Low Polynomial Exclusion of Planar Graph Patterns

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Abstract

The celebrated grid exclusion theorem states that for every h-vertex planar graph H, there is a constant c_h such that if a graph G does not contain H as a minor then G has treewidth at most c_h. We are looking for patterns of H where this bound can become a low degree polynomial. We provide such bounds for the following parameterized graphs: the wheel ($c_h = O(h)$), the double wheel ($c_h = O(h^2 \cdot \log^2 h)$), any graph of pathwidth at most 2 ($c_h = O(h^2)$), and the yurt graph ($c_h = O(h^4)$).

Keywords: Treewidth, Graph Minors

1 Introduction

Treewidth is one of the most important graph invariants in modern graph theory. It has been introduced in [21] by Robertson and Seymour as one of the cornerstones of their Graph Minors series. Apart from its huge combinatorial value, it has been extensively used in graph algorithm design (see [6] for an extensive survey on treewidth). On an intuitive level, treewidth expresses how close the topology of the graph is to that of a tree and, in a sense, can be seen as a measure of the "global connectivity" of a graph.

Formally, a *tree decomposition* of a graph G is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X}$ a family $(X_t)_{t \in V(T)}$ of subsets of $V(G)$ (called *bags*) indexed by elements of $V(T)$ and such that

(i) $\bigcup_{t \in V(T)} X_t = V(G)$;

(ii) for every edge e of G there is an element of $\mathcal{X}$ containing both ends of e;

(iii) for every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) \mid v \in X_t\}$ is connected.

The *width* of a tree decomposition is equal to $\max_{t \in V(T)} |X_t| - 1$, while the *treewidth* of G, written $\mathbf{tw}(G)$, is the minimum width of any of its tree decompositions. Similarly one may define the notions of path decomposition and pathwidth by additionally asking that T is a path (see Section 2).

We say that a graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by applying a series of edge contractions, and we denote this fact by $H \leqslant_{\mathrm{m}} G$.

The grid exclusion theorem. One of the most celebrated results from the Graph Minors series of Robertson and Seymour is the following result, also known as the *grid exclusion theorem*.

Proposition 1 ([22]). *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for every for every planar graph H on h vertices, every graph G that does not contain a minor isomorphic to H has treewidth at most $f(h)$.*

The original proof the the above result in [22] did not provided any explicit estimation for the function f. Later, in [23], Robertson, Seymour, and Thomas proved the same result for $f(h) = 2^{O(h^5)}$, while a less complicated proof appeared in [12]. The bound $f(h) \leqslant h - 2$ was also obtained in [2] in the case where H is required to be a forest.

For a long time, whether Proposition 1 can be proved for a polynomial f was an open problem. In [23], an $\Omega(h^2 \cdot \log h)$ lower bound was provided for the best possible estimation of f and was also conjectured that the optimal estimation should not be far away from this lower bound. In fact, a more precise variant of the same conjecture was given by Demaine, Hajiaghayi, and Kawarabayashi in [11] where they conjectured that Proposition 1 holds for $f(h) = O(h^3)$. The bounds of [23] were recently improved by Kawarabayashi and Kobayashi [16], where they show that Proposition 1 holds for $f(h) = 2^{O(h \cdot \log h)}$. The same bounds were obtained by Leaf and Seymour [18]. Until recently, this was the best known estimation of the function f.

Very recently, in a breakthrough result [9], Chekuri and Chuzhoy proved that Proposition 1 holds for $f(h) = O(h^{228})$. The remaining open question is whether the degree of this polynomial bound can be substantially reduced in general. In this direction, one may still consider restrictions either on the graph G or on the graph H that yield a low polynomial dependence between the treewidth and the size of the excluded minor. In the first direction, Demaine and Hajiaghayi proved in [10] that, when G is restricted to belong in some graph class excluding some fixed graph R as a minor, then Proposition 1 (optimally) holds for $f(h) = O(h)$. Similar results have been proved by Fomin, Saurabh, and Lokshtanov, in [15], for the case where G is either a unit disk graph or a map graph that does not contain a clique as a subgraph.

In a second direction, one may consider H to be some specific planar graph and find a good upper bound for the treewidth of the graphs that exclude it as a minor. More generally, we can consider a parametrized class of planar graphs $\mathcal{H}_k$ where each graph in $\mathcal{H}_k$ has size bounded by a polynomial in k, and prove that the following fragment of Proposition 1 holds for some low degree polynomial function $f : \mathbb{N} \to \mathbb{N}$:

$$\forall k \geqslant 0 \ \forall H \in \mathcal{H}_k, \ if H \not\leqslant_{\mathrm{m}} G \text{ then } \mathbf{tw}(G) \leqslant f(k). \tag{1}$$

The question can be stated as follows: find pairs $(\mathcal{H}_k, g(k))$ for which (1) holds for some $f(k) = O(g(k))$, where $\mathcal{H}_k$ is as general as possible and g is as small as possible (and certainly polynomial). It is known, for example, that (1) holds for the pair $(\{C_k\}, k)$, where C_k is the cycle or a path of k vertices (see e.g. [5,14]), and for the pair $(\{K_{2,k}\}, k)$, (see [8]). Two more results in the same direction that appeared recently are the following: according to the result of Birmelé, Bondy, and Reed in [4], (1) holds for the pair $(\mathcal{P}_k, k^2)$ where $\mathcal{P}_k$ contains all minors of $K_2 \times C_k$ (we denote by $K_2 \times C_k$ the Cartesian product of K_2 and the cycle of k vertices, also known as the k-*prism*). Finally, one of the consequences of the recent results of Leaf and Seymour in [18], implies that (1) holds for the pair $(\mathcal{F}_r, k)$, where $\mathcal{F}_r$ contains every graph on r vertices that contains a vertex that meets all its cycles.

Our results. In this paper we provide polynomially bounded minor exclusion theorems for the following parameterized graph classes:

$\mathcal{H}_k^0$: containing all graphs on k vertices that have pathwidth at most 2.

$\mathcal{H}_k^1$: containing all minors of a wheel on $k + 1$ vertices – see Figure 1.

$\mathcal{H}_k^2$: containing all minors of a double wheel on $k+2$ vertices – see Figure 1.

$\mathcal{H}_k^3$: containing all minors of the yurt graph on $2k+1$ vertices (i.e. the graph obtained it we take a $(2 \times k)$-grid and add a new vertex adjacent with all the vertices of its "upper layer" – see Figure 4).

Notice that none of the above classes is minor comparable with the classes $\mathcal{P}_k$ and $\mathcal{F}_k$ treated in [4] and [18]. Moreover, $\mathcal{H}_k^1 \subset \mathcal{H}_k^2 \subset \mathcal{H}_k^3$, while $\mathcal{H}_k^0$ is not minor comparable with the other three. In this paper we prove that (1) holds for the pairs:

- $(\mathcal{H}_k^0, k^2)$,

- $(\mathcal{H}_k^1, k)$,

- $(\mathcal{H}_k^2, k^2 \log^2 n)$, and

- $(\mathcal{H}_k^3, k^4)$.

The above results are presented in detail, without the O-notation, in Section 3. All of our proofs use as a departure point the results of Leaf and Seymour in [18].

2 Definitions

All graphs in this paper are finite and simple, *i.e.*, do not have loops nor multiple edges. We denote by $V(G)$ (resp. $E(G)$) the sets of vertices (resp. edges) of G. For every $i, j \in \mathbb{N}$, $i \leqslant j$, the notation $[\![i, j]\!]$ stands for the interval of integers $\{i, i+1, \ldots, j\}$. Logarithms are binary.

Definition 1 (path decomposition, pathwidth)**.** A *path decomposition* of a graph G is a tree decomposition T of G such that T is a path. Its width is the width of the tree decomposition T and the *pathwidth* of G, written $\mathbf{pw}(G)$, is the minimum width of any of its path decompositions. An *optimal path decomposition* is a path decomposition of minimum width.

Definition 2 (contraction and dissolution)**.** The *contraction* of an edge $\{u, v\}$ in a graph G is the operation which creates a new vertex adjacent to the neighbors of u and those of v, and deletes both u and v. The *dissolution* of a vertex of degree two is the contraction of one of the edges incident with it.

Definition 3 (minor model)**.** A *minor model* (sometimes abbreviated *model*) of a graph H in a graph G is a pair $(\mathcal{M}, \varphi)$ where $\mathcal{M}$ is a set of pairwise disjoint subsets of $V(G)$ such that $\forall X \in \mathcal{M}$, $G[X]$ is connected and $\varphi \colon V(H) \to \mathcal{M}$ is a bijection that satisfies $\forall \{u, v\} \in E(H), \exists u' \in \varphi(u), \exists v' \in \varphi(v), \; \{u', v'\} \in E(G)$. We say that a graph H is a *minor* of a graph G ($H \leqslant_\mathrm{m} G$) if there is a minor model of H in G. Notice that H is a minor of G if H can be obtained from subgraph of G by edges contractions.

Definition 4 (linked set)**.** Let G be a graph and $S \subseteq V(G)$. The set S is said to be *linked* in G if for every two subsets X_1, X_2 of S (not necessarily disjoint) such that $|X_1| = |X_2|$, there is a set Q of $|X_1|$ (vertex-)disjoint paths between X_1 and X_2 in G whose length is not one (but can be null) and whose endpoints only are in S.

Definition 5 (separation)**.** A pair (A, B) of subsets of $V(G)$ is a called a *separation of order k in G* if $k = |A \cap B|$, none of A, B is a subset of the other, and there is no edge of G between $A \setminus B$ and $B \setminus A$.

Definition 6 (left-contains, [18]). Let H be a graph on r vertices, G a graph and (A, B) a separation of order r in G. We say that (A, B) *left-contains* H if $G[A]$ contains a minor model $\mathcal{M}$ of H such that $\forall M \in \mathcal{M}, |M \cap (A \cap B)| = 1$

Definition 7 (Trees and cycles). Given a tree T we denote by $L(T)$ the set of its leaves, *i.e.* vertices of degree 1 and by $\mathbf{diam}(T)$ its diameter, that is the maximum length (in number of edges) of a path in T.

For every two vertices $u, v \in V(T)$, there is exactly one path in T between u and v, that we denote by uTv. Also, given that uTv has at least 2 vertices, we denote by $\mathring{u}Tv$ (resp. $uT\mathring{v}$) the path uTv with the vertex u (resp. v) deleted.

Let C be a cycle on which we fixed some orientation. Then, there is exactly one path following this orientation between any two vertices $u, v \in V(C)$. Similarly, we denote this path by uCv and we define $\mathring{u}Cv$ and $uC\mathring{v}$ as we did for the tree.

In a rooted tree T with root r, the *least common ancestor* of two vertices u and v, written $\mathbf{lca}_T(u, v)$ is the first common vertex of the paths uTr and vTr. We refer to the root of T by the notation $\mathbf{root}(T)$.

For every integer $h > 0$, we denote by B_h the complete binary tree of height h.

3 Results

We present in this paper bounds on the treewidth of graphs not containing the following parameterized graphs: the wheel of order k (section 5), the double wheel of order k (section 6), any graph on k vertices and pathwidth at most 2 (section 7) and the yurt graph of order k (section 8). The definitions of these graphs can be found in the corresponding sections. In section 4, we recall some propositions that we will use and we prove two lemmata which will be useful later. The theorems we then prove are the following.

Theorem 1. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 36k - 2$, then G contains a wheel of order k as minor.*

Theorem 2. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 12(8k \log(8k) + 2)^2 - 4$, then G contains a double wheel of order at least k as minor.*

Theorem 3. *Let $k > 0$ be an integer, G be a graph and H be a graph on k vertices and of pathwidth at most 2. If $\mathbf{tw}(G) \geqslant 3k(k - 4) + 8$ then G contains H as minor.*

Theorem 4. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 6k^4 - 24k^3 + 48k^2 - 48k + 23$, then G contains the yurt graph of order k as minor.*

4 Preliminaries

Proposition 2 ([18, (4.3)]). *Let $k > 0$ be an integer, let F be a forest on k vertices and let G be a graph. If $\mathbf{tw}(G) \geqslant \frac{3}{2}k - 1$, then G has a separation (A, B) of order k such that*

- *$G[B \setminus A]$ is connected;*
- *$A \cap B$ is linked in $G[B]$;*
- *(A, B) left-contains F.*

Proposition 3 (Erdős–Szekeres Theorem, [13]). *Let k and ℓ be two positive integers. Then any sequence of $(\ell - 1)(k - 1) + 1$ distinct integers contains either an increasing subsequence of length k or a decreasing subsequence of length ℓ.*

Lemma 1. *For every tree T, $|V(T)| \leqslant \frac{|L(T)| \cdot \mathbf{diam}(T)}{2} + 1$.*

Proof. Root T to a vertex $r \in V(T)$ that is halfway of a longest path of T. For each leaf $x \in L(T)$, we know that $|V(xT\mathring{r})| \leqslant \left\lfloor \frac{\mathbf{diam}(T)}{2} \right\rfloor$. Observe that $V(T) = \{r\} \cup \bigcup_{x \in L(T)} V(xT\mathring{r})$. Therefore,

$$|V(T)| \leqslant \sum_{x \in L(T)} |V(xT\mathring{r})| + 1$$

$$|V(T)| \leqslant |L(T)| \cdot \left\lfloor \frac{\mathbf{diam}(T)}{2} \right\rfloor + 1.$$

Notice that equality holds for the subdivided star (obtained from $K_{1,n}$ by subdividing k times every edge, for some $n, k \in \mathbb{N}$). $\qquad\square$

Definition 8 (The set $\Lambda(T)$)**.** Let T be a tree. We denote by $\Lambda(T)$ the set containing every graph obtained as follows: take the disjoint union of T, a path P where $|V(P)| \geqslant \sqrt{|L(T)|}$, and an extra vertex v_{new}, and add edges such that

(i) there is an edge between v_{new} and every vertex of P;

(ii) there are $|V(P)|$ disjoint edges between P and $L(T)$;

(iii) there are no more edges than the edges of T and P and the edges mentioned in (i) and (ii).

Lemma 2. *Let $n \geqslant 1$ be an integer, T be a tree on n vertices an let G be a graph. If $\mathbf{tw}(G) \geqslant 3n - 1$, then $H \leqslant_{\mathrm{m}} G$ for some $H \in \Lambda(T)$.*

Proof. Let n, T, and G be as in the statement of the lemma. Let l be the number of leaves of T, and let J be a path on l vertices. We consider the disjoint union of J and T.

The graph G has treewidth at least $\frac{3}{2}(n + l) - 1$, then by Proposition 2, G has a separation (A, B) of order $n + l$ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains the graph $J \cup T$.

Let $(\mathcal{M}, \varphi)$ be the a model of $J \cup T$ in $G[A]$ that witnesses (iii).

We call the vertices of $A \cap B$ that belong to $\varphi(v)$ for some $v \in V(J)$ the J-part, and vertices that belong to $\varphi(v)$ for some $v \in L(T)$ forms the $L(T)$-part. Notice that two distinct vertices of the J-part (resp. $L(T)$-part) will be contracted to distinct vertices by the model.

Let $\mathcal{P}$ a set of l disjoint paths with the one endpoint in the J-part and the other in the $L(T)$-part, and whose interior belongs to $B \setminus A$. The existence of such paths is given by (ii). For each $P \in \mathcal{P}$, we arbitrarily choose a vertex v_P of the interior of P, that is, $v_P \in V(P) \setminus A$. By (i), $G[B \setminus A]$ is connected: let Y be a smallest tree spanning the vertices $\{v_P\}_{P \in \mathcal{P}}$. Let $s = \sqrt{|L(T)|}$, and let Y^* be the tree obtained from Y by dissolving every vertex of degree two that is not v_P for some $P \in \mathcal{P}$. We are now facing two possible situations.

Case 1: Y^* has a path of length s. Let Q be the path of Y corresponding to a path of lenght s in Y^* and let S be the set of vertices $u \in V(Q)$ that are not of degree two or that are v_P for some $P \in \mathcal{P}$. Observe that from every $u \in S$, there is a path J_u to the $L(T)$-part and a path J'_u to the J-part. Indeed, if $u = v_P$ for some $P \in \mathcal{P}$, then u is a vertex of P linking (by definition) a vertex of the $L(T)$-part to a vertex of the J-part. Otherwise, u is of degree at least 3 in Y and every leaf of the subtrees of $Y \setminus Q$ (at least one of which is adjacent to u), is a v_P for

some $P \in \mathcal{P}$ (by minimality of Y), so is connected to the $L(T)$-part and the J-part as explained above. Furthermore, for every two distinct $u, v \in S$, the aforementioned path are disjoint.

Let us now summarize. $G\left[\cup_{v \in V(J)} \varphi(v)\right]$ is a connected subgraph of G, which is connected by the s disjoint paths $J'_{u \, u \in S}$ to the path Y. All the endpoints of the paths $J'_{u \, u \in S}$ on Y are connected by s disjoint paths $J_{u \, u \in S}$ to the $L(T)$-part, which correspond to the leaves in a model of T. Therefore this graph contains a member of $\Lambda(T)$ as a minor, as required.

Case 2: $\mathbf{diam}(Y^*) < s$. From Lemma 1, $|L(Y)| = |L(Y^*)| \geqslant s$. Observe that $L(Y) \subseteq \{v_P\}_{P \in \mathcal{P}}$ (this follows by the minimality of Y). Let $S = V(Y) \setminus L(Y)$. We consider the minor of G obtained by contracting, for every $P \in \mathcal{P}$ such that $v_P \in L(Y)$, every edge of the subpath connecting the J-part to a leaf of Y. In this graph, S induces a connected subgraph adjacent to at least s distinct vertices of the J-part. All these s vertices of the J-part are connected by s disjoint paths to distinct vertices of the $L(T)$-part. Thus this contains a member of $\Lambda(T)$ as a minor, and so do G.

$\square$

5   Excluding a wheel with a linear bound on treewidth

Definition 9 (wheel). Let $r > 2$ be an integer. The *wheel* of order r (denoted W_r) is a cycle of length r whose each vertex is adjacent to an extra vertex, in other words it is a the graph of the form

$$V(G) = \{o, w_1, \ldots, w_r\}$$
$$E(G) = \{\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{r-1}, w_r\}, \{w_r, w_1\}\} \cup \{\{o, w_1\}, \ldots, \{o, w_r\}\}$$
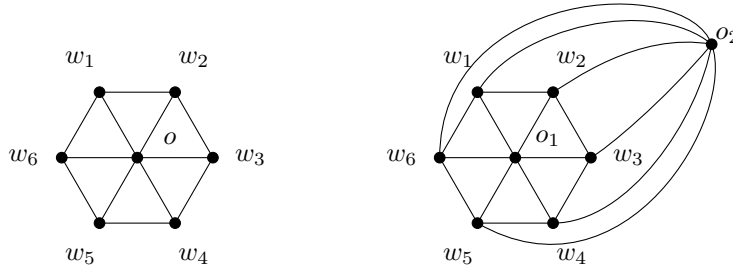
(see Figure 1 for an example).

Figure 1: A wheel of order six (left) and a double wheel of order 6 (right)

Lemma 3. *Let $h > 2$ be an integer. Let G be a graph obtained from the union of the tree $T = B_h$ and a path P by adding the edges $\{l, \psi(l)\} \in E(G)$ for every $l \in L(T)$, where $\psi \colon L(T) \to V(P)$ is a bijection. Then G contains a wheel of order $2^{h-2} + 1$ as a minor.*

Proof. Let h, ψ, T, $P = p_1 \ldots p_{2^h}$ and G be as above. Let r be the root of T.

In the arguments to follow, if $t \in V(T)$, we denote by T_t the subtree of T rooted at t (*i.e.* the subtree of T whose vertices are all the vertices $t' \in V(T)$ such that the path $t'Tr$ contains t).

We consider the vertices $u = \psi^{-1}(p_1) \in L(T)$ and $v = \psi^{-1}(p_{2^h}) \in L(T)$ and $w = \mathbf{lca}_T(u, v) \in V(T) \setminus L(T)$.

Let τ be a largest subtree of T which is disjoint from uTv. Let $L_\tau = L(\tau) \cap L(T)$ and let $Q = \psi(L_\tau) \subseteq P$. It is not hard to see that G contains $W_{|Q|+1}$ as a minor. Indeed, the paths P and uTv together with the edges $\{p_1, u\}$ and $\{p_{2h}, v\}$ form a cycle in G. Besides, the tree τ, which is disjoint from this cycle, has at least $|Q|+1$ vertices that are adjacent to distinct vertices of P: $|Q|$ of them are the elements of Q, and the other one is the (only) vertex of τ adjacent to uTv (which exists by maximality of τ). In the subgraph of G induced by $V(P) \cup V(uTv) \cup V(\tau)$, contracting τ to a vertex gives a vertex adjacent to at least $|Q|+1$ vertices of a (non necessarily induced) cycle, a graph containing $W_{|Q|+1}$ as subgraph.

Depending on G, $|Q|$ may take different values. However, we show that it is never less than 2^{h-2}. Remember, $|Q|$ is the number of leaves that a largest subtree of T that is disjoint from uTv shares with T. The root r of T has two children r_1 and r_2, inducing two subtrees T_{r_1} and T_{r_2} of T. Recall, $w = \mathbf{lca}_T(u, v)$.

Case 1. $w \neq r$. As $w \neq r$, w is a vertex of one of $\{T_{r_1}, T_{r_2}\}$, say T_{r_1}, which contains also u and v, and thus the path uTv. The other subtree T_{r_2} is then disjoint from uTv, it has height $h-1$ and is complete so it has 2^{h-1} leaves. Consequently, in this case $|Q| \geqslant 2^{h-1}$.

Case 2. $w = r$. In this case, the path uTv contains r (and $r \neq u$, $r \neq v$ as u and v are leaves) so u and v are not in the same subtree of $\{T_{r_1}, T_{r_2}\}$ and uTv contains the two edges $\{r, r_1\}$ and $\{r, r_2\}$. For every $i \in \{1, 2\}$, we denote by $r_{i,1}$ and $r_{i,2}$ the two children of r_i in T. We assume without loss of generality that $u \in V(T_{r_{1,1}})$ and $v \in V(T_{r_{2,1}})$ (if not, we just rename the r_i's ans $r_{i,j}$'s). Notice that the path uTv is the concatenation of the paths $uT_{r_1}r_1$, r_1Tr_2, $r_2T_{r_2}v$. Since the tree $T_{r_{1,2}}$ is disjoint from uTv, is complete and is of height $h-2$, it has 2^{h-2} leaves. Therefore we have $|Q| \geqslant 2^{h-2}$.

In both cases, $|Q| \geqslant 2^{h-2}$ and according to what we proved before, G contains a model of $W_{|Q|+1}$. As every wheel contains as a minor every smaller wheel, we proved that G contains a wheel of order at least 2^{h-2}. $\qquad\square$

Theorem 5. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 36k - 2$, then G contains a W_k-model.*

Proof. Let $k > 0$ be an integer, G be a graph such that $\mathbf{tw}(G) \geqslant 36k - 2$, and let $h = \lceil \log 4k \rceil$. Since every wheel contains a model of every smaller wheel, we have

$$W_k \leqslant_{\mathrm{m}} W_{2^{\lceil \log k \rceil}+1}$$
$$\leqslant_{\mathrm{m}} W_{2^{\lceil (\log 4k) - 2 \rceil}+1}$$
$$\leqslant_{\mathrm{m}} W_{2^{h-2}+1}$$

Therefore, if we prove that G contains a $W_{2^{h-2}+1}$-model, then we are done because the minor relation is transitive. Let Y_h^- be the graph of the following form: the disjoint union of the complete binary tree B_h of height h with leaves set Y_L and of the path Y_P on 2^h vertices, and let $\mathcal{Y}_h$ be the set of graphs of the same form, but with the extra edges $\{\{l, \phi(l)\}\}_{l \in Y_L}$, where $\phi : Y_L \to V(Y_P)$ is a bijection. As we proved in Lemma 3 that every graph in $\mathcal{Y}_h$ contains the wheel of order $2^{h-2} + 1$ as minor, showing that G contains an element of $\mathcal{Y}_h$ as minor suffices to prove this lemma. That is what we will do.

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From our initial assumption, we deduce the following.

$$\mathbf{tw}(G) \geqslant 36k - \frac{5}{2}$$
$$\geqslant \frac{3}{2}(3 \cdot 2^{\log 8k} - 1) - 1$$
$$\geqslant \frac{3}{2}(3 \cdot 2^{\lfloor \log 4k \rfloor + 1} - 1) - 1$$
$$\mathbf{tw}(G) \geqslant \frac{3}{2}(3 \cdot 2^h - 1) - 1$$

According to Proposition 2, G has a separation (A, B) of order $3 \cdot 2^h - 1$ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains the graph Y_h^-.

By definition of *left-contains*, $G[A]$ contains a model $(\mathcal{M}^-, \varphi^-)$ of Y_h^- and every element of $\mathcal{M}^-$ contains exactly one element of $A \cap B$. For every $x \in A \cap B$, we denote by M_x^- the element of $\mathcal{M}^-$ that contains x. Let L (resp. R) be the subset of $A \cap B$ of vertices that belong to an element of M related to the leaves of B_h in Y_h^- (resp. to the path P). We remark that these sets are both of cardinality 2^h.

Since $A \cap B$ is linked in $G[B]$ (see (ii)), there is a set $\mathcal{P}$ of 2^h disjoint paths between the vertices of L and the elements of R. Let $\psi \colon L \to V(P)$ be the function that match each element l of L with the (unique) element of R it is linked to by a path (that we call $\mathcal{P}_l$) of $\mathcal{P}$. Observe that ψ is a bijection. We set

$$\forall l \in L, \ M_l = M_l^- \cup V(l\mathcal{P}_l\mathring{\psi}(l))$$
$$\forall r \in (A \cap B) \setminus L, \ M_r = M_r^-$$

$$\mathcal{M} = \bigcup_{x \in A \cup B} M_x.$$

Let us show that $\mathcal{M}$ allows us to define a model of some $H \in \mathcal{Y}_h$. Let us consider the following mapping.

$$\varphi \colon \left\{ \begin{array}{ccc} V(Y_h^-) & \to & \mathcal{M} \\ x & \mapsto & M_x \end{array} \right.$$

We claim that $(\mathcal{M}, \varphi)$ is a model of H for some $H \in \mathcal{Y}_h$. This is a consequence of the following remarks.

Remark 1. Every element of $\mathcal{M}$ is either an element of $\mathcal{M}^-$, or the union of a element M of $\mathcal{M}^-$ and of the vertices of a path that start in M, thus every element of $\mathcal{M}$ induces a connected subgraph of G.

Remark 2. The paths of $\mathcal{P}$ are all disjoint and are disjoint from the elements of $\mathcal{M}^-$. Every interior of path of $\mathcal{P}$ is in but one element of $\mathcal{M}$, therefore the elements of $\mathcal{M}$ are disjoint.

Remark 3. The elements $\{m_l\}_{l \in L}$ are in bijection with the elements of $\{m_r\}_{r \in R}$ (thanks to the function ψ) and every two vertices $l \in L$ and $\psi(l) \in R$ are such that there is an edge between m_l and $m_{\psi(l)}$ (by definition of $\mathcal{M}^+$).

We just proved that $(\mathcal{M}, \varphi)$ is a model of a graph of $\mathcal{Y}_h$ in G. Finally, we apply Lemma 3 to find a model of the wheel of order $2^{h-2} + 1 = 2^{\lceil \log k \rceil - 2} + 1 \geqslant k$ in G and this concludes the proof. $\qquad \square$

6   Excluding a double wheel with a $(l \log l)^2$ bound on treewidth

Definition 10 (double wheel). Let $r > 2$ be an integer. The *double wheel* of order r (denoted W_r^2) is a cycle of length r whose each vertex is adjacent to two different extra vertices, in other words it is the graph of the form

$$
\begin{aligned}
V(G) =\,& \{o_1, o_2, w_1, \ldots, w_r\} \\
E(G) =\,& \{\{w_1, w_2\}, \{w_2, w_3\}, \ldots, \{w_{r-1}, w_r\}, \{w_r, w_1\}\} \\
& \cup \{\{o_1, w_1\}, \ldots, \{o_1, w_r\}\} \\
& \cup \{\{o_2, w_1\}, \ldots, \{o_2, w_r\}\}
\end{aligned}
$$

(see Figure 1 for an example).

Lemma 4. *Let G be a graph and $h > 0$ be an integer. If $tw(G) \geqslant 6 \cdot 2^h - 4$, then G contains as minor a double wheel of order at least $\frac{2^{\frac{h}{2}} - 2}{2h - 3}$.*

Proof. Let h and G be as above. Observe that $\mathbf{tw}(G) \geqslant 3(2^{h+1} - 1) - 1$. As the binary tree $T = B_h$ has $2^{h+1} - 1$ vertices, G contains a graph $H \in \Lambda(B_h)$ as minor (by Lemma 2). Let us show that any graph $H \in \Lambda(B_h)$ contains a double wheel of order at least $\frac{2^{\frac{h}{2}} - 2}{2h - 3}$ as minor.

Let P be the path of length at least $2^{\frac{h}{2}}$ in the definition of H. Let L be the set, of size at least $2^{\frac{h}{2}}$, of the leaves of T that are adjacent to P in H. Such a set exists by definition of $\Lambda(B_h)$. We also define u (resp. u') as the vertex of $L(T)$ that is adjacent to one end of P (resp. to the other end of P) and $Q = uTu'$.

As T is a binary tree of height h, Q has at most $2h - 1$ vertices. Each vertex of Q is of degree at most 3 in T except the two ends which are of degree 1. Consequently, $T \setminus Q$ has at most $2h - 3$ connected components that are subtrees of T. Notice that every vertex of the $2^{\frac{h}{2}}$ elements of L is either a leaf of one of these $2h - 3$ subtrees, or one of the two ends of Q. By the pigeonhole principle, one of these subtrees, which we call T_1, has at least $\frac{2^{\frac{h}{2}} - 2}{2h - 3}$ leaves that are elements of L.

Let M_{o_1} be the set of vertices of this subtree T_1. We also set $M_{o_2} = \{v_{\text{new}}\}$ (cf. Definition 8 for a definition of v_{new}). Let us consider the cycle C made by the concatenation of the paths uPu' and $u'Tu$ in H.

By definition of M_{o_1}, there are at least $\frac{2^{\frac{h}{2}} - 2}{2h - 3}$ vertices of C adjacent to vertices of M_{o_1}. Let $J = \{j_1, \ldots, j_{|J|}\}$ be the set of such vertices of C, in the same order as they appear in C (we then have $|J| \geqslant \frac{2^{\frac{h}{2}} - 2}{2h - 3}$).

We arbitrarily choose an orientation of C and define the sets of vertices $M_1, M_2, \ldots, M_{|J|}$ as follows.

$$
\begin{aligned}
\forall i \in [\![1, |J| - 1]\!], \; M_i &= V(j_i C \mathring{j}_{i+1}) \\
M_{|J|} &= V(j_{|J|} C \mathring{j}_1)
\end{aligned}
$$

Let $\mathcal{M} = \left\{ M_1, \ldots, M_{|J|}, M_{o_1}, M_{o_2} \right\}$ and $\psi \colon V(W_{|J|}^2) \to \mathcal{M}$ be the function defined by

$$
\begin{aligned}
\forall i \in [\![1, |J|]\!], \; \psi(w_i) &= M_i \\
\psi(o_1) &= M_{o_1} \\
\psi(o_2) &= M_{o_2}
\end{aligned}
$$

Notice that ψ maps the vertices of $\mathrm{W}^2_{\lceil J \rceil}$ to connected subgraphs of H such that $\forall (v, w) \in E(\mathrm{W}^2_{\lceil J \rceil})$, there is a vertex of $\psi(v)$ adjacent in H to a vertex of $\psi(w)$. Therefore, $(\mathcal{M}, \psi)$ is a $\mathrm{W}^2_{\lceil J \rceil}$-model in H.

Since $|J| \geqslant \frac{2^{\frac{h}{2}}-2}{2h-3}$, H contains a double wheel of order at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$, which is what we wanted to show. $\qquad\square$

Corollary 1. *Let $l > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 12l - 4$ then G contains a double wheel of order at least $\frac{\sqrt{l}-2}{2\log l - 5}$ as minor.*

Proof. Let l and G be as above. First remark that

$$\lceil \log l \rceil - 1 \leqslant \log l \leqslant \lceil \log l \rceil \tag{2}$$

Our initial assumption on $\mathbf{tw}(G)$ gives the following.

$$\begin{aligned}
\mathbf{tw}(G) &\geqslant 12l - 4 \\
&\geqslant 6 \cdot 2^{\log(2l)} - 4 \\
&\geqslant 6 \cdot 2^{\log l + 1} - 4 \\
&\geqslant 6 \cdot 2^{\lceil \log l \rceil} - 4 \qquad \text{by (2)}
\end{aligned}$$

By Lemma 4, G contains a double wheel of order at least

$$\begin{aligned}
q &= \frac{2^{\frac{\lceil \log l \rceil}{2}} - 2}{2 \lceil \log l \rceil - 3} \\
&\geqslant \frac{2^{\frac{1}{2}\log l} - 2}{2(\log l - 1) - 3} \qquad \text{by (2)} \\
&\geqslant \frac{\sqrt{l} - 2}{2\log l - 5}
\end{aligned}$$

Therefore, G contains a double wheel of order $q \geqslant \frac{\sqrt{l}-2}{2\log l - 5}$, as required. $\qquad\square$

Theorem 6 (follows from Corollary 1)**.** *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 12(8k\log(8k) + 2)^2 - 4$, then G contains a double wheel of order at least k as minor.*

Proof. Applying Corollary 1 for $l = (8k\log(8k) + 2)^2$ yields that G contains a double wheel of order at least

$$\begin{aligned}
q &\geqslant \frac{\sqrt{l} - 2}{2\log l - 5} \\
&\geqslant \frac{8k\log(8k)}{4\log(8k\log(8k) + 2) - 5} \\
&\geqslant \frac{8k\log(8k)}{4\log(8k\log(8k)) - 1} \\
&\geqslant \frac{8k\log(8k)}{4(\log(8k) + \log\log(8k)) - 1} \\
&\geqslant \frac{8k\log(8k)}{8\log(8k) - 1} \\
&\geqslant k
\end{aligned}$$

Consequently G contains a double wheel of order $q \geqslant k$ and we are done. $\qquad\square$

7 Excluding a graph of pathwidth at most 2 with a quadratic bound on treewidth

Definition 11 (graph Ξ_r). We define the graph Ξ_r as the graph of the following form (see figure 2).

$$\begin{cases} V(G) = \{x_0, \ldots, x_{r-1}, y_0, \ldots, y_{r-1}, z_0, \ldots, z_{r-1}\} \\ E(G) = \{\{x_i, x_{i+1}\}, \{z_i, z_{i+1}\}\}_{i \in [\![1,r-1]\!]} \cup \{\{x_i, y_i\}, \{y_i, z_i\}\}_{i \in [\![0,r-1]\!]} \end{cases}$$
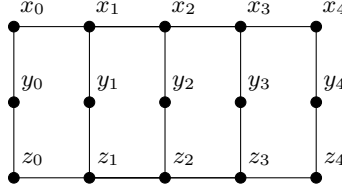

Figure 2: The graph Ξ_5

7.1 Graphs of pathwidth 2 in Ξ_r

Instead of proving that having a graph H of pathwidth 2 as minor forces a treewidth quadratic in $|V(H)|$, we prove that a Ξ_r-minor forces a treewidth quadratic in r and that every graph of pathwidth at most 2 on r vertices is minor of Ξ_r. For this, we first need some lemmata and remarks about path decompositions.

Definition 12 (nice path decompostion, [17]). A path decomposition $\left(p_1 p_2 \ldots p_k, \{X_{p_i}\}_{i \in [\![1,k]\!]}\right)$ of a graph G is said to be *nice* if $|X_{p_1}| = 1$ and

$$\forall i \in [\![2, k]\!], \ \left|(X_{p_i} \setminus X_{p_{i-1}}) \cup (X_{p_{i-1}} \setminus X_{p_i})\right| = 1$$

It is known [7] that every graph have an optimal path decomposition which is nice and that in such decomposition, every node X_i is either an *introduce node* (*i.e.* either $i = 1$ or $\left|X_{p_i} \setminus X_{p_{i-1}}\right| = 1$) or a *forget node* (*i.e.* $\left|X_{p_{i-1}} \setminus X_{p_i}\right| = 1$).

Remark 4. It is easy to observe that for every graph G on n vertices, there is an optimal path decomposition with n introduce nodes and n forget nodes (one of each for each vertex of G), thus of length $2n$.

Remark 5. Let G be a graph and let $(p_1 p_2 \ldots p_k, \mathcal{X})$, $\mathcal{X} = \{X_{p_i}\}_{i \in [\![1,k]\!]}$ be a nice (non necessarily optimal) path decomposition of G. Let w be the width of this decomposition.

For every $i \in [\![2, k-1]\!]$, if p_i is a forget node, $|X_{p_i}| \leqslant w - 1$ and p_{i+1} is an introduce node, then by setting

$$X'_{p_i} = X_{p_{i-1}} \cup X_{p_{i+1}}$$
$$\forall j \in [\![1, k]\!], \ j \neq i, \ X'_{p_j} = X_{p_j}$$
$$\mathcal{X}' = \left\{X'_{p_j}\right\}_{j \in [\![1,k]\!]}$$

we create from $(p_1 p_2 \ldots p_k, \mathcal{X}')$ a valid path decomposition of G, where p_i is now an introduce node and p_{i+1} a forget node. Observe that $\left|X'_{p_i}\right| \leqslant \left|X'_{p_i}\right| + 2 = w + 1$ Therefore the new path decomposition the same width as the original one. Note that the condition $|X_{p_i}| \leqslant w - 1$ holds, for instance, when p_{i-1} is required to be a forget node too (for $i \in [\![3, k-1]\!]$).

Remark 6. Let G be a graph and $P = (p_1 p_2 \ldots p_k, \mathcal{X})$ be a nice path decomposition of G. For every $i \in [\![1, k]\!]$, the path $p_1 \ldots p_i$ contains at most as many forget nodes as introduce nodes and the difference between these two numbers is at most $w + 1$ where w is the width of P.

Lemma 5. *Let G be a graph on n vertices . Then G has an optimal path decomposition P such that*

   (i) *every bag of P has size $\mathbf{pw}(G) + 1$;*

   (ii) *every two adjacent bags differs by exactly one element,* i.e. *for every two adjacent vertices u and v of P, $|X_u \setminus X_v| = |X_v \setminus X_u| = 1$.*

Proof. Let $P = (p_1 p_2 \ldots p_{2k}, \mathcal{X})$ with $\mathcal{X} = \{X_{p_i}\}_{i \in [\![1, 2k]\!]}$ be a nice optimal path decomposition of G with as many introduce nodes (resp. forget nodes) as there are vertices in G.

Let $s = \mathbf{pw}(G) + 1$. According to Remarks 5 and 6, P can be modified into a path decomposition of G of the same width and such that

(a) the s first vertices of P are introduce nodes and p_{s+1} is a forget node;

(b) the s last vertices of P are forget nodes and p_{2k-s} is an introduce node;

(c) for every $i \in [\![s, 2k - s]\!]$, p_i and p_{i+1} are nodes of different type.

In the arguments to follow, we assume that P satisfies this property.

Remark 7. Introduce nodes all have bags of cardinality s.

Remark 8. For every $i \in [\![0, k - s]\!]$, the node p_{s+2i} is an introduce node and the node p_{s+2i+1} is a forget node, which implies $X_{p_{s+2i}} \subsetneq X_{p_{s+2i+1}}$. Also note that for every $i \in [\![1, s - 1]\!]$, $X_{p_i} \subsetneq X_{p_s}$ and for every $i \in [\![2k - s + 1, 2k]\!]$, $X_{p_i} \subsetneq X_{p_{2k-s}}$.

Intuitively, every bag X that is included in one of its adjacent bags X' contains no more information than what X' already contains, so we will just remove it.

We thus define $P' = p_s p_{s+2} \ldots p_{s+2i} \ldots p_{2k-s}$ (a path made of all introduce nodes of P). Clearly, P and P' have the same width and as we deleted only redundant nodes, P' is still a valid path decomposition of G.

Since every two adjacent nodes of P' were introduce nodes separated by a forget node in P, they only differ by one element. According to Remark 7 and since every node of P' was an introduce node in P, every bag of P' have size $\mathbf{pw}(G) + 1$. Consequently, P' is an optimal path decomposition that satisfies the conditions of the lemma statement. $\square$

Remark 9. The path decomposition of Lemma 5 has length $V(G) - \mathbf{pw}(G)$.

Proof. Let $(P, \mathcal{X})$ be such a path decomposition. Remember that the first node of P has a bag of size $\mathbf{pw}(G) + 1$ and that every two adjacent nodes of P have bags which differs by exactly one element. Since every vertex of G is in a bag of P, in addition to the first bag containing $\mathbf{pw}(G) + 1$ vertices of G, P must have $V(G) - \mathbf{pw}(G) - 1$ other bags in order to contain all vertices of G. Therefore P has length $V(G) - \mathbf{pw}(G)$. $\square$

A proof of a slightly weaker version of the following lemma previously appeared [20].

Lemma 6. *For every graph G on n vertices and of pathwidth at most 2, there is a minor model of G in Ξ_{n-1}.*

Proof. Let G be as in the statement of the lemma. We assume that $\mathbf{pw}(G) = 2$ (if this is not the case we add edges to G in order to obtain a graph of pathwidth 2 which contains G as a minor). Let $r = V(G) - \mathbf{pw}(G) = n - 2$.

Let $P = (p_1 \dots p_r, \{X_{p_1}, \dots, X_{p_r}\})$ be an optimal path decomposition of G satisfying the properties of Lemma 5, of length r. Such decomposition exists according to Lemma 5 and Remark 9).

Using this decomposition, we will now define a labeling λ of the vertices of Ξ_{r+1}. When dealing with the vertices of Ξ_{r+1} we will use the notations defined in Definition 11. Let $\lambda \colon V(\Xi_{r+1}) \to V(G)$ be the function defined as follows:

(a) $\lambda(x_0)$ and $\lambda(y_0)$ are both equal to one (arbitrarily chosen) element of the set $X_{p_1} \cap X_{p_2}$;

(b) $\lambda(z_0)$ is equal to the only element of the set $X_{p_1} \cap X_{p_2} \setminus \{\lambda(x_1)\}$;

(c) $\forall i \in [\![2, r]\!]$, $\lambda(y_i) = X_{p_i} \setminus X_{p_{i-1}}$ and we consider two cases:

   Case 1: $X_{p_{i-1}} \cap X_{p_i} = X_{p_i} \cap X_{p_{i+1}}$
   $\qquad \lambda(x_i) = \lambda(x_{i-1})$ and $\lambda(z_i) = \lambda(z_{i-1})$;

   Case 2: $X_{p_{i-1}} \cap X_{p_i} \neq X_{p_i} \cap X_{p_{i+1}}$
   $\qquad$ if $X_{p_{i-1}} \cap X_{p_i} \cap X_{p_{i+1}} = \lambda(x_{i-1})$,
   $\qquad\qquad$ then $\lambda(x_i) = \lambda(x_{i-1})$ and $\lambda(z_i) = X_{p_i} \setminus X_{p_{i-1}}$;
   $\qquad\qquad$ else $\lambda(x_i) = X_{p_i} \setminus X_{p_{i-1}}$ and $\lambda(z_i) = \lambda(z_{i-1})$.

Thanks to this labeling, we are now able to present a minor model of G in Ξ_{r+1} :

$$\forall v \in V(G), \ M_v = \{u \in V(\Xi_{r+1}), \ \lambda(u) = v\}$$
$$\mathcal{M} = \{M_v\}_{v \in V(G)}$$

$$\varphi \colon \begin{cases} V(G) & \to & \mathcal{M} \\ u & \mapsto & M_u \end{cases}$$

To show that $(\mathcal{M}, \varphi)$ is a G-model in Ξ_{r+1}, we now check if it matches the definition of a minor model.

By definition, every element of $\mathcal{M}$ is a subset of $V(\Xi_{r+1})$. To show that every element of $\mathcal{M}$ induces a connected subgraph in G, it suffices to show that nodes of Ξ_{r+1} which have the same label induces a connected subgraph in G (by construction of the elements of $\mathcal{M}$). This can easily be seen by remarking that for every $i \in [\![2, r]\!]$, every vertex y_i of Ξ_{r+1} gets a new label and that every vertex x_i (resp. z_i) of Ξ_{r+1} receive either the same label as y_i, or the same label as x_{i-1} (resp. z_{i-1}).

Let us show that this labeling ensure that if two vertices u and v of G are in the same bag of P, there are two adjacent vertices of Ξ_{r+1} that respectively gets labels u and v. Let u, v be two vertices of G which are in the same bag of P. Let i be such that X_i is the first bag of P (with respect to the subscripts of the bags of P) which contains both u and v. The case $i = 1$ is trivial so we assume that $i > 1$. We also assume without loss of generality that $X_i \setminus X_{i-1} = \{v\}$, what gives $\lambda(y_i) = v$. Depending on in what case we are, either either $\lambda(x_i) = u$ (c1) or $\lambda(z_i) = u$ (c1 and c2). In both cases, u and v are the labels of two adjacent nodes of Ξ_{r+1}. By construction of the elements of $\mathcal{M}$, this implies that if $\{u, v\} \in E(G)$, then there are vertices $u' \in \varphi(u)$ and $v' \in \varphi(v)$ such that $\{u', v'\} \in E(\Xi_{r+1})$.

Therefore, $(\mathcal{M}, \varphi)$ is a G-model in Ξ_{n-1}, what we wanted to find.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

7.2 Exclusion of Ξ_r

Lemma 7. *For any graph, if* $\mathbf{tw}(G) \geqslant 3\ell - 1$ *then G contains as minor the following graph: a path $P = p_1 \ldots p_{2\ell}$ of length 2ℓ and a family Q of ℓ paths of length 2 such that every vertex of P is the end of exactly one path of Q and every path of Q has one end in $p_1 \ldots p_l$ (the first half of P) and the other end in $p_{l+1} \ldots p_{2l}$ (the second half of P) (see figure 3).*
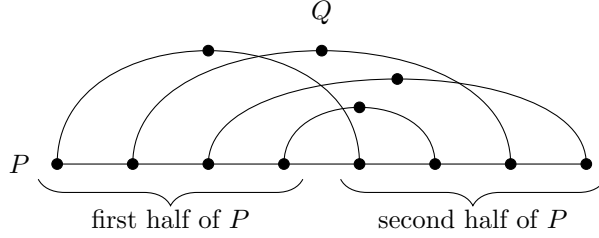

Figure 3: Example for Lemma 7

Proof. Let $\ell > 0$ be an integer and G be a graph of treewidth at least $3\ell - 1$. According to Proposition 2, G has a separation (A, B) of order 2ℓ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains a path $P = p_1 \ldots p_{2\ell}$ of length 2ℓ.

Let $(\mathcal{M}, \varphi)$ be a model of P in $G[A]$, with $\mathcal{M} = \{M_1, \ldots, M_{2\ell}\}$. We assume without loss of generality that φ maps p_i to M_i for every $i \in [\![1, 2\ell]\!]$.

As $A \cap B$ is linked in $G[B]$, there is a set Q of ℓ disjoint paths in $G[B]$ of length at least 2 and such that every path $q \in Q$ has one end in $(A \cap B) \cap \bigcup_{i \in [\![1, \ell]\!]} M_i$, the other end in $(A \cap B) \cap \bigcup_{i \in [\![\ell+1, 2\ell]\!]} M_i$ and its internal vertices are not in $A \cap B$.

Let G' be the graph obtained from $G\left[\left(\bigcup_{q \in Q} V(q)\right) \cup \left(\bigcup_{M \in \mathcal{M}} M\right)\right]$ after the following operations.

1. iteratively contract the edges of every path of Q until it reaches a length of 2. The paths of Q have length at least 2, so this is always possible.

2. for every $i \in [\![1, 2\ell]\!]$, contract M_i to a single vertex. The elements of a model are connected (by definition) thus this operation can always be performed.

As one can easily check, the graph G' is the graph we were looking for and it has been obtained by contracting some edges of a subgraph of G, therefore $G' \leqslant_{\mathrm{m}} G$. $\qquad \square$

Theorem 7. *Let G be a graph and H be a graph on h vertices satisfying $\mathbf{pw}(H) \leqslant 2$. If $\mathbf{tw}(G) \geqslant 3(h-2)^2 - 1$ then G contains H as a minor.*

Proof. Let G, H and h be as in the statement of the Lemma. According to Lemma 6, every graph of pathwidth at most two on n vertices is minor of Ξ_{n-1}. Therefore in order to show that $G \leqslant_{\mathrm{m}} H$ it is enough to prove that $G \leqslant_{\mathrm{m}} \Xi_{h-1}$. This is what we will do.

According to Lemma 7, G contains as minor two paths $P = p_1 \ldots p_{(h-2)^2}$ and $R = r_1 \ldots r_{h-2)^2}$ and a family Q of $(k-2)^2$ paths of length 2 such that every vertex of P or R is the end of exactly one path of Q and every path of Q has one end in P and the other end in R. For every $p \in P$, we denote by $\varphi(p)$ the (unique) vertex of R to which p is linked to by a path of Q. Observe that φ is a bijection. By Proposition 3, there is a subsequence $P' = (p'_1, p'_2, \ldots, p'_{h-1})$ of the vertices of P such that the vertices $\varphi(p'_1), \varphi(p'_2), \ldots, \varphi(p'_{h-1})$ appear in R either in this order or in the reverse order. Let $R' = (\varphi(p'_1), \varphi(p'_2), \ldots, \varphi(p'_{h-1}))$ and Q' be the set of inner vertices of the paths from p'_i to $\varphi(p'_i)$ for all $i \in [\![1, h-1]\!]$.

Iteratively contracting in G which have at most one end in P' (resp. in R') and removing the vertices that are not in P', R' or Q' gives the graph Ξ_{h-1}. The operations used to obtain it are vertices and edge deletions, and edge contractions, thus Ξ_{h-1} is a minor of G. This concludes the proof. $\qquad\square$

8 Excluding a yurt graph

Definition 13 (yurt graph of order r). Let $r > 0$ be an integer. In this paper, we call *yurt graph* of order r the graph Y_r of the form

$$V(Y_r) = \{x_1, \ldots, x_r, y_1, \ldots, y_r, o\}$$
$$E(Y_r) = \{\{x_i, x_{i+1}\}_{i \in [\![1, r-1]\!]}\}$$
$$\cup \{\{y_i, y_{i+1}\}_{i \in [\![1, r-1]\!]}\}$$
$$\cup \{\{x_i, y_i\}\}_{i \in [\![1, r]\!]}$$
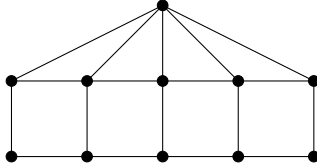$$\cup \{\{y_i, o\}\}_{i \in [\![1, r]\!]}$$

(see Figure 4 for an example).

Figure 4: The yurt graph of order 5, Y_5

For every $r > 0$, we define the *comb of order r* as the tree made from the path $p_1 p_2 \ldots p_r$ and the extra vertices $v_1, v_2, \ldots, v_r$ by adding an edge between p_i and v_i for every $i \in [\![1, r]\!]$.

Theorem 8. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geqslant 6k^4 - 24k^3 + 48k^2 - 48k + 23$, then G contains Y_k as minor.*

Proof. Let $k > 0$ be an integer and G be a graph such that $\mathbf{tw}(G) \geqslant 6k^4 - 24k^3 + 48k^2 - 48k + 23$. Let C be the comb with $l = k^4 - 4k^3 + 8k^2 - 8k + 4$ teeth. As $\mathbf{tw}(G) \geqslant 3|V(C)| - 1$, G contains some graph of $\Lambda(C)$ by Lemma 2.

Let us prove that every graph of $\Lambda(C)$ contains the yurt graph of order k. Let H be a graph of $\Lambda(C)$. We respectively call T, P and o the tree, path and extra vertex of $\Lambda(C)$. Let F be the subset of edges between P and the leaves of T

Let $L = l_0, \ldots, l_{k^2-2k+2}$ (resp. $Q = q_0, \ldots, q_{k^2-2k+2}$) be the leaves of T (resp. of P) that are the end of an edge of F We assume without loss of generality that they appears in this order.

According to Proposition 3, there is a subsequence Q' of Q of length k such that the corresponding vertices L' of L appear in the same order. As one can easily see, this graph contains the yurt of order k and we are done. □

9 Discussion and open problems

An natural question is whether the results of this paper for the classes $\mathcal{H}_k^i, i \in \{1, \ldots, 3\}$ are tight. This is indeed the case for the wheels in $\mathcal{H}_k^1$ as (1) does not hold for any pair of the form $(\mathcal{H}_k^1, f(k))$ where $f = o(k)$. To see this it is enough to observe that a clique K_k does not contain any wheel on $k + 1$ vertices as a minor while has treewidth $k - 1 = \Omega(k)$. Clearly, the same lower bound holds for $\mathcal{H}_k^2$ (i.e., the double wheels).

It is easy to prove that (1) does not hold for any pair of the form $(\mathcal{H}_k^0, f(k))$ or $(\mathcal{H}_k^3, f(k))$ where $f = o(k \log k)$. To see this, consider a n-vertex 3-regular Ramanujan graph R (see [19]). Such a graph has girth at least $c \log n$ for some universal constant c (see [3]), and satisfies $\mathbf{tw}(R) = \Omega(n)$ (cf. [1, Corollary 1]).

Let k' be the minimum integer such that $n < k' \cdot c \log n$ holds. Notice that $n = \Omega(k' \log k')$, thus $\mathbf{tw}(R) = \Omega(k' \log k')$. We will show that no graph of $\mathcal{H}_{2k'}^0 \cup \mathcal{H}_{2k'}^3$ is a minor of R. As every graph of $\mathcal{H}_{2k'}^0 \cup \mathcal{H}_{2k'}^3$ contains $k' \cdot K_3$ as a minor, it is enough to show that $k' \cdot K_3$ is not a minor of R. If $k' \cdot K_3$ is a minor of R, then R contains a collection of k' vertex-disjoint cycles. As the girth of R is at least $c \log n$, we have that $n \geqslant k' \cdot c \log n$, a contradiction.

The above observation implies that a function $f(k) = \Theta(k \log k)$ is the best for which (1) may hold for the pairs $(\mathcal{H}_k^0, f(k))$ and $(\mathcal{H}_k^3, f(k))$ and we conjecture that this is indeed the case. Observe that by the same remark, the lower bound $\Omega(k \log k)$ also holds for any class $\{H_k\}_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, $H_k \geqslant_m dk \cdot K_3$, for some universal constant $d \in \mathbb{R}$. Interestingly, the above proof does not apply for the double wheels in $\mathcal{H}_k^2$. This tempts us to conjecture that (1) holds (optimally) for the pair $(\mathcal{H}_k^2, k)$.

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