + 1$. The girth $g(G)$ is the length of a shortest cycle in $G$. Two vertices $x$ and $y$ are $p$-linked if there exists a path $x-v_1 \cdots v_p-y$ such that vertices $v_1, \ldots, v_p$ have degree 2, and $v_1 \cdots v_p$ is called a branch of $x$ (or $y$).

Borodin, Ivanova and Neustroeva \cite{1} studied sparse planar graphs, and prove the following result:

**Theorem 1.1** (\cite{1}) Every planar graph $G$ with $g(G) \geq 15$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

Note that this result was later extended to list-coloring \cite{2}.

Dolama and Sopena \cite{3} proved a more general result than Theorem 1.1, which is not restricted to planar graphs anymore. Theorem 1.2 however presents a slight loss in quality compared to Theorem 1.1: since for any planar graph $G$, $(\text{mad}(G) - 2)(g(G) - 2) < 4$, Theorem 1.2 implies only that Theorem 1.1 holds for $g(G) \geq 16$.

**Theorem 1.2** (\cite{3}) Every graph $G$ with $\text{mad}(G) < \frac{16}{7}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

We aim at making the upper bound on the maximum average degree optimal, and prove the following.

**Theorem 1.3** Every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) \geq 4$ admits a 2-distance $(\Delta(G) + 1)$-coloring.

The bound we obtain is optimal. Indeed, as pointed out by Montassier \cite{6}, there is a graph $G$ with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$ (see Figure 1).

![Fig. 1. A graph $G$ with $\text{mad}(G) = \frac{7}{3}$, $\Delta(G) = 4$ and $\chi^2(G) = 6$.](image)

When restricted to planar graphs, Theorem 1.3 is an improvement of Theorem 1.1 as it implies that Theorem 1.1 holds with $g(G) \geq 14$. It is not comparable to the more general result in \cite{2}, since we are not considering list-coloring.
We are going to use a discharging method to prove Theorem 1.3. We will prove that there are some configurations a minimal counter-example cannot contain, and, then use discharging rules to show that this graph does not exist.

2 Proof

In the figures, we draw in black a vertex that has no other neighbor than the ones already represented, in white a vertex that might have other neighbors than the ones represented. When there is a label inside a white vertex, it is an indication on the number of neighbors it has. The label ‘$i$’ means ”exactly $i$ neighbors”, the label ‘$i^+$’ (resp. ‘$i^-$’) means that it has at least (resp. at most) $i$ neighbors. Note that the white vertices may coincide with other vertices. The label ‘$T(v, a)$’ inside a vertex $v$ means that $T(v, a)$ exists, as defined below.

A configuration $T(v, a_4)$ (see Figure 2), is inductively defined as a vertex $v$ of degree 4 with neighbors $a_1, a_2, a_3, a_4$, where for $i \in \{1, 2, 3\}$, vertex $v$ is 2-linked by a path $v-a_i-b_i-w_i$ either to a vertex $w_i$ of degree at most 3 or to a configuration $T(w_i, b_i)$.

![Fig. 2. A $T(v, u_4)$](image)

Now we define configurations $(C_1)$ to $(C_5)$ (see Figure 3).

- $(C_1)$ is a vertex of degree 0 or 1.
- $(C_2)$ is a vertex 3-linked to a vertex not of maximal degree.
- $(C_3)$ is a vertex of degree 3 that is 2-linked to two vertices of degree 3, and 1-linked to a vertex of degree at most 3.
- $(C_4)$ is a vertex $u$ of degree at most 3 that is 2-linked by a path $u-y-x-v$ to a vertex $v$ such that $T(v, x)$ exists.
- $(C_5)$ is a vertex $u$ of degree 3 that is 2-linked to two vertices, and 1-linked by a path $u-x-v$ to a vertex $v$ such that $T(v, x)$ exists.
In the following lemma, we actually use $k$ instead of $\Delta(G)$ in order to ensure that any subgraph of $G$ admits a $(k+1)$-coloring even though $\Delta$ can decrease.

A graph is *minimal* for a property if it satisfies this property but none of its subgraphs does.

**Lemma 2.1** Let $k \geq 4$ and $G$ such that $\Delta(G) \leq k$ and $G$ admits no $2$-distance $(k+1)$-coloring, and $G$ is minimal for this property. Then $G$ does not contain any of Configurations $(C_1)$ to $(C_5)$.

The following lemma will ensure that the discharging rules we introduce later are well-defined.

**Lemma 2.2** In a graph $G$ where $(C_4)$ is forbidden, and $x$ and $y$ are two vertices of degree $4$ that are $2$-linked by a path $x-a-b-y$, at most one of $T(x, a)$ and $T(y, b)$ exists.

We design discharging rules $R_1$, $R_2$, $R_3$ (see Figure 4). We use them in the proof of Lemma 2.3, where the initial weight of a vertex equals its degree, and its final weight is shown to be at least $\frac{7}{3}$. For any two vertices $x$ and $y$ of degree at least $3$, with $d(x) \geq d(y)$,

- Rule $R_1$ is when $x$ and $y$ are $1$-linked by a path $x-a-y$.
  - $(R_{1,1})$ If $d(x) = d(y)$, then both $x$ and $y$ give $\frac{1}{5}$ to $a$.
  - $(R_{1,2})$ If $d(x) > d(y)$ and $T(x, a)$ exists, then both $x$ and $y$ give $\frac{1}{6}$ to $a$.
  - $(R_{1,3})$ If $d(x) > d(y)$ and $T(x, a)$ does not exist, then $x$ gives $\frac{1}{5}$ to $a$.

- Rule $R_2$ is when $x$ and $y$ are $2$-linked by a path $x-a-b-y$.
  - $(R_{2,1})$ If $d(x) = d(y)$ and neither $T(x, a)$ nor $T(y, b)$ exist, then $x$ (resp. $y$) gives $\frac{1}{3}$ to $a$ (resp. $b$).
\begin{itemize}
  \item \textbf{(R2.2)} If \(d(x) = d(y)\) and \(T(y, b)\) exists, then \(x\) gives \(\frac{1}{3}\) to \(a\) and both \(x\) and \(y\) give \(\frac{1}{6}\) to \(b\).
  \item \textbf{(R2.3)} If \(d(x) > d(y)\), then \(x\) gives \(\frac{1}{3}\) to \(a\) and both \(x\) and \(y\) give \(\frac{1}{6}\) to \(b\).
\end{itemize}

- Rule \(R_3\) is when \(x\) and \(y\), both of degree at least 4, are 3-linked by a path \(x - a - b - c - y\). Then \(x\) gives \(\frac{1}{3}\) to \(a\) and \(\frac{1}{6}\) to \(b\), and symmetrically for \(y\).

\textbf{Rule 1:} \(x\) and \(y\) are 1-linked \hspace{1cm} \textbf{Rule 2:} \(x\) and \(y\) are 2-linked

\textbf{Rule 3:} \(x\) and \(y\) are 3-linked.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Discharging rules \(R_1, R_2, R_3\).}
\end{figure}

We use these discharging rules to prove the following lemma:

**Lemma 2.3** A graph \(G\) that does not contain Configurations \((C_1)\) to \((C_5)\) verifies \(\operatorname{mad}(G) \geq \frac{7}{3}\).

**Proof of Theorem 1.3**

We prove a stronger version of Theorem 1.3 by contradiction. For \(k \geq 4\), let \(G\) be a minimal graph such that \(\Delta(G) \leq k\), \(\operatorname{mad}(G) < \frac{7}{3}\) and \(G\) does not admit a \((k + 1)\)-coloring. Graph \(G\) is also a minimal graph such that \(\Delta(G) \leq k\) and \(G\) does not admit a \((k + 1)\)-coloring (all its proper subgraphs verify \(\Delta \leq k\) and \(\operatorname{mad} < \frac{7}{3}\), so they admit a \((k + 1)\)-coloring). By Lemma 2.1, graph \(G\) cannot contain \((C_1)\) to \((C_5)\). Lemma 2.3 implies that \(\operatorname{mad}(G) \geq \frac{7}{3}\). Contradiction. \(\square\)
3 Conclusion

We actually proved a slightly stronger result than Theorem 1.3. However, the addition, namely that every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ and $\Delta(G) \leq 3$ admits a 2-distance 5-coloring, can be derived from a result of Dvořák, Škrekovski and Tancer [4].

Note that the proof of Theorem 1.3 also provides an $O(|V|^3)$ algorithm to find a 2-distance coloring of a graph $G$ with $\Delta(G) + 1$ colors if $G$ verifies the hypothesis of Theorem 1.3: indeed Lemma 2.3 proves that every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ contains $(C_1)$, $(C_2)$, ... or $(C_5)$. Consequently, we can find a $(C_i)$ in $G$, remove the corresponding vertices, and extend the coloring to the initial graph using the proof of Lemma 2.1.

As it was conjectured by Kostochka and Woodall [5] that 2-distance list-coloring requires exactly as many colors as 2-distance coloring, future work could aim at extending Theorem 1.3 to list-coloring.

References


