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Near-colorings: non-colorable graphs and NP-completeness

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Abstract
A graph $G$ is $(d_1, \ldots, d_l)$-colorable if the vertex set of $G$ can be partitioned into subsets $V_1, \ldots, V_l$ such that the graph $G[V_i]$ induced by the vertices of $V_i$ has maximum degree at most $d_i$ for all $1 \leq i \leq l$. In this paper, we focus on complexity aspects of such colorings when $l = 2, 3$. More precisely, we prove that, for any fixed integers $k, j, g$ with $(k, j) \neq (0, 0)$ and $g \geq 3$, either every planar graph with girth at least $g$ is $(k, j)$-colorable or it is NP-complete to determine whether a planar graph with girth at least $g$ is $(k, j)$-colorable. Also, for every fixed integer $k$, it is NP-complete to determine whether a planar graph that is either $(0, 0, 0)$-colorable or non-$(k, k, 1)$-colorable is $(0, 0, 0)$-colorable. Additionally, we exhibit non-$(3, 1)$-colorable planar graphs with girth 5 and non-$(2, 0)$-colorable planar graphs with girth 7.

1 Introduction
A graph $G$ is $(d_1, \ldots, d_k)$-colorable if the vertex set of $G$ can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph $G[V_i]$ induced by the vertices of $V_i$ has maximum degree at most $d_i$ for all $1 \leq i \leq k$. This notion generalizes those of proper $k$-coloring (when $d_1 = \cdots = d_k = 0$) and $d$-improper $k$-coloring (when $d_1 = \cdots = d_k = d \geq 1$).

Planar graphs are known to be $(0, 0, 0)$-colorable (Appel and Haken [1, 2]) and $(2, 2, 2)$-colorable (Cowen, Cowen, and Woodall [13]). The $(2, 2, 2)$-colorability is optimal (for any integer $k$, there exist non-$(k, k, 1)$-colorable planar graphs) and holds in the choosability case (Eaton and Hull [15] or Škrekovski [23]). Improper colorings have then been considered for planar graphs with large girth or graphs with low maximum average degree. We recall that the girth of a graph $G$, denoted by $g(G)$, is the length of a shortest cycle in

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and the maximum average degree of a graph $G$, denoted by $\text{mad}(G)$, is the maximum of the average degrees of all subgraphs of $G$, i.e. $\text{mad}(G) = \max \{2|E(H)|/|V(H)| \mid H \subseteq G\}$.

(1, 0)-coloring.
Glebov and Zambalâeva [20] proved that every planar graph with girth at least 16 is (1, 0)-colorable. This was then strengthened by Borodin and Ivanova [3] who proved that every graph $G$ with $\text{mad}(G) < \frac{7}{3}$ is (1, 0)-colorable. This implies that every planar graph $G$ with girth at least 14 is (1, 0)-colorable. Borodin and Kostochka [7] then proved that every graph $G$ with $\text{mad}(G) \leq \frac{12}{5}$ is (1, 0)-colorable. In particular, it follows that every planar graph with girth at least 12 is (1, 0)-colorable. On the other hand, they constructed graphs $G$ with $\text{mad}(G)$ arbitrarily close (from above) to $\frac{12}{5}$ that are not (1, 0)-colorable; hence their upper bound on the maximum average degree is best possible. The last result was strengthened for triangle-free graphs: Kim, Kostochka, and Zhu [22] proved that triangle-free graphs $G$ satisfying $11|V(G)| - 9|E(G)| \geq -4$ are (1, 0)-colorable. This implies that planar graphs with girth at least 11 are (1, 0)-colorable. On the other hand, Esperet, Montassier, Ochem, and Pinlou [16] proved that determining whether a planar graph with girth 9 is (1, 0)-colorable is NP-complete. To our knowledge, the question whether all planar graphs with girth at least 10 are (1, 0)-colorable is still open.

(k, 0)-coloring with $k \geq 2$.
Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] proved that every graph $G$ with $\text{mad}(G) < \frac{3k+4}{k+2}$ is $(k, 0)$-colorable. The proof in [4] extends the one in [3] but does not work for $k = 1$. Moreover, they exhibited a non-$(k, 0)$-colorable planar graph with girth 6. Finally, Borodin and Kostochka [8] proved that for $k \geq 2$, every graph $G$ with $\text{mad}(G) \leq \frac{3k+2}{k+1}$ is $(k, 0)$-colorable. This result is tight in terms of maximum average degree.

(k, 1)-coloring.
Recently, Borodin, Kostochka, and Yancey [9] proved that every graph with $\text{mad}(G) \leq \frac{14}{5}$ is (1, 1)-colorable, and the restriction on $\text{mad}(G)$ is sharp. In [5], it is proven that every graph $G$ with $\text{mad}(G) < \frac{10k+22}{3k+9}$ is $(k, 1)$-colorable for $k \geq 2$.

(k, j)-coloring.
A first step was made by Havet and Sereni [21] who showed that, for every $k \geq 0$, every graph $G$ with $\text{mad}(G) < \frac{4k+4}{k+2}$ is $(k, k)$-colorable (in fact $(k, k)$-choosable). More generally, they studied $k$-improper $l$-choosability and proved that every graph $G$ with $\text{mad}(G) < l + \frac{lk}{l+k}$ ($l \geq 2, k \geq 0$) is $k$-improper $l$-choosable; this implies that such graphs are $(k, \ldots, k)$-colorable (where the number of partite sets is $l$). Borodin, Ivanova, Montassier, and Raspaud [6] gave some sufficient conditions of $(k, j)$-colorability depending on the density of the graphs using linear programming. Finally, Borodin and Kostochka [8] solved the problem for a wide range of $j$ and $k$: let $j \geq 0$ and $k \geq 2j+2$; every graph $G$ with $\text{mad}(G) \leq 2(2 - \frac{k+2}{(j+2)(k+1)})$ is $(k, j)$-colorable. This result is tight in terms of the maximum average degree and improves some results in [4, 5, 6].
Using the fact that every planar graph $G$ with girth $g(G)$ has mad($G$) < $2g(G)/(g(G) - 2)$, the previous results give results for planar graphs. They are summarized in Table 1, which also shows the recent results that planar graphs with girth 5 are $(5,3)$-colorable (Choi and Raspaud [12]) and $(10,1)$-colorable (Choi, Choi, Jeong, and Suh [11]).

<table>
<thead>
<tr>
<th>girth</th>
<th>$(k, 0)$</th>
<th>$(k, 1)$</th>
<th>$(k, 2)$</th>
<th>$(k, 3)$</th>
<th>$(k, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3, 4</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>5</td>
<td>$\times$</td>
<td>(10, 1) [11]</td>
<td>(6, 2) [8]</td>
<td>(5, 3) [12]</td>
<td>(4, 4) [21]</td>
</tr>
<tr>
<td>6</td>
<td>$\times$</td>
<td>(4, 1) [8]</td>
<td>(2, 2) [21]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(4, 0) [8]</td>
<td>(1, 1) [9]</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>(2, 0) [8]</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>(1, 0) [22]</td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Table 1: The girth and the $(k, j)$-colorability of planar graphs. The symbol “$\times$” means that there exist non-$(k, j)$-colorable planar graphs for every $k$.

From the previous discussion, the following questions are natural:

**Question 1.**

1. Are planar graphs with girth 10 $(1, 0)$-colorable?
2. Are planar graphs with girth 7 $(3, 0)$-colorable?
3. Are planar graphs with girth 6 $(1, 1)$-colorable?
4. Are planar graphs with girth 5 $(4, 1)$-colorable?
5. Are planar graphs with girth 5 $(2, 2)$-colorable?

$(d_1, \ldots, d_k)$-coloring.

Finally we would like to mention two studies. Chang, Havet, Montassier, and Raspaud [10] gave some approximation results to Steinberg’s Conjecture using $(k, j, i)$-colorings. Dorbec, Kaiser, Montassier, and Raspaud [14] studied the particular case of $(d_1, \ldots, d_k)$-coloring where the value of each $d_i$ ($1 \leq i \leq k$) is either 0 or some value $d$, making the link between $(d, 0)$-coloring [8] and $(d, \ldots, d)$-coloring [21].

The aim of this paper is to provide complexity results on the subject and to obtain non-colorable planar graphs showing that some above-mentioned results are optimal.

**Claim 2.** There exist 2-degenerate planar graphs that are:

1. non-$(k, k)$-colorable with girth 4, for every $k \geq 0$,
2. non-$(3, 1)$-colorable with girth 5,
3. non-$(k, 0)$-colorable with girth 6,
4. non-(2, 0)-colorable with girth 7.

Claim 2.4 shows that the (2, 0)-colorability of planar graphs with girth at least 8 [8] is a tight result. Claim 2.3 has been obtained in [4] and the corresponding graph is depicted in Figure 1.

\[
\begin{array}{c}
\begin{array}{c}
2k + 1 \\
2k + 1 \\
2k + 1 \\
2k + 1
\end{array}
\end{array}
\]

Figure 1: A non-(k, 0)-colorable planar graph with girth 6 [4].

Theorem 3. Let \(k, j\), and \(g\) be fixed integers such that \((k, j) \neq (0, 0)\) and \(g \geq 3\). Either every planar graph with girth at least \(g\) is \((k, j)\)-colorable or it is NP-complete to determine whether a planar graph with girth at least \(g\) is \((k, j)\)-colorable.

Theorem 4. Let \(k\) be a fixed integer. It is NP-complete to determine whether a 3-degenerate planar graph that is either \((0, 0, 0)\)-colorable or non-(\(k, k, 1\))-colorable is \((0, 0, 0)\)-colorable.

We construct a non-(\(k, k\))-colorable planar graph with girth 4 in Section 2, a non-(3, 1)-colorable planar graph with girth 5 in Section 3, and a non-(2, 0)-colorable planar graph with girth 7 in Section 4. We prove Theorem 3 in Section 5 and we prove Theorem 4 in Section 6.

Notation.
In the following, when we consider a \((d_1, \ldots, d_k)\)-coloring of a graph \(G\), we color the vertices of \(V_i\) with color \(d_i\) for \(1 \leq i \leq k\): for example in a \((3, 0)\)-coloring, we will use color 3 to color the vertices of \(V_1\) inducing a subgraph with maximum degree 3 and use color 0 to color the vertices of \(V_2\) inducing a stable set. A vertex is said to be colored \(i^j\) if it is colored \(i\) and has \(j\) neighbors colored \(i\), that is, it has degree \(j\) in the subgraph induced by its color. A vertex is saturated if it is colored \(i^j\), that is, if it has maximum degree in the subgraph induced by its color. A cycle (resp. face) of length \(k\) is called a \(k\)-cycle (resp. \(k\)-face). A \(k\)-vertex (resp. \(k^\)-vertex, \(k^+\)-vertex) is a vertex of degree \(k\) (resp. at most \(k\), at least \(k\)). The minimum degree of a graph \(G\) is denoted by \(\delta(G)\).

2 A non-(\(k, k\))-colorable planar graph with girth 4

For every \(k \geq 0\), we construct a non-(\(k, k\))-colorable planar graph \(J_4\) with girth 4. Let \(H_{x,y}\) be a copy of \(K_{2, 2k+1}\), as depicted in Figure 2. In any \((k, k)\)-coloring of \(H_{x,y}\), the
vertices $x$ and $y$ must receive the same color. We obtain $J_4$ from a vertex $u$ and a star $S$ with center $v_0$ and $k + 1$ leaves $v_1, \ldots, v_{k+1}$ by linking $u$ to every vertex $v_i$ with a copy $H_{u,v_i}$ of $H_{x,y}$. The graph $J_4$ is not $(k,k)$-colorable: by the property of $H_{x,y}$, every vertex $v_i$ should get the same color as $u$. This gives a monochromatic $S$, which is forbidden. Notice that $J_4$ is a planar graph with girth 4 and is 2-degenerate.

Figure 2: A non-$(k,k)$-colorable planar graph with girth 4.

3 A non-$(3,1)$-colorable planar graph with girth 5

We construct a non-$(3,1)$-colorable planar graph $J_5$ with girth 5. Consider the graph $H_{x,y}$ depicted in Figure 3. If $x$ and $y$ are colored 3 but have no neighbor colored 3, then it is

Figure 3: A non-$(3,1)$-colorable planar graph with girth 5.
not possible to extend this partial coloring to $H_{x,y}$. Now, we construct the graph $S_z$ as follows. Let $z$ be a vertex and $t_1 t_2 t_3$ be a path on three vertices. Take 21 copies $H_{x_i,y_j}$ of $H_{x,y}$ with $1 \leq i \leq 7$ and $1 \leq j \leq 3$. Identify every $x_i$ with $z$ and identify every $y_i$ with $t_i$. Finally, we obtain $J_5$ from three copies $S_{z_1}$, $S_{z_2}$, and $S_{z_3}$ of $S_z$ by adding the edges $z_1 z_2$ and $z_2 z_3$ (Figure 3). Notice that $J_5$ is planar with girth 5 and is 2-degenerate. Let us show that $J_5$ is not $(3,1)$-colorable. In every $(3,1)$-coloring of $J_5$, the path $z_1 z_2 z_3$ contains a vertex $z$ colored 3. In the copy of $S_z$ corresponding to $z$, the path $t_1 t_2 t_3$ contains a vertex $t$ colored 3. Since $z$ (resp. $t$) has at most 3 neighbors colored 3, one of the seven copies of $H_{x,y}$ between $z$ and $t$, does not contain a neighbor of $z$ or $t$ colored 3. This copy of $H_{x,y}$ is not $(3,1)$-colorable, and thus $J_5$ is not $(3,1)$-colorable.

4 A non-(2,0)-colorable planar graph with girth 7

We construct of a non-(2,0)-colorable planar graph $J_7$ with girth 7. Consider the graphs $T_{x,y,z}$ and $S$ in Figure 4.

![Graphs T_{x,y,z} and S](image)

Figure 4: The graphs $T_{x,y,z}$ and $S$.

If the vertices $x$, $y$, and $z$ of $T_{x,y,z}$ are colored 2 and have no neighbor colored 2, then $w$ is colored $2^2$. Suppose that the vertices $a, b, c, d, e, f, g$ of $S$ are respectively colored 2, 0, 2, 2, 2, 0, and that $a$ has no neighbor colored 2. Using successively the property of $T_{x,y,z}$, we have that $w_1$, $w_2$, and $w_3$ must be colored $2^2$. It follows that $w_4$ is colored 0, $w_5$ is colored 2, and so $w_6$ is colored $2^2$. Again, by the property of $T_{x,y,z}$, $w_7$ must be colored $2^2$. Finally, $w_8$ must be colored 0 and there is no choice of color for $w_9$. Hence, such a coloring of the outer 7-cycle $abcdefg$ cannot be extended.
The graph $H_z$ depicted on the left of Figure 5 is obtained as follows.

![Figure 5: The graphs $H_z$ and $J_7$.](image)

We link a vertex $z$ to every vertex of a 7-cycle $v_1 \ldots v_7$ with a path of three edges. Then we embed the graph $S$ in every 7-face $F_i$ ($1 \leq i \leq 7$) incident to $z$ by identifying the outer 7-cycle of $S$ with the 7-cycle of $F_i$ such that $a$ is identified to $z$. Finally, the graph $J_7$ depicted on the right of Figure 5 is obtained from two adjacent vertices $u$ and $v$ and six copies $H_{z_1}, \ldots, H_{z_6}$ of $H_z$ by identifying $z_1, z_2, z_3$ with $u$ and $z_4, z_5, z_6$ with $v$. Notice that $J_7$ is planar with girth 7. Let us prove that $J_7$ is not $(2,0)$-colorable.

- We assume that $u$ is colored 2 since $u$ and $v$ cannot be both colored 0.
- In one of the three copies of $H_z$ attached to $u$, say $H_{z_1}$, $u$ has no neighbor colored 2.
- Since a 7-cycle is not 2-colorable, the 7-cycle $v_1 \ldots v_7$ of $H_{z_1}$ contains a monochromatic edge colored 2, say $v_1v_2$.
- The coloring of the face $F_2$ cannot be extended to the copy of $S$ embedded in $F_2$.

5 NP-completeness of $(k, j)$-colorings

Let $g_{k,j}$ be the largest integer $g$ such that there exists a planar graph with girth $g$ that is not $(k, j)$-colorable. Because of large odd cycles, $g_{0,0}$ is not defined. For $(k, j) \neq (0, 0)$, we have $4 \leq g_{k,j} \leq 10$ by the example in Figure 2 and the result that planar graphs with girth at least 11 are $(0, 1)$-colorable [22]. We prove this equivalent form of Theorem 3:

Theorem 5. Let $k$ and $j$ be fixed integers such that $(k, j) \neq (0, 0)$. It is NP-complete to determine whether a planar graph with girth $g_{k,j}$ is $(k, j)$-colorable.
Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of $3^+$-vertices in $G$. For any two graphs $G_1$ and $G_2$, we have $G_1 \prec G_2$ if and only if at least one of the following conditions holds:

- $|V(G_1)| < |V(G_2)|$ and $n_3(G_1) \leq n_3(G_2)$.
- $n_3(G_1) < n_3(G_2)$.

Note that the partial order $\preceq$ is well-defined and is a partial linear extension of the subgraph poset. The following lemma is useful.

**Lemma 6.** Let $k$ and $j$ be fixed integers such that $(k,j) \neq (0,0)$. There exists a planar graph $G_{k,j}$ with girth $g_{k,j}$, minimally non-$(k,j)$-colorable for the subgraph order, such that $\delta(G_{k,j}) = 2$.

**Proof.** We have $\delta(G_{k,j}) \geq 2$, since a non-$(k,j)$-colorable graph that is minimal for the subgraph order does not contain a $1^-$-vertex. Suppose that for some pair $(k,j)$, we construct a 2-degenerate non-$(k,j)$-colorable planar graph with girth $g_{k,j}$. Then this graph contains a (not necessarily proper) minimally non-$(k,j)$-colorable subgraph with minimum degree 2. Thus, we can prove the lemma for the following pairs $(k,j)$ by using Claim 2.

- Pairs $(k,j)$ such that $g_{k,j} \leq 4$: We actually have $g_{k,j} = 4$ by Claim 2.1.
- Pairs $(k,j)$ such that $g_{k,j} \geq 6$: Indeed, a planar graph with girth at least 6 is 2-degenerate. In particular, Claim 2.3 shows that $g_{k,0} \geq 6$, so the lemma is proved for all pairs $(k,0)$.
- Pairs $(k,1)$ such that $1 \leq k \leq 3$: If $g_{k,j} \geq 6$, then we are in a previous case. Otherwise, we have $g_{k,j} = 5$ by Claim 2.2.

The remaining pairs satisfy $g_{k,j} = 5$. There are two types of remaining pairs $(k,j)$:

- Type 1: $k \geq 4$ and $j = 1$.
- Type 2: $2 \leq j \leq k$.

Consider a planar graph $G$ with girth 5 that is non-$(k,j)$-colorable and is minimal for the order $\preceq$. Suppose for contradiction that $G$ does not contain a 2-vertex. Also, suppose that $G$ contains a 3-vertex $a$ adjacent to three $4^-$-vertices $a_1$, $a_2$, and $a_3$. For colorings of type 1, we can extend to $G$ a coloring of $G \setminus \{a\}$ by assigning to $a$ the color of improperty at least 4. For colorings of type 2, we consider the graph $G'$ obtained from $G \setminus \{a\}$ by adding three 2-vertices $b_1$, $b_2$, and $b_3$ adjacent to, respectively, $a_2$ and $a_3$, $a_1$ and $a_3$, $a_1$ and $a_2$. Notice that $G' \preceq G$, so $G'$ admits a coloring $c$ of type 2. We can extend to $G$ the coloring of $G \setminus \{a\}$ induced by $c$ as follows. If $a_1$, $a_2$, and $a_3$ have the same color, then we assign to $a$ the other color. Otherwise, we assign to $a$ the color that appears at least twice among the colors of $b_1$, $b_2$, and $b_3$. Now, since $G$ does not contain a 2-vertex nor a
3-vertex adjacent to three 4-vertices, we have mad(G) \geq \frac{10}{3}$. This can be seen using the discharging procedure such that the initial charge of each vertex is its degree and every 5'-vertex gives $\frac{1}{2}$ to each adjacent 3-vertex. The final charge of a 3-vertex is at least $3 + \frac{1}{3} = \frac{10}{3}$, the final charge of a 4-vertex is $4 > \frac{10}{3}$, and the final charge of a $k$-vertex with $k \geq 5$ is at least $k - k \times \frac{1}{3} = \frac{2k}{3} \geq \frac{10}{3}$. Now, mad(G) \geq \frac{10}{3} contradicts the fact that $G$ is a planar graph with girth 5, and this contradiction shows that $G$ contains a 2-vertex.

We are ready to prove Theorem 5. The case of $(1, 0)$-coloring is proved in a stronger form which takes into account restrictions on both the girth and the maximum degree of the input planar graph [16].

Proof of the case $(k, 0)$, $k \geq 2$.
We consider a graph $G_{k, 0}$ as described in Lemma 6, which contains a path $uxv$ where $x$ is a 2-vertex. By minimality, any $(k, 0)$-coloring of $G_{k, 0} \setminus \{x\}$ is such that $u$ and $v$ get distinct saturated colors. Let $G$ be the graph obtained from $G_{k, 0} \setminus \{x\}$ by adding three 2-vertices $x_1$, $x_2$, and $x_3$ to create the path $ux_1x_2x_3v$. So any $(k, 0)$-coloring of $G$ is such that $x_2$ is colored $k^1$. To prove the NP-completeness, we reduce the $(k, 0)$-coloring problem to the $(1, 0)$-coloring problem. Let $I$ be a planar graph with girth $g_{1, 0}$. For every vertex $s$ of $I$, add $(k - 1)$ copies of $G$ such that the vertex $x_2$ of each copy is adjacent to $s$, to obtain the graph $I'$. By construction, $I'$ is $(k, 0)$-colorable if and only if $I$ is $(1, 0)$-colorable. Moreover, $I'$ is planar, and since $g_{k, 0} \leq g_{1, 0}$, the girth of $I'$ is $g_{k, 0}$.

Proof of the case $(1, 1)$.
By Claim 2.2 and [9], $g_{1, 1}$ is either 5 or 6. There exist two independent proofs [17, 19] that $(1, 1)$-coloring is NP-complete for triangle-free planar graphs with maximum degree 4. We use a reduction from that problem to prove that $(1, 1)$-coloring is NP-complete for planar graphs with girth $g_{1, 1}$. We consider a graph $G_{1, 1}$ as described in Lemma 6, which contains a path $uxv$ where $x$ is a 2-vertex. By minimality, any $(1, 1)$-coloring of $G_{1, 1} \setminus \{x\}$ is such that $u$ and $v$ get distinct saturated colors. Let $G$ be the graph obtained from $G_{1, 1} \setminus \{x\}$ by adding a vertex $u'$ adjacent to $u$ and a vertex $v'$ adjacent to $v$. So any $(1, 1)$-coloring of $G$ is such that $u'$ and $v'$ get distinct colors and $u'$ (resp. $v'$) has a color distinct from the color of its (unique) neighbor. We construct the graph $E_{a,b}$ from two vertices $a$ and $b$ and two copies of $G$ such that $a$ is adjacent to the vertices $u'$ of both copies of $G$ and $b$ is adjacent to the vertices $v'$ of both copies of $G$. There exists a $(1, 1)$-coloring of $E_{a,b}$ such that $a$ and $b$ have distinct colors and neither $a$ nor $b$ is saturated. There exists a $(1, 1)$-coloring of $E_{a,b}$ such that $a$ and $b$ have the same color. Moreover, in every $(1, 1)$-coloring of $E_{a,b}$ such that $a$ and $b$ have the same color, both $a$ and $b$ are saturated.

The reduction is as follows. Let $I$ be a planar graph. For every edge $(p, q)$ of $I$, replace $(p, q)$ by a copy of $E_{a,b}$ such that $a$ is identified with $p$ and $b$ is identified with $q$, to obtain the graph $I'$. By the properties of $E_{a,b}$, $I$ is $(1, 1)$-colorable if and only if $I'$ is $(1, 1)$-colorable. Moreover, $I'$ is planar with girth $g_{1, 1}$.

Proof of the case $(k, j)$.
We consider a graph $G_{k,j}$ as described in Lemma 6, which contains a path $uxv$ where $x$
is a 2-vertex. By minimality, any \((k, j)\)-coloring of \(G_{k,j} \setminus \{x\}\) is such that \(u\) and \(v\) get distinct saturated colors. Let \(G\) be the graph obtained from \(G_{k,j} \setminus \{x\}\) by adding a vertex \(u'\) adjacent to \(u\) and a vertex \(v'\) adjacent to \(v\). So any \((k, j)\)-coloring of \(G\) is such that \(u'\) and \(v'\) get distinct colors and \(u'\) (resp. \(v'\)) has a color distinct from the color of its (unique) neighbor. Let \(t = \min(k-1, j)\). To prove the NP-completeness, we reduce the \((k, j)\)-coloring to the \((k-t, j-t)\)-coloring. Thus the case \((k, k)\) reduces to the case \((1, 1)\) which is NP-complete, and the case \((k, j)\) with \(j < k\) reduces to the case \((k-j, 0)\) which is NP-complete. The reduction is as follows. Let \(I\) be a planar graph with girth \(g_{k-t,j-t}\). For every vertex \(s\) of \(I\), add \(t\) copies of \(G\) such that the vertices \(u'\) and \(v'\) of each copy is adjacent to \(s\), to obtain the graph \(I'\). By construction, \(I'\) is \((k-t, j-t)\)-colorable if and only if \(I'\) is \((k, j)\)-colorable. Moreover, \(I'\) is planar, and since \(g_{k,j} < g_{k-t,j-t}\), the girth of \(I'\) is \(g_{k,j}\).

6 NP-completeness of \((k, j, i)\)-colorings

In this section, we prove Theorem 4 using a reduction from 3-colorability, i.e. \((0, 0, 0)\)-colorability, which is NP-complete for planar graphs [18].

Let \(E\) be the graph depicted in Fig 6. The graph \(E'\) is obtained from \(2k - 1\) copies of \(E\) by identifying the edge \(ab\) of all copies. Take 4 copies \(E'_1, E'_2, E'_3, \) and \(E'_4\) of \(E'\) and consider a triangle \(T\) formed by the vertices \(y_0, x_0, x_1\) in one copy of \(E\) in \(E'_1\). The graph \(E''\) is obtained by identifying the edge \(y_0x_0\) (resp. \(y_0x_1, x_0x_1\)) of \(T\) with the edge \(ab\) of \(E'_2\) (resp. \(E'_3, E'_4\)). The edge \(ab\) of \(E'_1\) is then said to be the edge \(ab\) of \(E''\).

Lemma 7.

1. \(E''\) admits a \((0, 0, 0)\)-coloring.

2. \(E'\) does not admit a \((k, k, 1)\)-coloring such that \(a\) and \(b\) have a same color of impropery \(k\).

3. \(E''\) does not admit a \((k, k, 1)\)-coloring such that \(a\) and \(b\) have the same color.

Proof.

1. The following \((0, 0, 0)\)-coloring \(c\) of \(E\) is unique up to permutation of colors: \(c(a) = c(x_i) = 1\) for even \(i\), \(c(b) = c(y_i) = 2\) for even \(i\), and \(c(x_i) = c(y_i) = 3\) for odd \(i\). This coloring can be extended into a \((0, 0, 0)\)-coloring of \(E'\) and \(E''\).

2. Let \(k_1, k_2\), and \(1\) denote the colors in a potential \((k, k, 1)\)-coloring \(c\) of \(E'\) such that \(c(a) = c(b) = k_1\). By the pigeon-hole principle, one of the \(2k - 1\) copies of \(E\) in \(E'\), say \(E^*\), is such that \(a\) and \(b\) are the only vertices with color \(k_1\). So, one of the vertices \(x_0, y_0, \) and \(x_{3k+3+t}\) in \(E^*\) must get color \(k_2\) since they cannot all get color \(1\). We thus have a vertex \(v_1 \in \{a, b\}\) colored \(k_1\) and vertex \(v_2 \in \{x_0, y_0, x_{3k+3+t}\}\) colored \(k_2\) in \(E^*\) which both dominate a path on at least \(3k + 3\) vertices. This path contains no vertex colored \(k_1\) since it is in \(E^*\). Moreover, it contains at most \(k\) vertices colored...
Figure 6: The graph $E$. We take $t = 0$ if $k$ is odd and $t = 1$ if $k$ is even, so that $3k + 3 + t$ is even.

$k_2$. On the other hand, every set of 3 consecutive vertices in this path contains at least one vertex colored $k_2$, so it contains at least $\frac{3k+3}{3} = k + 1$ vertices colored $k_2$. This contradiction shows that $E'$ does not admit a $(k, k, 1)$-coloring such that $a$ and $b$ have a same color of improperty $k$.

3. By the previous item and by construction of $E''$, if $E''$ admits a $(k, k, 1)$-coloring $c$ such that $c(a) = c(b)$, then $c(a) = c(b) = 1$. We thus have that $\{c(y_0), c(x_0), c(x_1)\} \subset \{k_1, k_2\}$. This implies that at least one edge of the triangle $T$ is monochromatic with a color of improperty $k$. By the previous item, the coloring $c$ cannot be extended to the copy of $E'$ attached to that monochromatic edge. This shows that $E''$ does not admit a $(k, k, 1)$-coloring such that $a$ and $b$ have the same color.

For every fixed integer $k$, we give a polynomial construction that transforms every planar graph $G$ into a planar graph $G'$ such that $G'$ is $(0, 0, 0)$-colorable if $G$ is $(0, 0, 0)$-colorable and $G'$ is not $(k, k, 1)$-colorable otherwise. The graph $G'$ is obtained from $G$ by identifying every edge of $G$ with the edge $ab$ of a copy of $E''$. If $G$ is $(0, 0, 0)$-colorable, then this coloring can be extended into a $(0, 0, 0)$-coloring of $G'$ by Lemma 7.1. If $G$ is not $(0, 0, 0)$-colorable, then every $(k, k, 1)$-coloring $G$ contains a monochromatic edge $uv$, and then the copy of $E''$ corresponding to $uv$ (and thus $G'$) does not admit a $(k, k, 1)$-coloring by Lemma 7.3. The instance graph $G$ in the proof that $(0, 0, 0)$-coloring is NP-complete [18] is 3-degenerate. Then by construction, $G'$ is also 3-degenerate.
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