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# Near-colorings: non-colorable graphs and NP-completeness

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## Abstract

A graph  $G$  is  $(d_1, \dots, d_l)$ -colorable if the vertex set of  $G$  can be partitioned into subsets  $V_1, \dots, V_l$  such that the graph  $G[V_i]$  induced by the vertices of  $V_i$  has maximum degree at most  $d_i$  for all  $1 \leq i \leq l$ . In this paper, we focus on complexity aspects of such colorings when  $l = 2, 3$ . More precisely, we prove that, for any fixed integers  $k, j, g$  with  $(k, j) \neq (0, 0)$  and  $g \geq 3$ , either every planar graph with girth at least  $g$  is  $(k, j)$ -colorable or it is NP-complete to determine whether a planar graph with girth at least  $g$  is  $(k, j)$ -colorable. Also, for every fixed integer  $k$ , it is NP-complete to determine whether a planar graph that is either  $(0, 0, 0)$ -colorable or non- $(k, k, 1)$ -colorable is  $(0, 0, 0)$ -colorable. Additionally, we exhibit non- $(3, 1)$ -colorable planar graphs with girth 5 and non- $(2, 0)$ -colorable planar graphs with girth 7.

## 1 Introduction

A graph  $G$  is  $(d_1, \dots, d_k)$ -colorable if the vertex set of  $G$  can be partitioned into subsets  $V_1, \dots, V_k$  such that the graph  $G[V_i]$  induced by the vertices of  $V_i$  has maximum degree at most  $d_i$  for all  $1 \leq i \leq k$ . This notion generalizes those of proper  $k$ -coloring (when  $d_1 = \dots = d_k = 0$ ) and  $d$ -improper  $k$ -coloring (when  $d_1 = \dots = d_k = d \geq 1$ ).

Planar graphs are known to be  $(0, 0, 0, 0)$ -colorable (Appel and Haken [1, 2]) and  $(2, 2, 2)$ -colorable (Cowen, Cowen, and Woodall [13]). The  $(2, 2, 2)$ -colorability is optimal (for any integer  $k$ , there exist non- $(k, k, 1)$ -colorable planar graphs) and holds in the choosability case (Eaton and Hull [15] or Škrekovski [23]). Improper colorings have then been considered for planar graphs with large girth or graphs with low maximum average degree. We recall that the girth of a graph  $G$ , denoted by  $g(G)$ , is the length of a shortest cycle in

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$G$ , and the maximum average degree of a graph  $G$ , denoted by  $\text{mad}(G)$ , is the maximum of the average degrees of all subgraphs of  $G$ , i.e.  $\text{mad}(G) = \max \{2|E(H)|/|V(H)|, H \subseteq G\}$ .

### **(1, 0)-coloring.**

Glebov and Zambalaeva [20] proved that every planar graph with girth at least 16 is  $(1, 0)$ -colorable. This was then strengthened by Borodin and Ivanova [3] who proved that every graph  $G$  with  $\text{mad}(G) < \frac{7}{3}$  is  $(1, 0)$ -colorable. This implies that every planar graph  $G$  with girth at least 14 is  $(1, 0)$ -colorable. Borodin and Kostochka [7] then proved that every graph  $G$  with  $\text{mad}(G) \leq \frac{12}{5}$  is  $(1, 0)$ -colorable. In particular, it follows that every planar graph with girth at least 12 is  $(1, 0)$ -colorable. On the other hand, they constructed graphs  $G$  with  $\text{mad}(G)$  arbitrarily close (from above) to  $\frac{12}{5}$  that are not  $(1, 0)$ -colorable; hence their upper bound on the maximum average degree is best possible. The last result was strengthened for triangle-free graphs: Kim, Kostochka, and Zhu [22] proved that triangle-free graphs  $G$  satisfying  $11|V(G)| - 9|E(G)| \geq -4$  are  $(1, 0)$ -colorable. This implies that planar graphs with girth at least 11 are  $(1, 0)$ -colorable. On the other hand, Esperet, Montassier, Ochem, and Pinlou [16] proved that determining whether a planar graph with girth 9 is  $(1, 0)$ -colorable is NP-complete. To our knowledge, the question whether all planar graphs with girth at least 10 are  $(1, 0)$ -colorable is still open.

### **( $k, 0$ )-coloring with $k \geq 2$ .**

Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] proved that every graph  $G$  with  $\text{mad}(G) < \frac{3k+4}{k+2}$  is  $(k, 0)$ -colorable. The proof in [4] extends the one in [3] but does not work for  $k = 1$ . Moreover, they exhibited a non- $(k, 0)$ -colorable planar graph with girth 6. Finally, Borodin and Kostochka [8] proved that for  $k \geq 2$ , every graph  $G$  with  $\text{mad}(G) \leq \frac{3k+2}{k+1}$  is  $(k, 0)$ -colorable. This result is tight in terms of maximum average degree.

### **( $k, 1$ )-coloring.**

Recently, Borodin, Kostochka, and Yancey [9] proved that every graph with  $\text{mad}(G) \leq \frac{14}{5}$  is  $(1, 1)$ -colorable, and the restriction on  $\text{mad}(G)$  is sharp. In [5], it is proven that every graph  $G$  with  $\text{mad}(G) < \frac{10k+22}{3k+9}$  is  $(k, 1)$ -colorable for  $k \geq 2$ .

### **( $k, j$ )-coloring.**

A first step was made by Havet and Sereni [21] who showed that, for every  $k \geq 0$ , every graph  $G$  with  $\text{mad}(G) < \frac{4k+4}{k+2}$  is  $(k, k)$ -colorable (in fact  $(k, k)$ -choosable). More generally, they studied  $k$ -improper  $l$ -choosability and proved that every graph  $G$  with  $\text{mad}(G) < l + \frac{lk}{l+k}$  ( $l \geq 2, k \geq 0$ ) is  $k$ -improper  $l$ -choosable; this implies that such graphs are  $(k, \dots, k)$ -colorable (where the number of partite sets is  $l$ ). Borodin, Ivanova, Montassier, and Raspaud [6] gave some sufficient conditions of  $(k, j)$ -colorability depending on the density of the graphs using linear programming. Finally, Borodin and Kostochka [8] solved the problem for a wide range of  $j$  and  $k$ : let  $j \geq 0$  and  $k \geq 2j + 2$ ; every graph  $G$  with  $\text{mad}(G) \leq 2(2 - \frac{k+2}{(j+2)(k+1)})$  is  $(k, j)$ -colorable. This result is tight in terms of the maximum average degree and improves some results in [4, 5, 6].

Using the fact that every planar graph  $G$  with girth  $g(G)$  has  $\text{mad}(G) < 2g(G)/(g(G)-2)$ , the previous results give results for planar graphs. They are summarized in Table 1, which also shows the recent results that planar graphs with girth 5 are  $(5, 3)$ -colorable (Choi and Raspaud [12]) and  $(10, 1)$ -colorable (Choi, Choi, Jeong, and Suh [11]).

girth	$(k, 0)$	$(k, 1)$	$(k, 2)$	$(k, 3)$	$(k, 4)$
3,4	×	×	×	×	×
5	×	$(10, 1)$ [11]	$(6, 2)$ [8]	$(5, 3)$ [12]	$(4, 4)$ [21]
6	×	$(4, 1)$ [8]	$(2, 2)$ [21]		
7	$(4, 0)$ [8]	$(1, 1)$ [9]			
8	$(2, 0)$ [8]				
11	$(1, 0)$ [22]				

Table 1: The girth and the  $(k, j)$ -colorability of planar graphs. The symbol “ $\times$ ” means that there exist non- $(k, j)$ -colorable planar graphs for every  $k$ .

From the previous discussion, the following questions are natural:

**Question 1.**

1. Are planar graphs with girth 10  $(1, 0)$ -colorable?
2. Are planar graphs with girth 7  $(3, 0)$ -colorable?
3. Are planar graphs with girth 6  $(1, 1)$ -colorable?
4. Are planar graphs with girth 5  $(4, 1)$ -colorable?
5. Are planar graphs with girth 5  $(2, 2)$ -colorable?

**$(d_1, \dots, d_k)$ -coloring.**

Finally we would like to mention two studies. Chang, Havet, Montassier, and Raspaud [10] gave some approximation results to Steinberg’s Conjecture using  $(k, j, i)$ -colorings. Dorbec, Kaiser, Montassier, and Raspaud [14] studied the particular case of  $(d_1, \dots, d_k)$ -coloring where the value of each  $d_i$  ( $1 \leq i \leq k$ ) is either 0 or some value  $d$ , making the link between  $(d, 0)$ -coloring [8] and  $(d, \dots, d)$ -coloring [21].

The aim of this paper is to provide complexity results on the subject and to obtain non-colorable planar graphs showing that some above-mentioned results are optimal.

**Claim 2.** *There exist 2-degenerate planar graphs that are:*

1. non- $(k, k)$ -colorable with girth 4, for every  $k \geq 0$ ,
2. non- $(3, 1)$ -colorable with girth 5,
3. non- $(k, 0)$ -colorable with girth 6,

4. *non-(2, 0)-colorable with girth 7.*

Claim 2.4 shows that the  $(2, 0)$ -colorability of planar graphs with girth at least 8 [8] is a tight result. Claim 2.3 has been obtained in [4] and the corresponding graph is depicted in Figure 1.

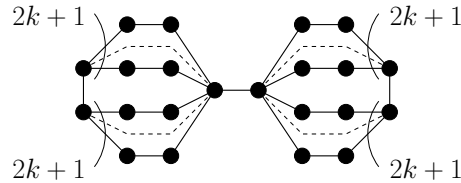


Figure 1: A non- $(k, 0)$ -colorable planar graph with girth 6 [4].

**Theorem 3.** *Let  $k, j$ , and  $g$  be fixed integers such that  $(k, j) \neq (0, 0)$  and  $g \geq 3$ . Either every planar graph with girth at least  $g$  is  $(k, j)$ -colorable or it is NP-complete to determine whether a planar graph with girth at least  $g$  is  $(k, j)$ -colorable.*

**Theorem 4.** *Let  $k$  be a fixed integer. It is NP-complete to determine whether a 3-degenerate planar graph that is either  $(0, 0, 0)$ -colorable or non- $(k, k, 1)$ -colorable is  $(0, 0, 0)$ -colorable.*

We construct a non- $(k, k)$ -colorable planar graph with girth 4 in Section 2, a non- $(3, 1)$ -colorable planar graph with girth 5 in Section 3, and a non- $(2, 0)$ -colorable planar graph with girth 7 in Section 4. We prove Theorem 3 in Section 5 and we prove Theorem 4 in Section 6.

**Notation.**

In the following, when we consider a  $(d_1, \dots, d_k)$ -coloring of a graph  $G$ , we color the vertices of  $V_i$  with color  $d_i$  for  $1 \leq i \leq k$ : for example in a  $(3, 0)$ -coloring, we will use color 3 to color the vertices of  $V_1$  inducing a subgraph with maximum degree 3 and use color 0 to color the vertices of  $V_2$  inducing a stable set. A vertex is said to be *colored  $i^j$*  if it is colored  $i$  and has  $j$  neighbors colored  $i$ , that is, it has degree  $j$  in the subgraph induced by its color. A vertex is *saturated* if it is colored  $i^i$ , that is, if it has maximum degree in the subgraph induced by its color. A cycle (resp. face) of length  $k$  is called a  $k$ -cycle (resp.  $k$ -face). A  $k$ -vertex (resp.  $k^-$ -vertex,  $k^+$ -vertex) is a vertex of degree  $k$  (resp. at most  $k$ , at least  $k$ ). The minimum degree of a graph  $G$  is denoted by  $\delta(G)$ .

## 2 A non- $(k, k)$ -colorable planar graph with girth 4

For every  $k \geq 0$ , we construct a non- $(k, k)$ -colorable planar graph  $J_4$  with girth 4. Let  $H_{x,y}$  be a copy of  $K_{2,2k+1}$ , as depicted in Figure 2. In any  $(k, k)$ -coloring of  $H_{x,y}$ , the

vertices  $x$  and  $y$  must receive the same color. We obtain  $J_4$  from a vertex  $u$  and a star  $S$  with center  $v_0$  and  $k + 1$  leaves  $v_1, \dots, v_{k+1}$  by linking  $u$  to every vertex  $v_i$  with a copy  $H_{u,v_i}$  of  $H_{x,y}$ . The graph  $J_4$  is not  $(k, k)$ -colorable: by the property of  $H_{x,y}$ , every vertex  $v_i$  should get the same color as  $u$ . This gives a monochromatic  $S$ , which is forbidden. Notice that  $J_4$  is a planar graph with girth 4 and is 2-degenerate.

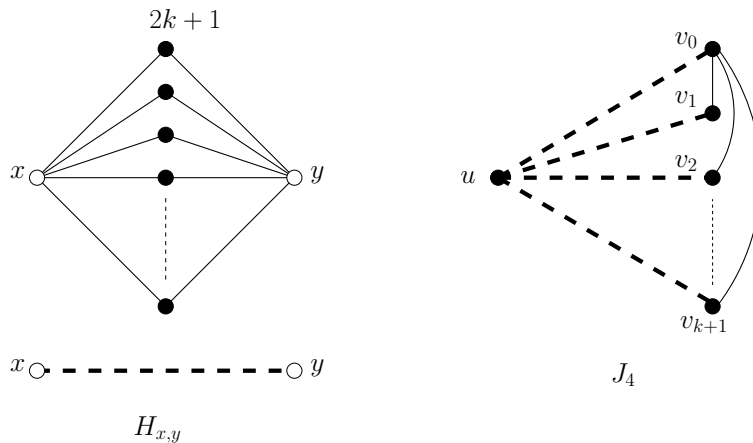


Figure 2: A non- $(k, k)$ -colorable planar graph with girth 4.

### 3 A non- $(3, 1)$ -colorable planar graph with girth 5

We construct a non- $(3, 1)$ -colorable planar graph  $J_5$  with girth 5. Consider the graph  $H_{x,y}$  depicted in Figure 3. If  $x$  and  $y$  are colored 3 but have no neighbor colored 3, then it is

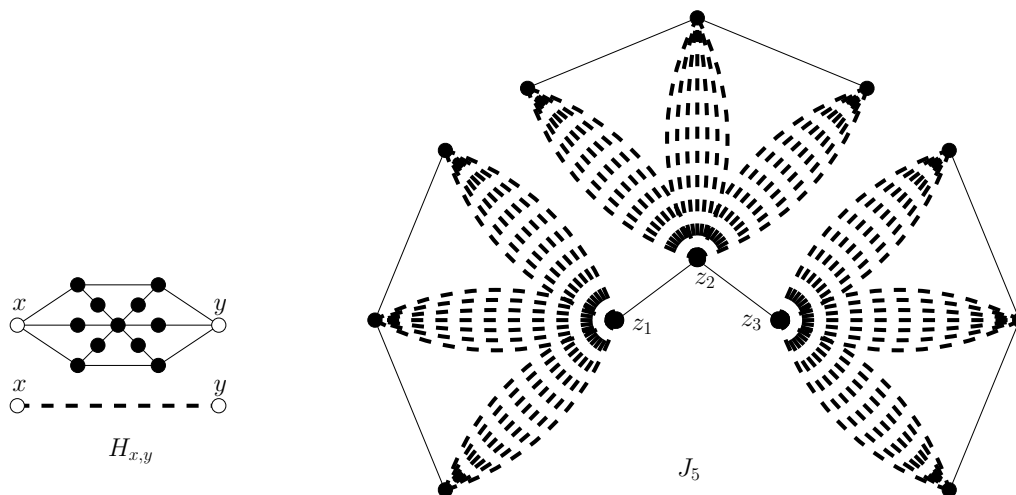


Figure 3: A non- $(3, 1)$ -colorable planar graph with girth 5.

not possible to extend this partial coloring to  $H_{x,y}$ . Now, we construct the graph  $S_z$  as follows. Let  $z$  be a vertex and  $t_1t_2t_3$  be a path on three vertices. Take 21 copies  $H_{x_i,y_j}$  of  $H_{x,y}$  with  $1 \leq i \leq 7$  and  $1 \leq j \leq 3$ . Identify every  $x_i$  with  $z$  and identify every  $y_i$  with  $t_i$ . Finally, we obtain  $J_5$  from three copies  $S_{z_1}, S_{z_2},$  and  $S_{z_3}$  of  $S_z$  by adding the edges  $z_1z_2$  and  $z_2z_3$  (Figure 3). Notice that  $J_5$  is planar with girth 5 and is 2-degenerate. Let us show that  $J_5$  is not  $(3, 1)$ -colorable. In every  $(3, 1)$ -coloring of  $J_5$ , the path  $z_1z_2z_3$  contains a vertex  $z$  colored 3. In the copy of  $S_z$  corresponding to  $z$ , the path  $t_1t_2t_3$  contains a vertex  $t$  colored 3. Since  $z$  (resp.  $t$ ) has at most 3 neighbors colored 3, one of the seven copies of  $H_{x,y}$  between  $z$  and  $t$ , does not contain a neighbor of  $z$  or  $t$  colored 3. This copy of  $H_{x,y}$  is not  $(3, 1)$ -colorable, and thus  $J_5$  is not  $(3, 1)$ -colorable.

#### 4 A non-(2, 0)-colorable planar graph with girth 7

We construct of a non-(2, 0)-colorable planar graph  $J_7$  with girth 7. Consider the graphs  $T_{x,y,z}$  and  $S$  in Figure 4.

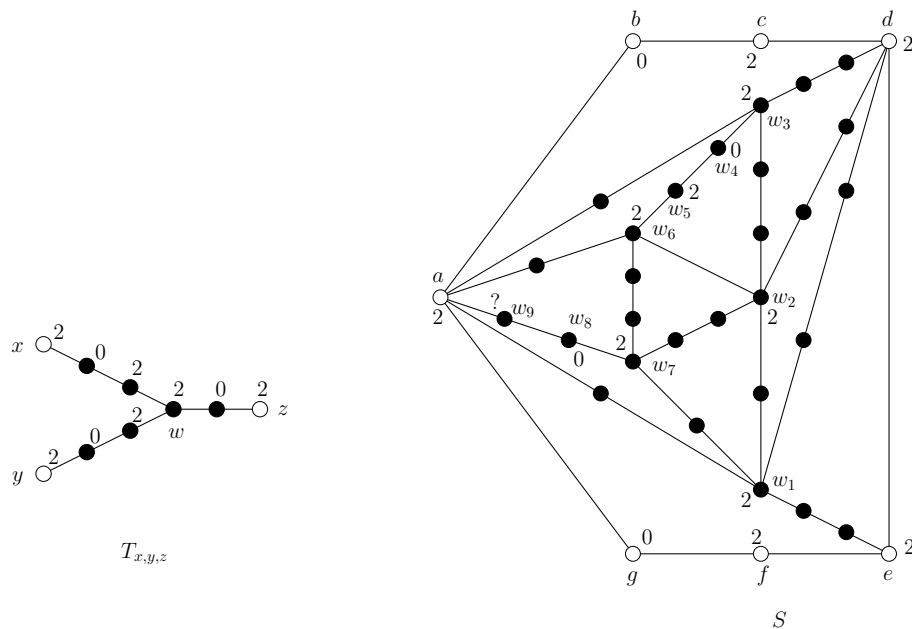


Figure 4: The graphs  $T_{x,y,z}$  and  $S$ .

If the vertices  $x, y,$  and  $z$  of  $T_{x,y,z}$  are colored 2 and have no neighbor colored 2, then  $w$  is colored  $2^2$ . Suppose that the vertices  $a, b, c, d, e, f, g$  of  $S$  are respectively colored 2, 0, 2, 2, 2, 2, 0, and that  $a$  has no neighbor colored 2. Using successively the property of  $T_{x,y,z}$ , we have that  $w_1, w_2,$  and  $w_3$  must be colored  $2^2$ . It follows that  $w_4$  is colored 0,  $w_5$  is colored 2, and so  $w_6$  is colored  $2^2$ . Again, by the property of  $T_{x,y,z}$ ,  $w_7$  must be colored  $2^2$ . Finally,  $w_8$  must be colored 0 and there is no choice of color for  $w_9$ . Hence, such a coloring of the outer 7-cycle  $abcdefg$  cannot be extended.

The graph  $H_z$  depicted on the left of Figure 5 is obtained as follows.

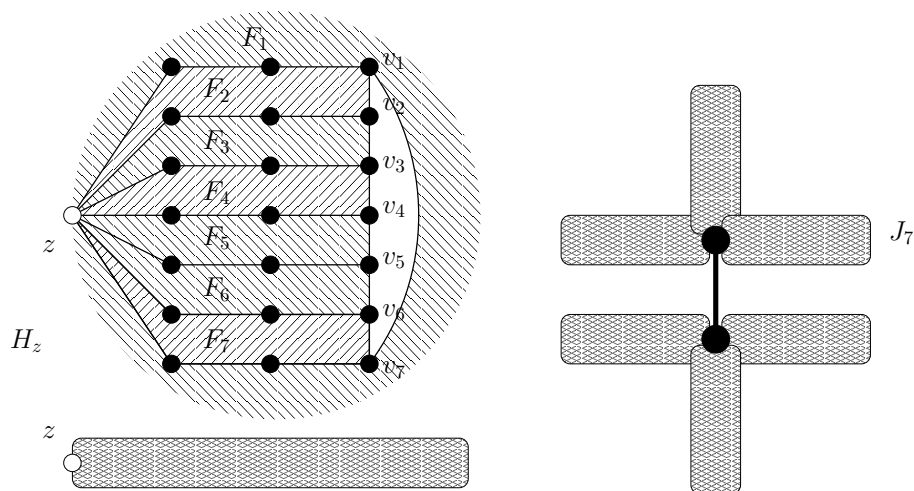


Figure 5: The graphs  $H_z$  and  $J_7$ .

We link a vertex  $z$  to every vertex of a 7-cycle  $v_1 \dots v_7$  with a path of three edges. Then we embed the graph  $S$  in every 7-face  $F_i$  ( $1 \leq i \leq 7$ ) incident to  $z$  by identifying the outer 7-cycle of  $S$  with the 7-cycle of  $F_i$  such that  $a$  is identified to  $z$ . Finally, the graph  $J_7$  depicted on the right of Figure 5 is obtained from two adjacent vertices  $u$  and  $v$  and six copies  $H_{z_1}, \dots, H_{z_6}$  of  $H_z$  by identifying  $z_1, z_2, z_3$  with  $u$  and  $z_4, z_5, z_6$  with  $v$ . Notice that  $J_7$  is planar with has girth 7. Let us prove that  $J_7$  is not  $(2, 0)$ -colorable.

- We assume that  $u$  is colored 2 since  $u$  and  $v$  cannot be both colored 0.
- In one of the three copies of  $H_z$  attached to  $u$ , say  $H_{z_1}$ ,  $u$  has no neighbor colored 2.
- Since a 7-cycle is not 2-colorable, the 7-cycle  $v_1 \dots v_7$  of  $H_{z_1}$  contains a monochromatic edge colored 2, say  $v_1 v_2$ .
- The coloring of the face  $F_2$  cannot be extended to the copy of  $S$  embedded in  $F_2$ .

## 5 NP-completeness of $(k, j)$ -colorings

Let  $g_{k,j}$  be the largest integer  $g$  such that there exists a planar graph with girth  $g$  that is not  $(k, j)$ -colorable. Because of large odd cycles,  $g_{0,0}$  is not defined. For  $(k, j) \neq (0, 0)$ , we have  $4 \leq g_{k,j} \leq 10$  by the example in Figure 2 and the result that planar graphs with girth at least 11 are  $(0, 1)$ -colorable [22]. We prove this equivalent form of Theorem 3:

**Theorem 5.** *Let  $k$  and  $j$  be fixed integers such that  $(k, j) \neq (0, 0)$ . It is NP-complete to determine whether a planar graph with girth  $g_{k,j}$  is  $(k, j)$ -colorable.*



Let us define the partial order  $\preceq$ . Let  $n_3(G)$  be the number of  $3^+$ -vertices in  $G$ . For any two graphs  $G_1$  and  $G_2$ , we have  $G_1 \prec G_2$  if and only if at least one of the following conditions holds:

- $|V(G_1)| < |V(G_2)|$  and  $n_3(G_1) \leq n_3(G_2)$ .
- $n_3(G_1) < n_3(G_2)$ .

Note that the partial order  $\preceq$  is well-defined and is a partial linear extension of the subgraph poset. The following lemma is useful.

**Lemma 6.** *Let  $k$  and  $j$  be fixed integers such that  $(k, j) \neq (0, 0)$ . There exists a planar graph  $G_{k,j}$  with girth  $g_{k,j}$ , minimally non- $(k, j)$ -colorable for the subgraph order, such that  $\delta(G_{k,j}) = 2$ .*

*Proof.* We have  $\delta(G_{k,j}) \geq 2$ , since a non- $(k, j)$ -colorable graph that is minimal for the subgraph order does not contain a  $1^-$ -vertex. Suppose that for some pair  $(k, j)$ , we construct a 2-degenerate non- $(k, j)$ -colorable planar graph with girth  $g_{k,j}$ . Then this graph contains a (not necessarily proper) minimally non- $(k, j)$ -colorable subgraph with minimum degree 2. Thus, we can prove the lemma for the following pairs  $(k, j)$  by using Claim 2.

- Pairs  $(k, j)$  such that  $g_{k,j} \leq 4$ : We actually have  $g_{k,j} = 4$  by Claim 2.1.
- Pairs  $(k, j)$  such that  $g_{k,j} \geq 6$ : Indeed, a planar graph with girth at least 6 is 2-degenerate. In particular, Claim 2.3 shows that  $g_{k,0} \geq 6$ , so the lemma is proved for all pairs  $(k, 0)$ .
- Pairs  $(k, 1)$  such that  $1 \leq k \leq 3$ : If  $g_{k,j} \geq 6$ , then we are in a previous case. Otherwise, we have  $g_{k,j} = 5$  by Claim 2.2.

The remaining pairs satisfy  $g_{k,j} = 5$ . There are two types of remaining pairs  $(k, j)$ :

- Type 1:  $k \geq 4$  and  $j = 1$ .
- Type 2:  $2 \leq j \leq k$ .

Consider a planar graph  $G$  with girth 5 that is non- $(k, j)$ -colorable and is minimal for the order  $\preceq$ . Suppose for contradiction that  $G$  does not contain a 2-vertex. Also, suppose that  $G$  contains a 3-vertex  $a$  adjacent to three  $4^-$ -vertices  $a_1, a_2$ , and  $a_3$ . For colorings of type 1, we can extend to  $G$  a coloring of  $G \setminus \{a\}$  by assigning to  $a$  the color of impropriety at least 4. For colorings of type 2, we consider the graph  $G'$  obtained from  $G \setminus \{a\}$  by adding three 2-vertices  $b_1, b_2$ , and  $b_3$  adjacent to, respectively,  $a_2$  and  $a_3$ ,  $a_1$  and  $a_3$ ,  $a_1$  and  $a_2$ . Notice that  $G' \preceq G$ , so  $G'$  admits a coloring  $c$  of type 2. We can extend to  $G$  the coloring of  $G \setminus \{a\}$  induced by  $c$  as follows. If  $a_1, a_2$ , and  $a_3$  have the same color, then we assign to  $a$  the other color. Otherwise, we assign to  $a$  the color that appears at least twice among the colors of  $b_1, b_2$ , and  $b_3$ . Now, since  $G$  does not contain a 2-vertex nor a

3-vertex adjacent to three  $4^-$ -vertices, we have  $\text{mad}(G) \geq \frac{10}{3}$ . This can be seen using the discharging procedure such that the initial charge of each vertex is its degree and every  $5^+$ -vertex gives  $\frac{1}{3}$  to each adjacent 3-vertex. The final charge of a 3-vertex is at least  $3 + \frac{1}{3} = \frac{10}{3}$ , the final charge of a 4-vertex is  $4 > \frac{10}{3}$ , and the final charge of a  $k$ -vertex with  $k \geq 5$  is at least  $k - k \times \frac{1}{3} = \frac{2k}{3} \geq \frac{10}{3}$ . Now,  $\text{mad}(G) \geq \frac{10}{3}$  contradicts the fact that  $G$  is a planar graph with girth 5, and this contradiction shows that  $G$  contains a 2-vertex.  $\square$

We are ready to prove Theorem 5. The case of  $(1, 0)$ -coloring is proved in a stronger form which takes into account restrictions on both the girth and the maximum degree of the input planar graph [16].

Proof of the case  $(k, 0)$ ,  $k \geq 2$ .

We consider a graph  $G_{k,0}$  as described in Lemma 6, which contains a path  $uxv$  where  $x$  is a 2-vertex. By minimality, any  $(k, 0)$ -coloring of  $G_{k,0} \setminus \{x\}$  is such that  $u$  and  $v$  get distinct saturated colors. Let  $G$  be the graph obtained from  $G_{k,0} \setminus \{x\}$  by adding three 2-vertices  $x_1, x_2$ , and  $x_3$  to create the path  $ux_1x_2x_3v$ . So any  $(k, 0)$ -coloring of  $G$  is such that  $x_2$  is colored  $k^1$ . To prove the NP-completeness, we reduce the  $(k, 0)$ -coloring problem to the  $(1, 0)$ -coloring problem. Let  $I$  be a planar graph with girth  $g_{1,0}$ . For every vertex  $s$  of  $I$ , add  $(k - 1)$  copies of  $G$  such that the vertex  $x_2$  of each copy is adjacent to  $s$ , to obtain the graph  $I'$ . By construction,  $I'$  is  $(k, 0)$ -colorable if and only if  $I$  is  $(1, 0)$ -colorable. Moreover,  $I'$  is planar, and since  $g_{k,0} \leq g_{1,0}$ , the girth of  $I'$  is  $g_{k,0}$ .

Proof of the case  $(1, 1)$ .

By Claim 2.2 and [9],  $g_{1,1}$  is either 5 or 6. There exist two independent proofs [17, 19] that  $(1, 1)$ -coloring is NP-complete for triangle-free planar graphs with maximum degree 4. We use a reduction from that problem to prove that  $(1, 1)$ -coloring is NP-complete for planar graphs with girth  $g_{1,1}$ . We consider a graph  $G_{1,1}$  as described in Lemma 6, which contains a path  $uxv$  where  $x$  is a 2-vertex. By minimality, any  $(1, 1)$ -coloring of  $G_{1,1} \setminus \{x\}$  is such that  $u$  and  $v$  get distinct saturated colors. Let  $G$  be the graph obtained from  $G_{1,1} \setminus \{x\}$  by adding a vertex  $u'$  adjacent to  $u$  and a vertex  $v'$  adjacent to  $v$ . So any  $(1, 1)$ -coloring of  $G$  is such that  $u'$  and  $v'$  get distinct colors and  $u'$  (resp.  $v'$ ) has a color distinct from the color of its (unique) neighbor. We construct the graph  $E_{a,b}$  from two vertices  $a$  and  $b$  and two copies of  $G$  such that  $a$  is adjacent to the vertices  $u'$  of both copies of  $G$  and  $b$  is adjacent to the vertices  $v'$  of both copies of  $G$ . There exists a  $(1, 1)$ -coloring of  $E_{a,b}$  such that  $a$  and  $b$  have distinct colors and neither  $a$  nor  $b$  is saturated. There exists a  $(1, 1)$ -coloring of  $E_{a,b}$  such that  $a$  and  $b$  have the same color. Moreover, in every  $(1, 1)$ -coloring of  $E_{a,b}$  such that  $a$  and  $b$  have the same color, both  $a$  and  $b$  are saturated.

The reduction is as follows. Let  $I$  be a planar graph. For every edge  $(p, q)$  of  $I$ , replace  $(p, q)$  by a copy of  $E_{a,b}$  such that  $a$  is identified with  $p$  and  $b$  is identified with  $q$ , to obtain the graph  $I'$ . By the properties of  $E_{a,b}$ ,  $I$  is  $(1, 1)$ -colorable if and only if  $I'$  is  $(1, 1)$ -colorable. Moreover,  $I'$  is planar with girth  $g_{1,1}$ .

Proof of the case  $(k, j)$ .

We consider a graph  $G_{k,j}$  as described in Lemma 6, which contains a path  $uxv$  where  $x$

is a 2-vertex. By minimality, any  $(k, j)$ -coloring of  $G_{k,j} \setminus \{x\}$  is such that  $u$  and  $v$  get distinct saturated colors. Let  $G$  be the graph obtained from  $G_{k,j} \setminus \{x\}$  by adding a vertex  $u'$  adjacent to  $u$  and a vertex  $v'$  adjacent to  $v$ . So any  $(k, j)$ -coloring of  $G$  is such that  $u'$  and  $v'$  get distinct colors and  $u'$  (resp.  $v'$ ) has a color distinct from the color of its (unique) neighbor. Let  $t = \min(k - 1, j)$ . To prove the NP-completeness, we reduce the  $(k, j)$ -coloring to the  $(k - t, j - t)$ -coloring. Thus the case  $(k, k)$  reduces to the case  $(1, 1)$  which is NP-complete, and the case  $(k, j)$  with  $j < k$  reduces to the case  $(k - j, 0)$  which is NP-complete. The reduction is as follows. Let  $I$  be a planar graph with girth  $g_{k-t, j-t}$ . For every vertex  $s$  of  $I$ , add  $t$  copies of  $G$  such that the vertices  $u'$  and  $v'$  of each copy is adjacent to  $s$ , to obtain the graph  $I'$ . By construction,  $I$  is  $(k - t, j - t)$ -colorable if and only if  $I'$  is  $(k, j)$ -colorable. Moreover,  $I'$  is planar, and since  $g_{k,j} \leq g_{k-t, j-t}$ , the girth of  $I'$  is  $g_{k,j}$ .

## 6 NP-completeness of $(k, j, i)$ -colorings

In this section, we prove Theorem 4 using a reduction from 3-colorability, i.e.  $(0, 0, 0)$ -colorability, which is NP-complete for planar graphs [18].

Let  $E$  be the graph depicted in Fig 6. The graph  $E'$  is obtained from  $2k - 1$  copies of  $E$  by identifying the edge  $ab$  of all copies. Take 4 copies  $E'_1, E'_2, E'_3,$  and  $E'_4$  of  $E'$  and consider a triangle  $T$  formed by the vertices  $y_0, x_0, x_1$  in one copy of  $E$  in  $E'_1$ . The graph  $E''$  is obtained by identifying the edge  $y_0x_0$  (resp.  $y_0x_1, x_0x_1$ ) of  $T$  with the edge  $ab$  of  $E'_2$  (resp.  $E'_3, E'_4$ ). The edge  $ab$  of  $E'_1$  is then said to be the edge  $ab$  of  $E''$ .

### Lemma 7.

1.  $E''$  admits a  $(0, 0, 0)$ -coloring.
2.  $E'$  does not admit a  $(k, k, 1)$ -coloring such that  $a$  and  $b$  have a same color of *improperty*  $k$ .
3.  $E''$  does not admit a  $(k, k, 1)$ -coloring such that  $a$  and  $b$  have the same color.

*Proof.*

1. The following  $(0, 0, 0)$ -coloring  $c$  of  $E$  is unique up to permutation of colors:  $c(a) = c(x_i) = 1$  for even  $i$ ,  $c(b) = c(y_i) = 2$  for even  $i$ , and  $c(x_i) = c(y_i) = 3$  for odd  $i$ . This coloring can be extended into a  $(0, 0, 0)$ -coloring of  $E'$  and  $E''$ .
2. Let  $k_1, k_2,$  and  $1$  denote the colors in a potential  $(k, k, 1)$ -coloring  $c$  of  $E'$  such that  $c(a) = c(b) = k_1$ . By the pigeon-hole principle, one of the  $2k - 1$  copies of  $E$  in  $E'$ , say  $E^*$ , is such that  $a$  and  $b$  are the only vertices with color  $k_1$ . So, one of the vertices  $x_0, y_0,$  and  $x_{3k+3+t}$  in  $E^*$  must get color  $k_2$  since they cannot all get color  $1$ . We thus have a vertex  $v_1 \in \{a, b\}$  colored  $k_1$  and vertex  $v_2 \in \{x_0, y_0, x_{3k+3+t}\}$  colored  $k_2$  in  $E^*$  which both dominate a path on at least  $3k + 3$  vertices. This path contains no vertex colored  $k_1$  since it is in  $E^*$ . Moreover, it contains at most  $k$  vertices colored

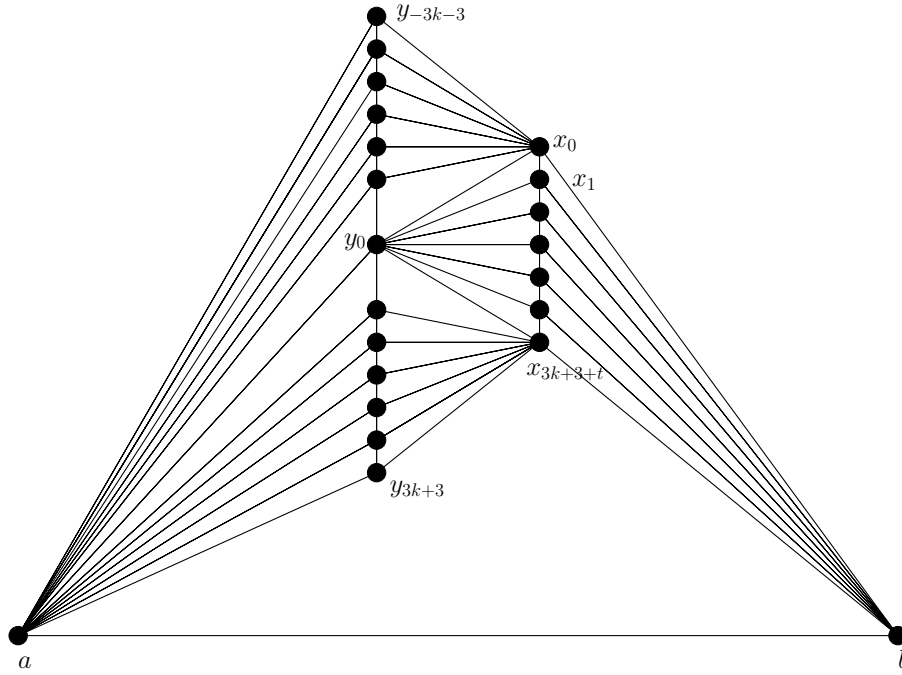


Figure 6: The graph  $E$ . We take  $t = 0$  if  $k$  is odd and  $t = 1$  if  $k$  is even, so that  $3k + 3 + t$  is even.

$k_2$ . On the other hand, every set of 3 consecutive vertices in this path contains at least one vertex colored  $k_2$ , so it contains at least  $\frac{3k+3}{3} = k + 1$  vertices colored  $k_2$ . This contradiction shows that  $E'$  does not admit a  $(k, k, 1)$ -coloring such that  $a$  and  $b$  have a same color of impropriety  $k$ .

3. By the previous item and by construction of  $E''$ , if  $E''$  admits a  $(k, k, 1)$ -coloring  $c$  such that  $c(a) = c(b)$ , then  $c(a) = c(b) = 1$ . We thus have that  $\{c(y_0), c(x_0), c(x_1)\} \subset \{k_1, k_2\}$ . This implies that at least one edge of the triangle  $T$  is monochromatic with a color of impropriety  $k$ . By the previous item, the coloring  $c$  cannot be extended to the copy of  $E'$  attached to that monochromatic edge. This shows that  $E''$  does not admit a  $(k, k, 1)$ -coloring such that  $a$  and  $b$  have the same color.

□

For every fixed integer  $k$ , we give a polynomial construction that transforms every planar graph  $G$  into a planar graph  $G'$  such that  $G'$  is  $(0, 0, 0)$ -colorable if  $G$  is  $(0, 0, 0)$ -colorable and  $G'$  is not  $(k, k, 1)$ -colorable otherwise. The graph  $G'$  is obtained from  $G$  by identifying every edge of  $G$  with the edge  $ab$  of a copy of  $E''$ . If  $G$  is  $(0, 0, 0)$ -colorable, then this coloring can be extended into a  $(0, 0, 0)$ -coloring of  $G'$  by Lemma 7.1. If  $G$  is not  $(0, 0, 0)$ -colorable, then every  $(k, k, 1)$ -coloring  $G$  contains a monochromatic edge  $uv$ , and then the copy of  $E''$  corresponding to  $uv$  (and thus  $G'$ ) does not admit a  $(k, k, 1)$ -coloring by Lemma 7.3. The instance graph  $G$  in the proof that  $(0, 0, 0)$ -coloring is NP-complete [18] is 3-degenerate. Then by construction,  $G'$  is also 3-degenerate.

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