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# Toroidal maps : Schnyder woods, orthogonal surfaces and straight-line representations\*

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## Abstract

A Schnyder wood is an orientation and coloring of the edges of a planar map satisfying a simple local property. We propose a generalization of Schnyder woods to graphs embedded on the torus with application to graph drawing. We prove several properties on this new object. Among all we prove that a graph embedded on the torus admits such a Schnyder wood if and only if it is an essentially 3-connected toroidal map. We show that these Schnyder woods can be used to embed the universal cover of an essentially 3-connected toroidal map on an infinite and periodic orthogonal surface. Finally we use this embedding to obtain a straight-line flat torus representation of any toroidal map in a polynomial size grid.

## 1 Introduction

A closed curve on a surface is *contractible* if it can be continuously transformed into a single point. Given a graph embedded on the torus, a *contractible loop* is an edge forming a contractible cycle. Two *homotopic multiple edges* are two edges with the same extremities such that their union forms a contractible cycle. In this paper, we will almost always consider graphs embedded on the torus with no contractible loop and no homotopic multiple edges. We call these graphs *toroidal graphs* for short and keep the distinction with *graph embedded on the torus* that may have contractible loops or homotopic multiple edges. A *map* on a surface is a graph embedded on this surface where every face is homeomorphic to an open disk. A *map embedded on the torus* is a graph embedded on the torus that is a map (it may contains contractible loops or homotopic multiple edges). A *toroidal map* is a toroidal graph that is a map (it has no contractible loop and no homotopic multiple edges). A *toroidal triangulation* is a

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toroidal map where every face has size three. A general graph (i.e. not embedded on a surface) is *simple* if it contains no loop and no multiple edges. Since some loops and multiple edges are allowed in toroidal graphs, the class of toroidal graphs is larger than the class of simple toroidal graphs.

The torus is represented by a parallelogram in the plane whose opposite sides are pairwise identified. This representation is called the *flat torus*. The *universal cover*  $G^\infty$  of a graph  $G$  embedded on the torus is the infinite planar graph obtained by replicating a flat torus representation of  $G$  to tile the plane (the tiling is obtained by translating the flat torus along two vectors corresponding to the sides of the parallelogram). Note that a graph  $G$  embedded on the torus has no contractible loop and no homotopic multiple edges if and only if  $G^\infty$  is simple.

Given a general graph  $G$ , let  $n$  be the number of vertices and  $m$  the number of edges. Given a graph embedded on a surface, let  $f$  be the number of faces. Euler's formula says that any map on a surface of genus  $g$  satisfies  $n - m + f = 2 - 2g$ . Where the plane is the surface of genus 0, and the torus the surface of genus 1.

Schnyder woods were originally defined for planar triangulations by Schnyder [26]:

**Definition 1 (Schnyder wood, Schnyder property)** *Given a planar triangulation  $G$ , a Schnyder wood is an orientation and coloring of the edges of  $G$  with the colors 0, 1, 2 where each inner vertex  $v$  satisfies the Schnyder property, (see Figure 1 where each color is represented by a different type of arrow):*

- Vertex  $v$  has out-degree one in each color.
- The edges  $e_0(v)$ ,  $e_1(v)$ ,  $e_2(v)$  leaving  $v$  in colors 0, 1, 2, respectively, occur in counterclockwise order.
- Each edge entering  $v$  in color  $i$  enters  $v$  in the counterclockwise sector from  $e_{i+1}(v)$  to  $e_{i-1}(v)$  (where  $i + 1$  and  $i - 1$  are understood modulo 3).

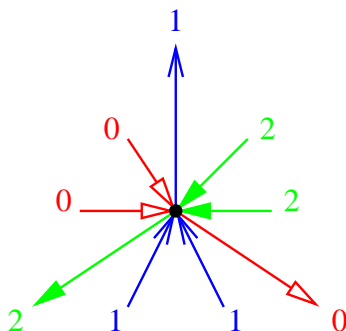


Figure 1: Schnyder property

For higher genus triangulated surfaces, a generalization of Schnyder wood has been proposed by Castelli Aleardi et al. [3], with applications to encoding. Unfortunately, in

this definition, the simplicity and the symmetry of the original Schnyder wood are lost. Here we propose an alternative generalization of Schnyder woods for toroidal graphs, with application to graph drawings.

By Euler's formula, a planar triangulation satisfies  $m = 3n - 6$ . Thus there is not enough edges in the graph for all vertices to be of out-degree three. This explains why just some vertices (inner ones) are required to verify the Schnyder property. For a toroidal triangulation, Euler's formula gives exactly  $m = 3n$  so there is hope for a nice object satisfying the Schnyder property for every vertex. This paper shows that such an object exists. Here we do not restrict ourselves to triangulations and we directly define Schnyder woods in a more general framework.

Felsner [8, 9] (see also [20]) has generalized Schnyder woods to 3-connected planar maps by allowing edges to be oriented in one direction or in two opposite directions. We also allow edges to be oriented in two directions in our definition:

**Definition 2 (Toroidal Schnyder wood)** *Given a toroidal graph  $G$ , a (toroidal) Schnyder wood of  $G$  is an orientation and coloring of the edges of  $G$  with the colors 0, 1, 2, where every edge  $e$  is oriented in one direction or in two opposite directions (each direction having a distinct color), satisfying the following (see example of Figure 2):*

- (T1) *Every vertex  $v$  satisfies the Schnyder property (see Definition 1)*
- (T2) *Every monochromatic cycle of color  $i$  intersects at least one monochromatic cycle of color  $i - 1$  and at least one monochromatic cycle of color  $i + 1$ .*

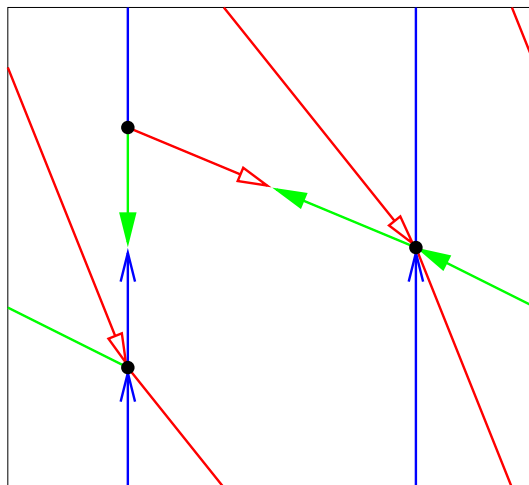


Figure 2: Example of a Schnyder wood of a toroidal graph.

In the case of toroidal triangulations,  $m = 3n$  implies that there are too many edges to have bi-oriented edges. Thus, we can use this general definition of Schnyder wood

for toroidal graphs and keep in mind that when restricted to toroidal triangulations all edges are oriented in one direction only.

Extending the notion of essentially 2-connectedness [23], we say that a toroidal graph  $G$  is *essentially  $k$ -connected* if its universal cover is  $k$ -connected. Note that an essentially 1-connected toroidal graph is a toroidal map. We prove that essentially 3-connected toroidal maps are characterized by existence of Schnyder woods.

**Theorem 1** *A toroidal graph admits a Schnyder wood if and only if it is an essentially 3-connected toroidal map.*

In our definition of Schnyder woods, two properties are required : a local one (T1) and a global one (T2). This second property is important to use Schnyder woods to embed toroidal graphs on orthogonal surfaces like it has been done in the plane by Miller [20] (see also [9]).

**Theorem 2** *The universal cover of an essentially 3-connected toroidal map admits a geodesic embedding on an infinite and periodic orthogonal surface.*

A *straight-line flat torus representation* of a toroidal map  $G$  is the restriction to a flat torus of a periodic straight-line representation of  $G^\infty$ . The problem to find a straight line flat torus representation of a toroidal map was previously solved on exponential size grids [21]. There are several works to represent a toroidal map inside a parallelogram in a polynomial size grid [5, 6], but in these representations the opposite sides of the parallelogram do not perfectly match. In the embeddings obtained by Theorem 2, vertices are not coplanar but we prove that for toroidal triangulations one can project the vertices on a plane to obtain a periodic straight-line representation of  $G^\infty$ . This gives the first straight-line flat torus representation of any toroidal map in a polynomial size grid.

**Theorem 3** *A toroidal graph admits a straight-line flat torus representation in a polynomial size grid.*

In Section 2, we explain how our definition of Schnyder woods in the torus generalize the planar case. In Section 3, we show that our Schnyder woods are of two fundamentally different types. In Section 4, we study the behavior of Schnyder woods in the universal cover, we define the notion of regions and show that the existence of Schnyder woods for a toroidal graph implies that the graph is an essentially 3-connected toroidal map. In Section 5, we define the dual of a Schnyder wood. In Section 6, we show how the definition of Schnyder woods can be relax for one of the two types of Schnyder wood. This relaxation is used in the next sections for proving existence of Schnyder wood. In Section 7, we use a result of Fijavz [14] on existence of non homotopic cycles in simple toroidal triangulations to obtain a short proof of existence of Schnyder wood for simple triangulations. In Section 8, we prove a technical lemma showing how a Schnyder

wood of a graph  $G$  can be derived from a Schnyder wood of the graph  $G'$ , where  $G'$  is obtained from  $G$  by contracting an edge. This lemma is then used in Section 9 to prove the existence of Schnyder woods for any essentially 3-connected toroidal maps. In Section 10, we use Schnyder woods to embed the universal cover of essentially 3-connected toroidal maps on periodic and infinite orthogonal surfaces by generalizing the region vector method defined in the plane. In Section 11, we show that the dual map can also be embedded on this orthogonal surface. In Section 12, we show that, in the case of toroidal triangulations, this orthogonal surface can be projected on a plane to obtain a straight-line flat torus representation.

## 2 Generalization of the planar case

Felsner [8, 9] has generalized planar Schnyder woods by allowing edges to be oriented in one direction or in two opposite directions. The formal definition is the following:

**Definition 3 (Planar Schnyder wood)** *Given a planar map  $G$ . Let  $x_0, x_1, x_2$  be three distinct vertices occurring in counterclockwise order on the outer face of  $G$ . The suspension  $G^\sigma$  is obtained by attaching a half-edge that reaches into the outer face to each of these special vertices. A (planar) Schnyder wood rooted at  $x_0, x_1, x_2$  is an orientation and coloring of the edges of  $G^\sigma$  with the colors 0, 1, 2, where every edge  $e$  is oriented in one direction or in two opposite directions (each direction having a distinct color), satisfying the following (see example of Figure 3):*

- (P1) *Every vertex  $v$  satisfies the Schnyder property and the half-edge at  $x_i$  is directed outwards and colored  $i$*
- (P2) *There is no monochromatic cycle.*

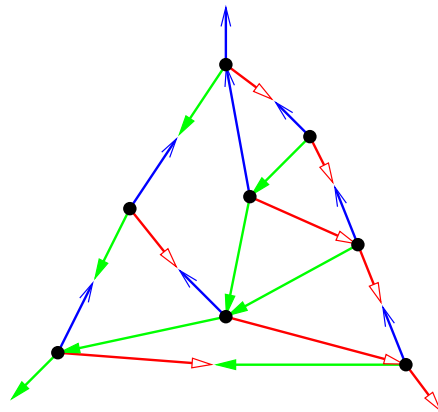


Figure 3: Example of a Schnyder wood of a planar map.

In the definition given by Felsner [9], property (P2) is in fact replaced by “There is no interior face the boundary of which is a monochromatic cycle”, but the two are equivalent by results of [8, 9].

With our definition of Schnyder wood for toroidal graph, the goal is to generalize the definition of Felsner. In the torus, property (P1) can be simplified as every vertex plays the same role: there is no special outer vertices with a half-edge reaching into the outer face. This explain property (T1) in our definition. Then if one asks that every vertex satisfies the Schnyder property, there is necessarily monochromatic cycles and (P2) is not satisfied. This explain why (P2) has been replaced by (T2) in our generalization to the torus.

It would have been possible to replace (P2) by “there is no contractible monochromatic cycles” but this is no enough to suit our needs. Our goal is to use Schnyder woods to embed universal cover of toroidal graphs on orthogonal surfaces like it has been done in the plane by Miller [20] (see also [9]). The difference being that our surface is infinite and periodic. In such a representation the three colors 0, 1, 2 corresponds to the three directions of the space. Thus the monochromatic cycles with different colors have to intersect each other in a particular way. This explains why property (T2) is required. Figure 4 gives an example of an orientation and coloring of the edges of a toroidal triangulation satisfying (T1) but not (T2) as there is no pair of intersecting monochromatic cycles.

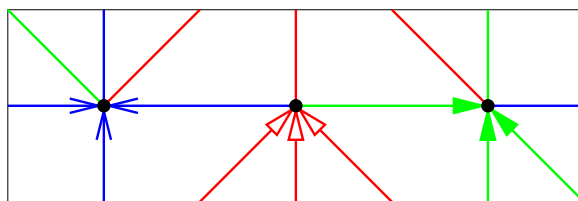


Figure 4: An orientation and coloring of the edges of a toroidal triangulation satisfying (T1) but not (T2) as there is no pair of intersecting monochromatic cycles.

Let  $G$  be a toroidal graph given with a Schnyder wood. Let  $G_i$  be the directed graph induced by the edges of color  $i$ . This definition includes edges that are half-colored  $i$ , and in this case, the edges gets only the direction corresponding to color  $i$ . Each graph  $G_i$  has exactly  $n$  edges, so it does not induce a rooted tree (contrarily to planar Schnyder woods) and the term “wood” has to be handle with care here. Note also that  $G_i$  is not necessarily connected (for example in the graph of Figure 5, every Schnyder wood has one color which corresponding subgraph is not connected). But each components of  $G_i$  has exactly one outgoing arc for each of its vertices. Thus each connected component of  $G_i$  has exactly one directed cycle that is a *monochromatic cycle* of color  $i$ , or  *$i$ -cycle* for short. Note that monochromatic cycles can contain edges oriented in two directions with different colors, but the *orientation* of a  $i$ -cycle is the orientation given by the (half-)edges of color  $i$ . The graph  $G_i^{-1}$  is the graph obtained from  $G_i$  by reversing all its edges. The graph  $G_i \cup G_{i-1}^{-1} \cup G_{i+1}^{-1}$  is obtained from the graph  $G$  by orienting edges

in one or two direction depending on whether this orientation is present in  $G_i$ ,  $G_{i-1}^{-1}$  or  $G_{i+1}^{-1}$ . The following Lemma shows that our property (T2) in fact implies that there is no contractible monochromatic cycles.

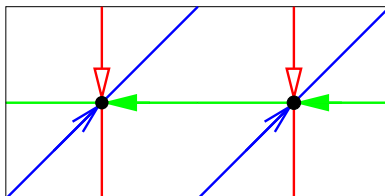


Figure 5: A toroidal graph where every Schnyder wood has one color which corresponding subgraph is not connected.

**Lemma 1** *The graph  $G_i \cup G_{i-1}^{-1} \cup G_{i+1}^{-1}$  contains no contractible directed cycle.*

*Proof.* Suppose there is a contractible directed cycle in  $G_i \cup G_{i-1}^{-1} \cup G_{i+1}^{-1}$ . Let  $C$  be such a cycle containing the minimum number of faces in the closed disk  $D$  bounded by  $C$ . Suppose by symmetry that  $C$  turns around  $D$  clockwise. Then, by (T1), there is no edge of color  $i - 1$  leaving the closed disk  $D$ . So there is a  $(i - 1)$ -cycle in  $D$  and this cycle is  $C$  by minimality of  $C$ . Then, by (T1), there is no edge of color  $i$  leaving  $D$ . So, again by minimality of  $C$ , the cycle  $C$  is a  $i$ -cycle. Thus all the edges of  $C$  are oriented in color  $i$  clockwise and in color  $i - 1$  counterclockwise. Then, by (T1), all the edges of color  $i + 1$  incident to  $C$  have to leave  $D$ . Thus there is no  $(i + 1)$ -cycle intersecting  $C$ , a contradiction to property (T2).  $\square$

Let  $G$  be a planar map and  $x_0, x_1, x_2$  be three distinct vertices occurring in counter-clockwise order on the outer face of  $G$ . One can transform  $G^\sigma$  into the following toroidal map  $G^+$  (see Figure 6): Add a vertex  $v$  in the outer face of  $G$ . Add three non-parallel and non-contractible loops on  $v$ . Connect the three half edges leaving  $x_i$  to  $v$  such that there is no two such edge entering  $v$  consecutively. Then we have the following.

**Theorem 4** *The Schnyder woods of a planar map  $G$  rooted at  $x_0, x_1, x_2$  are in bijection with the Schnyder woods of the toroidal map  $G^+$ .*

*Proof.* ( $\implies$ ) Given a Schnyder wood of the planar graph  $G$ , rooted at  $x_0, x_1, x_2$ . Orient and color the graph  $G^+$  as in the example of Figure 6, i.e. the edges of the original graph  $G$  have the same color and orientation as in  $G^\sigma$ , the edge from  $x_i$  to  $v$  is colored  $i$  and leaving  $x_i$ , the three loops around  $v$  are colored and oriented appropriately so that  $v$  satisfies the Schnyder property. Then it is clear that all the vertices of  $G^+$  satisfy (T1). By (P2), we know that  $G^\sigma$  has no monochromatic cycles. All the edges between  $G$  and  $v$  are leaving  $G$ , so there is no monochromatic cycle of  $G^+$  involving vertices of  $G$ . Thus the only monochromatic cycles of  $G^+$  are the three loops around  $v$  and they satisfy (T2).



( $\Leftarrow$ ) Given a Schnyder wood of  $G^+$ , the restriction of the orientation and coloring to  $G$  and the three edges leaving  $v$  gives a Schnyder wood of  $G^\sigma$ . The three loops around  $v$  are three monochromatic cycles corresponding to three edges leaving  $v$ , thus they have different colors by (T1). Thus the three edges between  $G$  and  $v$  are entering  $v$  with three different colors. The three loops around  $v$  have to leave  $v$  in counterclockwise order 0, 1, 2 and we can assume by maybe permuting the colors that the edge leaving  $x_i$  is colored  $i$ . Clearly all the vertices of  $G^\sigma$  satisfies (P1). By Lemma 1, there is no contractible monochromatic cycles in  $G^+$ , so  $G^\sigma$  satisfies (P2).  $\square$

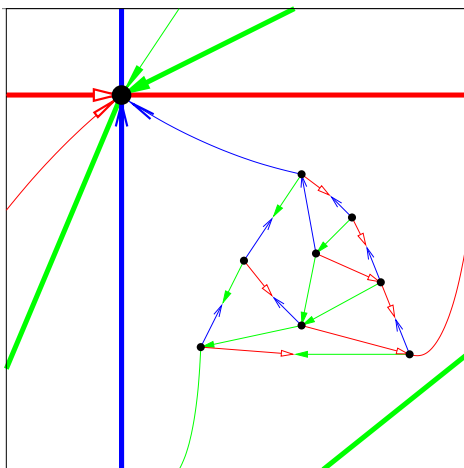


Figure 6: The toroidal Schnyder wood corresponding to the planar Schnyder wood of Figure 3.

A planar map  $G$  is *internally 3-connected* if there exists three vertices on the outer face such that the graph obtained from  $G$  by adding a vertex adjacent to the three vertices is 3-connected. Miller [20] (see also [8]) proved that a planar map admits a Schnyder wood if and only if it is internally 3-connected. The following results show that the notion of essentially 3-connected is the natural generalization of internally 3-connected to the torus.

**Theorem 5** *A planar map  $G$  is internally 3-connected if and only if there exists three vertices on the outer face of  $G$  such that  $G^+$  is an essentially 3-connected toroidal map.*

*Proof.* ( $\implies$ ) Let  $G$  be an internally 3-connected planar map. By definition, there exists three vertices  $x_0, x_1, x_2$  on the outer face such that the graph  $G'$  obtained from  $G$  by adding a vertex adjacent to these three vertices is 3-connected. Let  $G''$  be the graph obtained from  $G$  by adding three vertices  $y_0, y_1, y_2$  that form a triangle and by adding the three edges  $x_i y_i$ . It is not difficult to check that  $G''$  is 3-connected. Since  $G^\infty$  can be obtained from the (infinite) triangular grid, which is 3-connected, by gluing copies of  $G''$  along triangles,  $G^\infty$  is clearly 3-connected. Thus  $G^+$  is an essentially 3-connected toroidal map.

( $\Leftarrow$ ) Suppose there exists three vertices on the outer face of  $G$  such that  $G^+$  is an essentially 3-connected toroidal map, i.e.  $G^\infty$  is 3-connected. A copy of  $G$  is contained in a triangle  $y_0y_1y_2$  of  $G^\infty$ . Let  $G''$  be the subgraph of  $G^\infty$  induced by this copy plus the triangle, and let  $x_i$  be the unique neighbor of  $y_i$  in the copy of  $G$ . Since  $G''$  is connected to the rest of  $G^\infty$  by a triangle,  $G''$  is also 3-connected. Let us now prove that this implies that  $G$  is internally 3-connected for  $x_0, x_1$  and  $x_2$ . This is equivalent to say that the graph  $G'$ , obtained by adding a vertex  $z$  connected to  $x_0, x_1$  and  $x_2$ , is 3-connected. If  $G'$  had a separator  $\{a, b\}$  or  $\{a, z\}$ , with  $a, b \in V(G') \setminus \{z\}$ , then  $\{a, b\}$  or  $\{a, y_i\}$ , for some  $i \in [0, 2]$ , would be a separator of  $G''$ . This would contradict the 3-connectedness of  $G''$ . So  $G$  is internally 3-connected.  $\square$

### 3 Two different types of Schnyder woods

Two non contractible closed curves are *homotopic* if one can be continuously transformed into the other. The following are general useful lemmas on the torus.

**Lemma 2** *Let  $C_1, C_2$  be two non contractible closed curve on the Torus. If  $C_1, C_2$  are not homotopic, then their intersection is non empty.*

**Lemma 3** *Let  $C_1, C_2, C_3$  be three non contractible closed curve on the Torus. If  $C_1, C_2$  are homotopic and  $C_2, C_3$  are homotopic, then  $C_1, C_3$  are homotopic.*

**Lemma 4** *Let  $C_1, C_2, C_3$  be three non contractible closed curve on the Torus. If  $C_1, C_2$  are homotopic and  $C_1, C_3$  are not homotopic. Then  $C_2, C_3$  are not homotopic and thus their intersection is non empty.*

Two non contractible oriented closed curves on the torus are *fully-homotopic* if one can be continuously transformed into the other by preserving the orientation. We say that two monochromatic directed cycles  $C_i, C_j$  of different colors are *reversal* if one is obtained from the other by reversing all the edges ( $C_i = C_j^{-1}$ ). We say that two monochromatic cycle are *crossing* if they intersects but are not reversal. We define the *right side* of a  $i$ -cycle  $C_i$ , as the right side while “walking” along the directed cycle by following the orientation given by the edges colored  $i$ .

Let  $G$  be a toroidal graph given with a Schnyder wood.

**Lemma 5** *All  $i$ -cycles are non contractible, non intersecting and fully-homotopic.*

*Proof.* By Lemma 1, all  $i$ -cycles are non contractible. If there exists two such distinct  $i$ -cycles that are intersecting. Then there is a vertex that has two outgoing edge of color  $i$ , a contradiction to (T1). So the  $i$ -cycles are non intersecting. Then, by Lemma 2, they are homotopic.

Suppose that there exists two  $i$ -cycles  $C_i, C'_i$  that are not fully-homotopic. By the first part of the proof, cycles  $C_i, C'_i$  are non contractible, non intersecting and homotopic. Let  $R$  be the region between  $C_i$  and  $C'_i$  situated on the right of  $C_i$ . Suppose by symmetry that  $C_i^{-1}$  is not a  $(i+1)$ -cycle. By (T2), there exists a cycle  $C_{i+1}$  intersecting  $C_i$  and thus  $C_{i+1}$  is crossing  $C_i$ . By Property (T1),  $C_{i+1}$  is entering  $C_i$  from its right side and so it is leaving the region  $R$  when it crosses  $C_i$ . To enter the region  $R$ , the cycle  $C_{i+1}$  has to enter  $C_i$  or  $C'_i$  from their left side, a contradiction to property (T1).  $\square$

**Lemma 6** *If two monochromatic cycles are crossing then they are of different colors and they are not homotopic.*

*Proof.* By Lemma 5, two crossing monochromatic cycles are not of the same color. Suppose that there exists two monochromatic cycles  $C_{i-1}$  and  $C_{i+1}$ , of color  $i-1$  and  $i+1$ , that are crossing and homotopic. By Lemma 1, the cycles  $C_{i-1}$  and  $C_{i+1}$  are not contractible. Since  $C_{i-1} \neq C_{i+1}^{-1}$  and  $C_{i-1} \cap C_{i+1} \neq \emptyset$ , the cycle  $C_{i+1}$  is leaving  $C_{i-1}$ . It is leaving  $C_{i-1}$  on its right side by (T1). Since  $C_{i-1}$  and  $C_{i+1}$  are homotopic, the cycle  $C_{i+1}$  is entering  $C_{i-1}$  at least once from its right side. This is in contradiction with (T1).  $\square$

Let  $\mathcal{C}_i$  be the set of  $i$ -cycles of  $G$ . Let  $(\mathcal{C}_i)^{-1}$  denote the set of cycles obtained by reversing all the cycles of  $\mathcal{C}_i$ . By Lemma 5, the cycles of  $\mathcal{C}_i$  are non contractible, non intersecting and fully-homotopic. So we can order them as follow  $\mathcal{C}_i = \{C_i^0, \dots, C_i^{k_i-1}\}$ ,  $k_i \geq 1$ , such that, for  $0 \leq j \leq k_i - 1$ , there is no  $i$ -cycle in the region  $R(C_i^j, C_i^{j+1})$  between  $C_i^j$  and  $C_i^{j+1}$  containing the right side of  $C_i^j$  (superscript understood modulo  $k_i$ ).

We show that Schnyder woods are of two different types (see figure 7):

**Theorem 6** *Let  $G$  be a toroidal graph given with a Schnyder wood, then all  $i$ -cycles are non contractible, non intersecting and fully-homotopic and either:*

- *For every pair of two monochromatic cycles  $C_i, C_j$  of different colors  $i, j$ , the two cycles  $C_i$  and  $C_j$  are not homotopic and thus intersect (We say the Schnyder wood is of Type 1).*

or

- *There exists a color  $i$  such that  $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$  and for any pair of monochromatic cycles  $C_i, C_j$  of colors  $i, j$ , with  $j \neq i$ , the two cycles  $C_i$  and  $C_j$  are not homotopic and thus intersect (We say the Schnyder wood is of Type 2, or Type 2.i if we want to specify the color  $i$ ).*

*Moreover, if  $G$  is a toroidal triangulation, then there is no edges oriented in two directions and the Schnyder wood is of Type 1.*

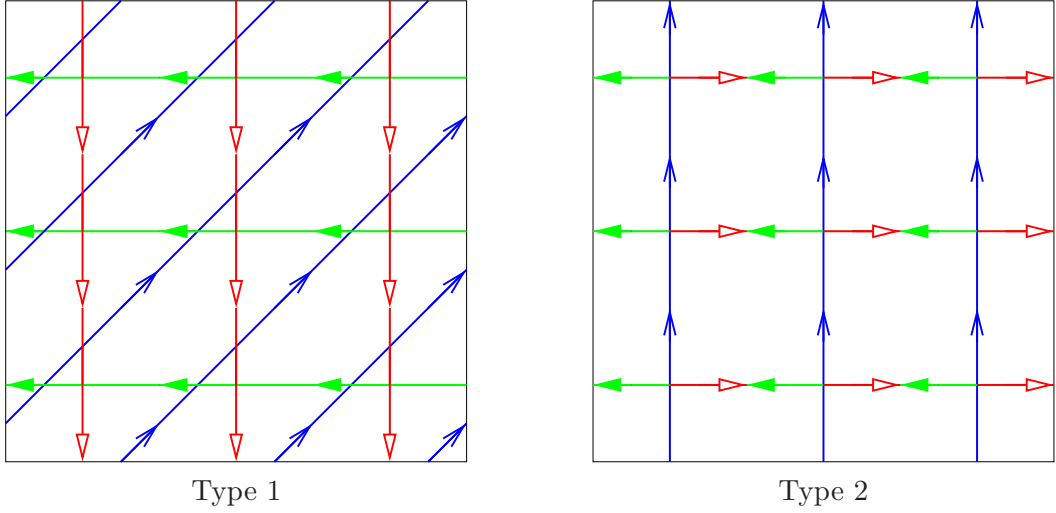


Figure 7: The two types of Schnyder woods on toroidal graphs

*Proof.* By Lemma 5, all  $i$ -cycles are non contractible, non intersecting and fully-homotopic. Suppose that there exists a  $(i - 1)$ -cycle  $C_{i-1}$  and a  $(i + 1)$ -cycle  $C_{i+1}$  that are homotopic, i.e.  $C_{i-1} = C_{i+1}^{-1}$ . We prove that the Schnyder wood is of Type 2.i. We first prove that  $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$ . Let  $C'_{i-1}$  be any  $(i - 1)$ -cycle. By (T2),  $C'_{i-1}$  intersects a  $(i + 1)$ -cycle  $C'_{i+1}$ . By Lemma 5,  $C'_{i-1}$  (resp.  $C'_{i+1}$ ) is homotopic to  $C_{i-1}$  (resp.  $C_{i+1}$ ). So, by Lemma 3,  $C'_{i-1}$  and  $C'_{i+1}$  are homotopic. By Lemma 6,  $C'_{i-1}$  and  $C'_{i+1}$  are reversal. Thus  $\mathcal{C}_{i-1} \subseteq (\mathcal{C}_{i+1})^{-1}$  and so by symmetry  $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$ . Now we prove that for any pair of monochromatic cycles  $C'_i, C'_j$  of colors  $i, j$ , with  $j \neq i$ , the two cycles  $C'_i$  and  $C'_j$  are not homotopic. By (T2),  $C'_j$  intersects a  $i$ -cycle  $C_i$ . Since  $\mathcal{C}_{i-1} = (\mathcal{C}_{i+1})^{-1}$ , cycle  $C'_j$  is bi-oriented in color  $i - 1$  and  $i + 1$ , thus we cannot have  $C'_j = C_i^{-1}$ . So  $C'_j$  and  $C_i$  are crossing and by Lemma 6, they are not homotopic. By Lemma 5,  $C'_i$  and  $C_i$  are homotopic. Thus, by Lemma 4,  $C'_j$  and  $C'_i$  are not homotopic. Thus the Schnyder wood is of Type 2.i.

If there is no two monochromatic cycles of different colors that are homotopic, then the Schnyder wood is of Type 1.

For toroidal triangulation,  $m = 3n$  by Euler's formula, so there is no edges oriented in two directions, so only Type 1 is possible.  $\square$

Note that in a Schnyder wood of Type 1, we may have edges that are in two monochromatic cycles of different colors (see Figure 2).

We do not know if the set of Schnyder woods of a given toroidal graph has a kind of lattice structure like in the planar case [10]. De Fraysseix et al. [15] proved that Schnyder woods of a planar triangulation are in one-to-one correspondence with orientation of the edges of the graph where each inner vertex has out-degree three. It is possible to retrieve the coloring of the edges of a Schnyder wood from the orientation. The situation is

different for toroidal triangulations. There exists orientations of toroidal triangulations where each vertex has out-degree three but there is no corresponding Schnyder wood. For example, if one consider a toroidal triangulation with just one vertex, the orientations of edges that satisfies (T1) are the orientations where there is no three consecutive edges leaving the vertex (see Figure 8).

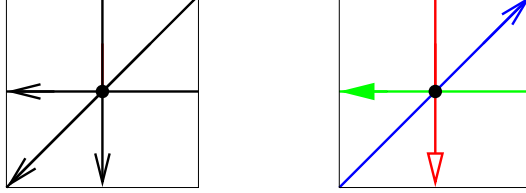


Figure 8: Two different orientations of a toroidal triangulation. Only the second one corresponds to a Schnyder wood.

## 4 Schnyder woods in the universal cover

Let  $G$  be a toroidal graph given with a Schnyder wood. Consider the orientation and coloring of the edges of  $G^\infty$  that corresponds to the Schnyder wood of  $G$ .

**Lemma 7** *The orientation and coloring of the edges of  $G^\infty$  satisfies the following:*

- (U1) *Every vertex of  $G^\infty$  verifies the Schnyder property*
- (U2) *There is no monochromatic cycle in  $G^\infty$ .*

*Proof.* Clearly, (U1) is satisfied. Now we prove (U2). Suppose by contradiction that there is a monochromatic cycle  $U$  of color  $i$  in  $G^\infty$ . Let  $C$  be the closed curve of  $G$  corresponding to edges of  $U$ . If  $C$  self intersects, then there is a vertex of  $G$  with two edges leaving  $v$  in color  $i$ , a contradiction to (T1). So  $C$  is a monochromatic cycle of  $G$ . Since  $C$  corresponds to a cycle of  $G^\infty$ , it is a contractible cycle of  $G$ , a contradiction to Lemma 1.  $\square$

One can remark that properties (U1) and (U2) are the same as in the definition of Schnyder wood for 3-connected planar graphs (properties (P1) and (P2)). Note that if the orientation and coloring of the edges of  $G^\infty$ , corresponding to an orientation and coloring of the edges of  $G$ , satisfies properties (U1) and (U2), we do not necessarily have a Schnyder wood of  $G$ . For example the graph  $G^\infty$  obtained by replicating the graph  $G$  of Figure 4 satisfies (U1) and (U2) whereas the orientation and coloring of  $G$  is not a Schnyder wood as (T2) is not satisfied.

Recall that the notation  $\mathcal{C}_i = \{C_i^0, \dots, C_i^{k_i-1}\}$  denotes the set of  $i$ -cycles of  $G$  such that there is no  $i$ -cycle in the region  $R(C_i^j, C_i^{j+1})$ . As monochromatic cycles are not

contractible by Lemma 1, a directed monochromatic cycles  $C_i^j$  corresponds to a family of infinite directed monochromatic paths of  $G^\infty$  (infinite in both directions of the path). This family is denoted  $\mathcal{L}_i^j$ . Each element of  $\mathcal{L}_i^j$  is called a *monochromatic line* of color  $i$ , or  *$i$ -line* for short. By Lemma 5, all  $i$ -lines are non intersecting and oriented in the same direction. Given any two  $i$ -lines  $L, L'$ , the unbounded region between  $L$  and  $L'$  is noted  $R(L, L')$ . We say that two  $i$ -lines  $L, L'$  are *consecutive* if there is no  $i$ -lines contained in  $R(L, L')$ .

Let  $v$  be a vertex of  $G^\infty$ . For each color  $i$ , vertex  $v$  is the starting vertex of a unique infinite directed monochromatic path of color  $i$ , denoted  $P_i(v)$ . Indeed this is a path since there is no monochromatic cycle in  $G^\infty$  by Property (U2), and it is infinite (in one direction of the path only) because every reached vertex of  $G^\infty$  has exactly one edge leaving in color  $i$  by Property (U1). As  $P_i(v)$  is infinite, it necessarily contains two vertices  $u, u'$  of  $G^\infty$  that are copies of the same vertex of  $G$ . The subpath of  $P_i(v)$  between  $u$  and  $u'$  corresponds to a  $i$ -cycle of  $G$  and thus is part of a  $i$ -line of  $G^\infty$ . Let  $L_i(v)$  be the  $i$ -line intersecting  $P_i(v)$ .

**Lemma 8** *The graph  $G_i^\infty \cup (G_{i-1}^\infty)^{-1} \cup (G_{i+1}^\infty)^{-1}$  contains no directed cycle.*

*Proof.* Suppose there is a contractible directed cycle  $C$  in  $G_i^\infty \cup (G_{i-1}^\infty)^{-1} \cup (G_{i+1}^\infty)^{-1}$ . Let  $D$  be the closed disk bounded by  $C$ . Suppose by symmetry that  $C$  turns around  $D$  clockwise. Then, by (U1), there is no edge of color  $i-1$  leaving the closed disk  $D$ . So there is a  $(i-1)$ -cycle in  $D$ , a contradiction to (U2).  $\square$

**Lemma 9** *For every vertex  $v$  and color  $i$ , the two paths  $P_{i-1}(v)$  and  $P_{i+1}(v)$  have  $v$  as only common vertex.*

*Proof.* If  $P_{i-1}(v)$  and  $P_{i+1}(v)$  intersect on two vertices, then  $G_{i-1}^\infty \cup (G_{i+1}^\infty)^{-1}$  contains a cycle, contradicting Lemma 8.  $\square$

By Lemma 9, for every vertex  $v$ , the three paths  $P_0(v), P_1(v), P_2(v)$  divide  $G^\infty$  into three unbounded regions  $R_0(v), R_1(v)$  and  $R_2(v)$ , where  $R_i(v)$  denotes the region delimited by the two paths  $P_{i-1}(v)$  and  $P_{i+1}(v)$ . Let  $R_i^\circ(v) = R_i(v) \setminus (P_{i-1}(v) \cup P_{i+1}(v))$ .

**Lemma 10** *For all distinct vertices  $u, v$ , we have:*

- (i) *If  $u \in R_i(v)$ , then  $R_i(u) \subseteq R_i(v)$ .*
- (ii) *If  $u \in R_i^\circ(v)$ , then  $R_i(u) \subsetneq R_i(v)$ .*
- (iii) *There exists  $i$  and  $j$  with  $R_i(u) \subsetneq R_i(v)$  and  $R_j(v) \subsetneq R_j(u)$ .*

*Proof.* (i) Suppose by symmetry that the Schnyder wood is not of Type 2.(i+1). Then in  $G$ ,  $i$ -cycles are not homotopic to  $(i-1)$ -cycles. Thus in  $G^\infty$ , every  $i$ -line crosses every

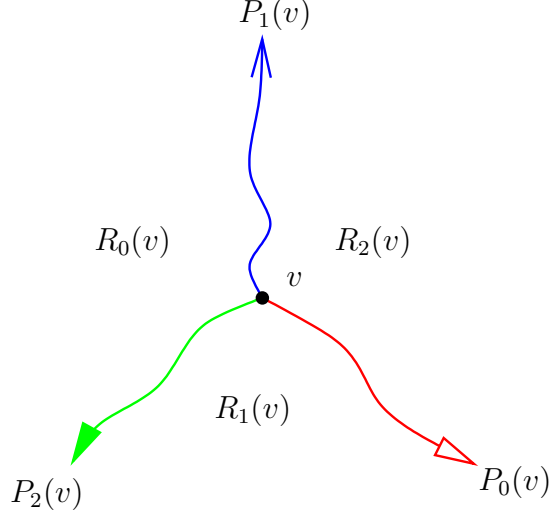


Figure 9: Regions corresponding to a vertex

$(i - 1)$ -line. Moreover a  $i$ -line crosses a  $(i - 1)$ -line exactly once and from its right side to its left side by (U1). Vertex  $v$  is between two consecutive monochromatic  $(i - 1)$ -lines  $L_{i-1}, L'_{i-1}$ , with  $L'_{i-1}$  situated on the right of  $L_{i-1}$ . Let  $R$  be the region situated on the right of  $L_{i-1}$ , so  $v \in R$ .

- (1) For any vertex  $w$  of  $R$ , the path  $P_i(w)$  leaves the region  $R$ .

The  $i$ -line  $L_i(w)$  has to cross  $L_{i-1}$  exactly once and from right to left, thus  $P_i(w)$  leaves the region  $R$ . This proves claim (1).  $\diamond$

The path  $P_{i+1}(v)$  cannot leave the region  $R$  as this would contradict (U1). Thus by Claim 1 for  $w = v$ , we have  $R_i(v) \subseteq R$  and so  $u \in R$ . Moreover the paths  $P_{i-1}(u)$  and  $P_{i+1}(u)$  cannot leave region  $R_i(v)$  as this would contradict (U1). Thus by Claim 1 for  $w = u$ , the path  $P_i(u)$  leaves the region  $R_i(v)$  and so  $R_i(u) \subseteq R_i(v)$ .

(ii) By (i),  $R_i(u) \subseteq R_i(v)$ , so the paths  $P_{i-1}(u)$  and  $P_{i+1}(u)$  are contained in  $R_i(v)$ . Then none of them can contain  $v$  as this would contradict (U1). So all the faces of  $R_i(v)$  incident to  $v$  are not in  $R_i(u)$  (and there is at least one such face).

(iii) By symmetry, we prove that there exists  $i$  with  $R_i(u) \subsetneq R_i(v)$ . If  $u \in R_i^\circ(v)$  for some color  $i$ , then  $R_i(u) \subsetneq R_i(v)$  by (ii). Suppose now that  $u \in P_i(v)$  for some  $i$ . By Lemma 9, at least one of the two paths  $P_{i-1}(u)$  and  $P_{i+1}(u)$  does not contain  $v$ . Suppose by symmetry that  $P_{i-1}(u)$  does not contain  $v$ . As  $u \in P_i(v) \subseteq R_{i+1}(v)$ , we have  $R_{i+1}(u) \subseteq R_{i+1}(v)$  by (i), and as none of  $P_{i-1}(u)$  and  $P_i(u)$  contains  $v$ , we have  $R_{i+1}(u) \subsetneq R_{i+1}(v)$ .  $\square$

**Lemma 11** *If a toroidal graph  $G$  admits a Schnyder wood, then  $G$  is essentially 3-connected.*

*Proof.* Let  $u, v, x, y$  be any four distinct vertices of  $G^\infty$ . Let us prove that there exists a path between  $u$  and  $v$  in  $G^\infty \setminus \{x, y\}$ . Suppose by symmetry, that the Schnyder wood is of Type 1 or Type 2.1. Then the monochromatic lines of color 0 and 2 form a kind of grid, i.e. the 0-lines intersect all the 2-lines. Let  $L_0, L'_0$  be 0-lines and  $L_2, L'_2$  be 2-lines, such that  $u, v, x, y$  are all in the interior of the bounded region  $R(L_0, L'_0) \cap R(L_2, L'_2)$ .

By Lemma 9, the three paths  $P_i(v)$ , for  $0 \leq i \leq 2$ , are disjoint except on  $v$ . Thus there exists  $i$ , such that  $P_i(v) \cap \{x, y\} = \emptyset$ . Similarly there exists  $j$ , such that  $P_j(u) \cap \{x, y\} = \emptyset$ . The two paths  $P_i(v)$  and  $P_j(u)$  are infinite, so they intersect the boundary of  $R(L_0, L'_0) \cap R(L_2, L'_2)$ . Thus  $P_i(v) \cup P_j(u) \cup L_0 \cup L'_0 \cup L_2 \cup L'_2$  contains a path from  $u$  to  $v$  in  $G^\infty \setminus \{x, y\}$ .  $\square$

By Lemma 11, if  $G$  admits a Schnyder wood, then it is essentially 3-connected, so it is a map and each face is a disk.

Note that if (T2) is not required in the definition of Schnyder wood, then Lemma 11 is false. Figure 10 gives an example of an orientation and coloring of the edges of a toroidal graph satisfying (T1), such that there is no contractible monochromatic cycles but where (T2) is not satisfied as there is a 0-cycle not intersecting any 2-cycle. This graph is not essentially 3-connected, indeed  $G^\infty$  is not connected.

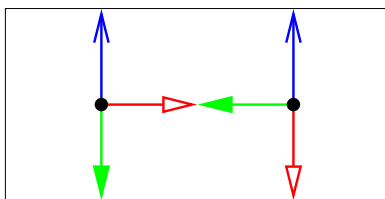


Figure 10: An orientation and coloring of the edges of a toroidal graph satisfying (T1) but that is not essentially 3-connected.

## 5 Duality of Schnyder woods

Given a planar map  $G$ , and  $x_0, x_1, x_2$  three distinct vertices occurring in counterclockwise order on the outer face of  $G$ . A *Schnyder angle labeling* [8] of  $G$  with respect to  $x_0, x_1, x_2$  is a labeling of the angles of  $G^\sigma$  satisfying the following:

- (L1) The label of the angles at each vertex form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. The two angles at the half-edge at  $x_i$  have labels  $i + 1$  and  $i - 1$
- (L2) The label of the angles at each inner face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. At the outer-face the same is true in clockwise order.



Felsner [9] proved that, for planar maps, Schnyder woods are in bijection with Schnyder angle labellings. In the toroidal case, we do not see a simple definition of Schnyder angle labeling that would be equivalent to our definition of Schnyder woods. This is due to the fact that contrarily to (P2) that is local and can be verified just by considering faces, (T2) is global. Nevertheless we have one implication.

The *angle labeling corresponding to* a Schnyder wood of a toroidal map  $G$  is a labeling of the angles of  $G$  such that the angles at a vertex  $v$  in the counterclockwise sector between  $e_{i+1}(v)$  and  $e_{i-1}(v)$  are labeled  $i$  (see Figure 11).

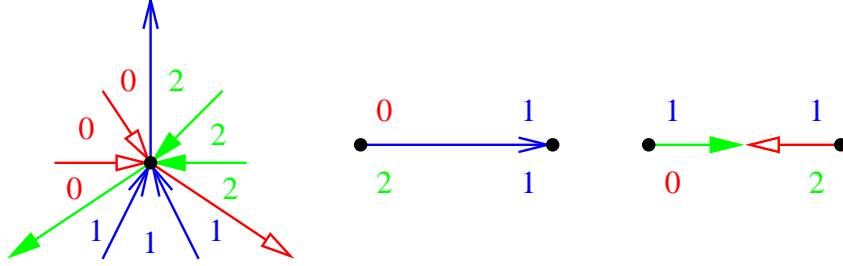


Figure 11: Angle labeling around vertices and edges

**Lemma 12** *The angle labeling corresponding to a Schnyder wood of a toroidal map satisfies the following: the angles at each vertex and at each face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's.*

*Proof.* Clearly the property is true at each vertex by (T1). To prove that the property is true at each faces we count the number of color changes around vertices, faces and edges. This number of changes is denoted  $d$ . For a vertex  $v$  there are exactly three changes, so  $d(v) = 3$  (see Figure 11). For an edge  $e$ , that can be either oriented in one or two direction, there are also exactly three changes, so  $d(e) = 3$  (see Figure 11). Now consider a face  $F$ . Suppose we cycle counterclockwise around  $F$ , then an angle colored  $i$  is always followed by an angle colored  $i$  or  $i + 1$ . Consequently,  $d(F)$  must be a multiple of three. Suppose that  $d(F) = 0$ , then all its angles are colored with one color  $i$ . In that case the cycle around face  $F$  would be completely oriented in counterclockwise order in color  $i + 1$  (and in clockwise order in color  $i - 1$ ). This cycle being contractible this would contradict Lemma 1. So  $d(F) \geq 3$ .

The sum of the changes around edges must be equal to the sum of the changes around faces and vertices. Thus  $3m = \sum_e d(e) = \sum_v d(v) + \sum_F d(F) = 3n + \sum_F d(F)$ . Euler's formula gives  $m = n + f$ , so  $\sum_F d(F) = 3f$  and this is possible only if  $d(F) = 3$  for every face  $F$ .  $\square$

There is no converse to Lemma 12. Figure 4 gives an example of a coloring and orientation of the edges of a toroidal triangulation not satisfying (T2) but where the angles at each vertex and at each face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's.

Let  $G$  be a toroidal graph given with a Schnyder wood. By Lemma 11,  $G$  is an essentially 3-connected toroidal map, thus the dual  $G^*$  of  $G$  has no contractible loop and no homotopic multiple edges. Let  $\tilde{G}$  be a simultaneous drawing of  $G$  and  $G^*$  such that only dual edges intersect.

The *dual* of the Schnyder wood is the orientation and coloring of the edges of  $G^*$  obtained by the following method (see Figure 12 and 13): Let  $e$  be an edge of  $G$  and  $e^*$  the dual edge of  $e$ . If  $e$  is oriented in one direction only and colored  $i$ , then  $e^*$  is oriented in two direction, entering  $e$  from the right side in color  $i - 1$  and from the left side in color  $i + 1$  (the right side of  $e$  is the right side while following the orientation of  $e$ ). Symmetrically, if  $e$  is oriented in two direction in color  $i + 1$  and  $i - 1$ , then  $e^*$  is oriented in one direction only and colored  $i$  such that  $e$  is entering  $e^*$  from its right side in color  $i - 1$ .

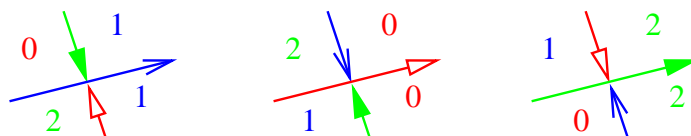


Figure 12: Rules for the dual Schnyder wood and angle labeling.

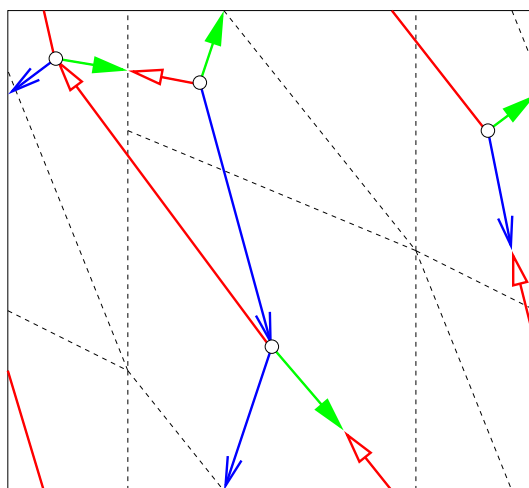


Figure 13: Dual Schnyder wood of the Schnyder wood of Figure 2.

**Lemma 13** *Let  $G$  be a toroidal map. The dual of a Schnyder wood of a toroidal map  $G$  is a Schnyder wood of the dual  $G^*$ . Moreover we have :*

- (i) *On the simultaneous drawing  $\tilde{G}$  of  $G$  and  $G^*$ , the  $i$ -cycles of the dual Schnyder wood are homotopic to the  $i$ -cycles of the primal Schnyder wood and oriented in opposite direction.*

(ii) *The dual of a Schnyder wood is of Type 2.i if and only if the primal Schnyder wood is of Type 2.i.*

*Proof.* In every face of  $\tilde{G}$ , there is exactly one angle of  $G$  and one angle of  $G^*$ . Thus a Schnyder angle labeling of  $G$  corresponds to an angle labeling of  $G^*$ . The dual of the Schnyder wood is defined such that an edge  $e$  is leaving  $F$  in color  $i$  if and only if the angle at  $F$  on the left of  $e$  is labeled  $i - 1$  and the angle at  $F$  on the right of  $e$  is labeled  $i + 1$ , and such that an edge  $e$  is entering  $F$  in color  $i$  if and only if at least one of the angles at  $F$  incident to  $e$  is labeled  $i$  (see Figure 12). By Lemma 12, the angles at a face form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. Thus the edges around a vertex of  $G^*$  satisfy property (T1).

Consider  $\tilde{G}$  with the orientation and coloring of primal and dual edges.

Let  $C$  be a  $i$ -cycle of  $G^*$ . Suppose, by contradiction, that  $C$  is contractible. Let  $D$  be the disk delimited by  $C$ . Suppose by symmetry that  $C$  is going anticlockwise around  $D$ . Then all the edges of  $G$  that are dual to edges of  $C$  are entering  $D$  in color  $i - 1$ . Thus  $D$  contains a  $i - 1$ -cycle of  $G$ , a contradiction to Lemma 1. Thus every monochromatic cycle of  $G^*$  is non contractible.

The dual of the Schnyder wood is defined in such a way that an edge of  $G$  and an edge of  $G^*$  of the same color never intersect in  $\tilde{G}$ . Thus the  $i$ -cycles of  $G^*$  are homotopic to  $i$ -cycles of  $G$ . Consider a  $i$ -cycle  $C_i$  (resp.  $C_i^*$ ) of  $G$  (resp.  $G^*$ ). The two cycles  $C_i$  and  $C_i^*$  are homotopic. By symmetry, we assume that the primal Schnyder wood is not of Type 2.(i-1). Let  $C_{i+1}$  be a  $(i + 1)$ -cycle of  $G$ . The two cycles  $C_i$  and  $C_{i+1}$  are not homotopic and  $C_i$  is entering  $C_{i+1}$  on its left side. Thus, by Lemma 4, the two cycles  $C_i^*$  and  $C_{i+1}$  are not homotopic and by the dual rules  $C_i^*$  is entering  $C_{i+1}$  on its right side. So  $C_i$  and  $C_i^*$  are homotopic and going in opposite directions.

Suppose the Schnyder wood of  $G$  is of Type 1. Then two monochromatic cycles of  $G$  of different colors are not homotopic. Thus the same is true for monochromatic cycles of the dual. So (T2) is satisfied and the dual of the Schnyder wood is a Schnyder wood of Type 1.

Suppose now that the Schnyder wood of  $G$  is of Type 2. Assume by symmetry that it is of Type 2.i. Then all monochromatic cycles of color  $i$  and  $j$ , with  $j \in \{i - 1, i + 1\}$ , intersect. Now suppose, by contradiction, that there is a  $j$ -cycle  $C^*$ , with  $j \in \{i - 1, i + 1\}$ , that is not equal to a monochromatic cycle of color in  $\{i - 1, i + 1\} \setminus \{j\}$ . By symmetry we can assume that  $C^*$  is of color  $i - 1$ . Let  $C$  be the  $(i - 1)$ -cycle of the primal that is the first on the right side of  $C^*$  in  $\tilde{G}$ . By definition of Type 2.i,  $C^{-1}$  is a  $(i + 1)$ -cycle of  $G$ . Let  $R$  be the region delimited by  $C^*$  and  $C$  situated on the right side of  $C^*$ . Cycle  $C^*$  is not a  $(i + 1)$ -cycle so there is at least one edge of color  $i + 1$  leaving a vertex of  $C^*$ . By (T1) in the dual, this edge is entering the interior of the region  $R$ . An edge of  $G^*$  of color  $i + 1$  cannot intersect  $C$  and cannot enter  $C^*$  from its right side. So in the interior of the region  $R$  there is at least one  $(i + 1)$ -cycle  $C_{i+1}^*$  of  $G^*$ . Cycle  $C_{i+1}^*$  is homotopic to  $C^*$  and going in opposite direction (i.e.  $C_{i+1}^*$  and  $C^*$  are not fully-homotopic). If  $C_{i+1}^*$  is not a  $(i + 1)$ -cycle, then we can define  $R' \subsetneq R$  the region delimited by  $C_{i+1}^*$  and  $C$

situated on the left side of  $C_{i+1}^*$  and as before we can prove that there is a  $(i-1)$ -cycle of  $G^*$  in the interior of  $R'$ . So in any case, there is a  $(i-1)$ -cycle  $C_{i-1}^*$  of  $G^*$  in the interior of  $R$  and  $C_{i-1}^*$  is fully-homotopic to  $C^*$ . Let  $R'' \subsetneq R$  be the region delimited by  $C^*$  and  $C_{i-1}^*$  situated on the right side of  $C^*$ . Clearly  $R''$  does not contain  $C$ . Thus by definition of  $C$ , the region  $R''$  does not contain any  $(i-1)$ -cycle of  $G$ . But  $R''$  is non empty and contains at least one vertex  $v$  of  $G$ . The path  $P_{i-1}(v)$  cannot leave  $R''$ , a contradiction. So (T2) is satisfied and the dual Schnyder wood is of Type 2.i.  $\square$

By Lemma 13, we have the following:

**Theorem 7** *There is a bijection between Schnyder woods of a toroidal map and Schnyder woods of its dual.*

## 6 Relaxing the definition

In the plane, the proof of existence of Schnyder woods can be done without too much difficulty as the properties to be satisfied are only local. In the toroidal case, things are much more complicated as property (T2) is global. The following lemma shows that property (T2) can be relaxed a bit in the case of Schnyder woods of Type 1.

**Lemma 14** *Let  $G$  be a toroidal graph given with an orientation and coloring of the edges of  $G$  with the colors 0, 1, 2, where every edge  $e$  is oriented in one direction or in two opposite directions. The orientation and coloring is a Schnyder wood of Type 1 if and only if it satisfies the following:*

(T1') *Every vertex  $v$  satisfies the Schnyder property.*

(T2') *For each pair  $i, j$  of different colors, there exists a  $i$ -cycle intersecting a  $j$ -cycle.*

(T3') *There is no monochromatic cycles  $C_i, C_j$  of different colors  $i, j$  such that  $C_i = C_j^{-1}$ .*

*Proof.* ( $\implies$ ) If we have a Schnyder wood of Type 1, then Property (T1') is satisfied as it is equal to Property (T1). Property (T1) implies that there always exist a monochromatic cycles of each color, thus Property (T2') is a relaxation of (T2). Property (T3') is implied by definition of Type 1 (see Theorem 6).

( $\impliedby$ ) Conversely, suppose we have an orientation and coloring satisfying (T1'), (T2'), (T3'). We prove several properties.

(2) *All  $i$ -cycles are non contractible, non intersecting and homotopic.*

Suppose there is a contractible monochromatic cycle. Let  $C$  be such a cycle containing the minimum number of faces in the closed disk  $D$  bounded by  $C$ . Suppose by symmetry

that  $C$  turns around  $D$  clockwise. Let  $i$  be the color of  $C$ . Then, by (T1'), there is no edge of color  $i - 1$  leaving the closed disk  $D$ . So there is a  $(i - 1)$ -cycle in  $D$  and this cycle is  $C$  by minimality of  $C$ , a contradiction to (T3').

If there exists two distinct  $i$ -cycles that are intersecting. Then there is vertex that has two outgoing edge of color  $i$ , a contradiction to (T1'). So the  $i$ -cycles are non intersecting. Then, by Lemma 2, they are homotopic. This proves claim (2).  $\diamond$

(3) *If two monochromatic cycles are intersecting then they are not homotopic.*

Suppose by contradiction that there exists  $C, C'$  two distinct directed monochromatic cycles that are homotopic and intersecting. By Claim (2), they are not contractible and of different color. Suppose  $C$  is a  $(i - 1)$ -cycle and  $C'$  a  $(i + 1)$ -cycle. By (T1'),  $C'$  is leaving  $C$  on its right side. Since  $C, C'$  are homotopic, the cycle  $C'$  is entering  $C$  at least once from its right side, a contradiction with (T1'). This proves claim (3).  $\diamond$

We are now able to prove that (T2) is satisfied. Let  $C_i$  be any  $i$ -cycle of color  $i$ . We have to prove that  $C_i$  intersects at least one  $(i - 1)$ -cycle and at least one  $(i + 1)$ -cycle. Let  $j$  be either  $i - 1$  or  $i + 1$ . By (T2'), there exists a  $i$ -cycle  $C'_i$  intersecting a  $j$ -cycle  $C'_j$  of color  $j$ . The two cycles  $C'_i, C'_j$  are not reversal by (T3'), thus they are crossing. By Claim (3),  $C'_i$  and  $C'_j$  are not homotopic. By Claim (2),  $C_i$  and  $C'_i$  are homotopic. Thus by Lemma 4,  $C_i$  and  $C'_j$  are intersecting.

Thus (T1) and (T2) are satisfied and thus the orientation and coloring is a Schnyder wood. By (T3') and Theorem 6 it is a Schnyder of Type 1.  $\square$

Note that for toroidal triangulations, there is no edges oriented in two directions in an orientation and coloring of the edges satisfying (T1'), by Euler's formula. So (T3') is automatically satisfied. Thus in the case of toroidal triangulations it is sufficient to have properties (T1') and (T2') to have a Schnyder wood. This is not true in general as shown by the example of Figure 14 that satisfies (T1') and (T2') but that is not a Schnyder wood. There is a monochromatic cycle of color 1 that is not intersecting any monochromatic cycle of color 2 so (T2) is not satisfied.

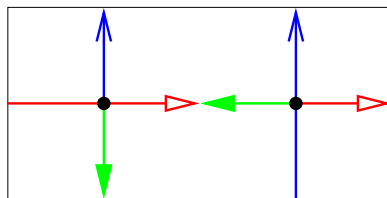


Figure 14: An orientation and coloring of the edges of toroidal graph satisfying (T1') and (T2') but that is not a Schnyder wood.

## 7 Existence for simple triangulations

In this section we present a short proof of existence of Schnyder woods for simple triangulations. Sections 8 and 9 contain the full proof of existence for essentially 3-connected toroidal maps.

Fijavz [14] proved a useful result concerning existence of particular non homotopic cycles in toroidal triangulations with no loop and no multiple edges. (Recall that in this paper we are less restrictive as we allow non contractible loops and non homotopic multiple edges.)

**Theorem 8 ([14])** *A simple toroidal triangulation contains three non contractible and non homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.*

Theorem 8 is not true for all toroidal triangulations as shown by the example on Figure 15.

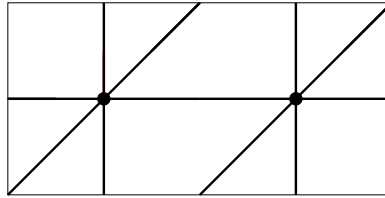


Figure 15: A toroidal triangulation that does not contain three non contractible and non homotopic cycles that all intersect on one vertex and that are pairwise disjoint otherwise.

Theorem 8 can be used to prove existence of particular Schnyder woods for simple toroidal triangulations.

**Theorem 9** *A simple toroidal triangulation admits a Schnyder wood with three monochromatic cycles of different colors all intersecting on one vertex and that are pairwise disjoint otherwise.*

*Proof.* Let  $G$  be a simple toroidal triangulation. By Theorem 8, let  $C_0, C_1, C_2$  be three non contractible and non homotopic cycles of  $G$  that all intersect on one vertex  $x$  and that are pairwise disjoint otherwise. By eventually shortening the cycles  $C_i$ , we can assume that the three cycles  $C_i$  are homotopically chordless (i.e. there is no edge between two vertices of  $C_i$  that can be continuously transformed into a part of  $C_i$ ). By symmetry, we can assume that the six edges  $e_i, e'_i$  of the cycles  $C_i$  incident to  $x$  appear around  $x$  in the counterclockwise order  $e_0, e'_2, e_1, e'_0, e_2, e'_1$ . The cycles  $C_i$  divide  $G$  into two regions, denoted  $R_1, R_2$  such that  $R_1$  is the region situated in the counterclockwise sector between  $e_0$  and  $e'_2$  of  $x$  and  $R_2$  is the region situated in the counterclockwise sector between  $e'_2$  and  $e_1$  of  $x$ . Let  $G_i$  be the subgraph of  $G$  contained in the region

$R_i$  (including the three cycles  $C_i$ ). Let  $G'_1$  (resp.  $G'_2$ ) be the graph obtained from  $G_1$  (resp.  $G_2$ ) by replacing  $x$  by three vertices  $x_1, x_2, x_3$ , such that  $x_i$  is incident to the edges in the counterclockwise sector between  $e_{i+1}$  and  $e'_i$  (resp.  $e'_i$  and  $e_{i-1}$ ). The cycles  $C_i$  being homotopically chordless, the two graphs  $G'_1$  and  $G'_2$  are internally 3-connected planar maps (for vertices  $x_i$  chosen on their outer face). The vertices  $x_0, x_1, x_2$  appear in counterclockwise order on the outer face of  $G'_1$  and  $G'_2$ . By a result of Miller [20] (see also [8, 9]), the two graphs  $G'_i$  admit planar Schnyder woods rooted at  $x_0, x_1, x_2$ . Orient and color the edges of  $G$  that intersect the interior of  $R_i$  by giving them the same orientation and coloring as in a planar Schnyder wood of  $G'_i$ . Orient and color the cycle  $C_i$  in color  $i$  such that it is entering  $x$  by edge  $e'_i$  and leaving  $x$  by edge  $e_i$ . We claim that the orientation and coloring that is obtained is a toroidal Schnyder wood of  $G$ .

Clearly, any interior vertex of the region  $R_i$  satisfies (T1). Let us show that (T1) is also satisfied for any vertex  $v$  of a cycle  $C_i$  distinct from  $x$ . In a Schnyder wood of  $G'_1$ , the cycle  $C_i$  is oriented in two direction, from  $x_{i-1}$  to  $x_i$  in color  $i$  and from  $x_i$  to  $x_{i-1}$  in color  $i-1$ . Thus the edge leaving  $v$  in color  $i+1$  is an inner edge of  $G'_1$  and vertex  $v$  has no edges entering in color  $i+1$ . Symmetrically, in  $G'_2$  the edge leaving  $v$  in color  $i-1$  is an inner edge of  $G'_2$  and vertex  $v$  has no edges entering in color  $i-1$ . Then one can paste  $G'_1$  and  $G'_2$  along  $C_i$ , orient  $C_i$  in color  $i$  and see that  $v$  satisfies (T1). The definition of the  $G'_i$ , and the orientation of the cycles is done so that  $x$  satisfies (T1). The cycles  $C_i$  being pairwise intersecting, (T2') is satisfied, so by Lemma 14, the orientation and coloring is a Schnyder wood.  $\square$

Note that in the Schnyder wood obtained by Theorem 9, we do not know if there are several monochromatic cycles of one color or not. So given any three monochromatic cycles of different color, they might not all intersect on one vertex. But for any two monochromatic cycles of different color, we know that they intersect exactly once. We wonder whether Theorem 9 can be modified as follow : Does a simple toroidal triangulation admits a Schnyder wood such that there is just one monochromatic cycle per color ? Moreover can one require that the monochromatic cycles of different colors pairwise intersect exactly once ? Or like in Theorem 9 that they all intersect on one vertex and that they are pairwise disjoint otherwise ?

## 8 The contraction Lemma

We prove existence of Schnyder wood for essentially 3-connected toroidal maps by contracting edges until we obtain a graph with just few vertices. Then the graph can be decontracted step by step to obtain a Schnyder wood of the original graph.

Given a toroidal map  $G$ , the *contraction* of a non-loop-edge  $e$  of  $G$  is the operation consisting of continuously contracting  $e$  until merging its two ends. We note  $G/e$  the obtained graph. On Figure 16 the contraction of an edge  $e$  is represented. We consider three different cases corresponding to whether the faces adjacent to the edge  $e$  are triangles or not. Note that only one edge of each set of homotopic multiple edges that is

possibly created is preserved.

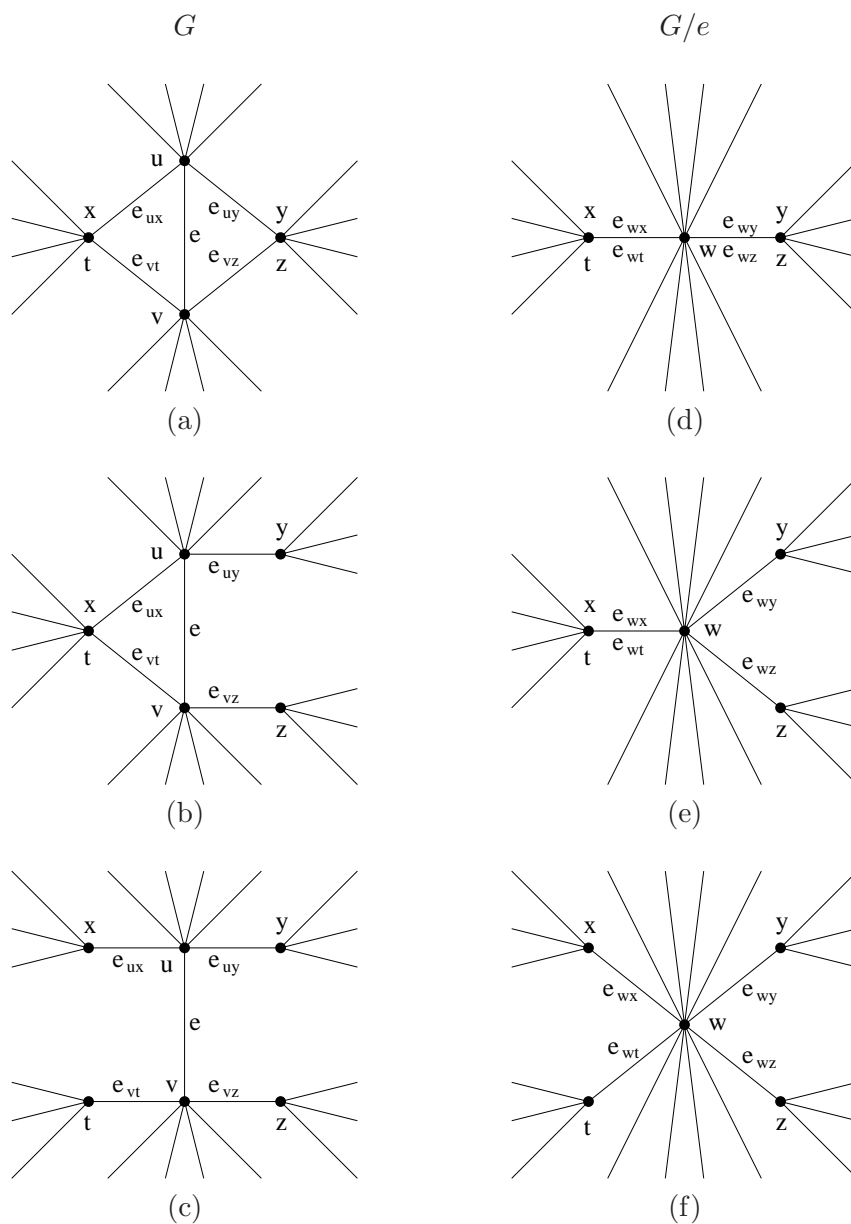


Figure 16: The contraction operation

The goal of this section is to prove the following lemma, that plays a key role in the proof of Section 9.

**Lemma 15** *If  $G$  is a toroidal map given with a non-loop edge  $e$  whose extremities are of degree at least three and such that  $G/e$  admits a Schnyder wood of Type 1, then  $G$  admits a Schnyder wood of type 1.*



The proof of Lemma 15 is long and technical. In the planar case, an analogous lemma can be proved without too much difficulty (see Section 2.6 of [13]) as there are special outer-vertices where the contraction can be done to reduce the case analysis and as Properties (P1) and (P2) are local and not too difficult to preserve during the decontraction process.

In the toroidal case there is a huge case analysis for the following reasons. One has to consider the three different kind of contraction depicted on Figure 16. For each of these cases, one has to consider the different ways that the edges can be oriented and color (around the contraction point) in  $G/e$ . For each of these cases, one has to show that the Schnyder wood  $G/e$  can be extended to a Schnyder wood of  $G$ . This would be quite easy if one just has to satisfy (T1) that is a local property, but satisfying (T2) is much more complicated. Instead of proving (T2), that is considering intersections between every monochromatic cycles, we prove (T2') and (T3'), that is equivalent for our purpose by Lemma 14. Property (T2') is simpler than (T2) as it is considering just one intersection for each pair of colors instead of all the intersections. Even with this simplification proving (T2') is the main difficulty of the proof. For each considered cases, one has to analyze the different ways that the monochromatic cycles go through the contracted vertex or not and to show that there always exists a coloring and orientation of  $G$  where (T2') is satisfied. Some cases are non trivial and involved the use of lemmas like Lemmas 16 and 17.

**Lemma 16** *Let  $G$  be a toroidal map given with a Schnyder wood and  $y, w$  be two vertices of  $G$  such that  $e_i(y)$  is entering  $w$ . Suppose that there is a directed path  $Q_{i-1}$  of color  $i-1$  from  $y$  to  $w$ , and a directed path  $Q_{i+1}$  of color  $i+1$  from  $y$  to  $w$ . Consider the two directed cycles  $C_{i-1} = Q_{i-1} \cup \{e_i(y)\}^{-1}$  and  $C_{i+1} = Q_{i+1} \cup \{e_i(y)\}^{-1}$ . Then  $C_{i-1}$  and  $C_{i+1}$  are not homotopic.*

*Proof.* By Lemma 1, the cycles  $C_{i-1}$  and  $C_{i+1}$  are not contractible. Suppose that  $C_{i-1}$  and  $C_{i+1}$  are homotopic. The path  $Q_{i+1}$  is leaving  $C_{i-1}$  at  $y$  on the right side of  $C_{i-1}$ . Since  $C_{i-1}$  and  $C_{i+1}$  are homotopic, the path  $Q_{i+1}$  is entering  $C_{i-1}$  at least once from its right side. This is in contradiction with (T1).  $\square$

The sector  $[e_1, e_2]$  of a vertex  $w$ , for  $e_1$  and  $e_2$  two edges incident to  $w$ , is the counterclockwise sector of  $w$  between  $e_1$  and  $e_2$ , including the edge  $e_1$  and  $e_2$ . The sector  $]e_1, e_2]$ ,  $[e_1, e_2[$ , and  $]e_1, e_2[$  are defined analogously by excluding the corresponding edges from the sectors.

**Lemma 17** *Let  $G$  be a toroidal map given with a Schnyder wood and  $w, x, y$  be three vertices such that  $e_{i-1}(x)$  and  $e_{i+1}(y)$  are entering  $w$ . Suppose that there is a directed path  $Q_{i-1}$  of color  $i-1$  from  $y$  to  $w$ , entering  $w$  in the sector  $[e_i(w), e_{i-1}(x)]$ , and a directed path  $Q_{i+1}$  of color  $i+1$  from  $x$  to  $w$ , entering  $w$  in the sector  $[e_{i+1}(y), e_i(w)]$ . Consider the two directed cycles  $C_{i-1} = Q_{i-1} \cup \{e_{i+1}(y)\}^{-1}$  and  $C_{i+1} = Q_{i+1} \cup \{e_{i-1}(x)\}^{-1}$ . Then either  $C_{i-1}$  and  $C_{i+1}$  are not homotopic or  $C_{i-1} = C_{i+1}^{-1}$ .*

*Proof.* By Lemma 1, the cycles  $C_{i-1}$  and  $C_{i+1}$  are not contractible. Suppose that  $C_{i-1}$  and  $C_{i+1}$  are homotopic and that  $C_{i-1} \neq C_{i+1}^{-1}$ . Since  $C_{i-1} \neq C_{i+1}^{-1}$ , the cycle  $C_{i+1}$  is leaving  $C_{i-1}$ . By (T1) and the assumption on the sectors,  $C_{i+1}$  is leaving  $C_{i-1}$  on its right side. Since  $C_{i-1}$  and  $C_{i+1}$  are homotopic, the path  $Q_{i+1}$  is entering  $C_{i-1}$  at least once from its right side. This is in contradiction with (T1).  $\square$

We are now able to prove Lemma 15.

*Proof of Lemma 15.* Let  $u, v$  be the two extremities of  $e$ . Vertices  $u$  and  $v$  are of degree at least three. Let  $x, y$  (resp.  $z, t$ ) be the neighbors of  $u$  (resp.  $v$ ) such that  $x, y, y$  (resp.  $z, u, t$ ) appear consecutively and in counterclockwise order around  $u$  (resp.  $v$ ) (see Figure 16.(c)). Note that  $u$  and  $v$  are distinct by definition of edge contraction but that  $x, y, z, t$  are not necessarily distinct, nor necessarily distinct from  $u$  and  $v$ . Depending on whether the faces incident to  $e$  are triangles or not, we are, by symmetry, in one of the three cases of Figure 16.(a).(b).(c). Let  $G' = G/e$  and consider a Schnyder wood of Type 1 of  $G'$ . Let  $w$  be the vertex of  $G'$  resulting from the contraction of  $e$ .

For each case (a), (b), (c). There are different cases corresponding to the different possibilities of orientation and coloring of the edges  $e_{wx}, e_{wy}, e_{wz}, e_{wt}$  in  $G'$ . For example for case (a), there should be 6 cases depending on if  $e_{wx}$  and  $e_{wy}$  are both entering  $w$ , both leaving  $w$  or one entering  $w$  and one leaving  $w$  (3 cases), multiplied by the coloring, both of the same or not (2 cases). The case where  $w$  has two edges leaving in the same color is impossible by (T1). So, by symmetry, only 5 cases remain represented by Figure  $a.k.0$ , for  $k = 1, \dots, 5$ , on Figure 17 (in the notation  $\alpha.k.l$ ,  $a.k$  indicates the line on the figures and  $l$  the column). For cases (b) and (c), there are more cases to consider but the analysis is similar. These cases are represented in the first columns of Figures 18 and 19. On these figure, a dotted half-edge represent the possibility for an edge to be uni- or bi-directed. In the last case of each figure, we have indicated where is the edge leaving in color 1 as there are two possibilities (up or down).

In each case  $\alpha.k$ ,  $\alpha \in \{a, b, c\}$ , we show how one can color and orient the edges of  $G$  to obtain a Schnyder wood of  $G$  from the Schnyder wood of  $G'$ . Just the edges  $e, e_{ux}, e_{uy}, e_{vt}, e_{vz}$  of  $G$  have to be specified, all the other edges of  $G$  keep the orientation and coloring of their corresponding edge in  $G'$ . In each case  $\alpha.k$ , there might be several possibilities to color and orient these edges to satisfy (T1). Just some of these possibilities, the one that are useful for our purpose, are represented on Figures  $\alpha.k.l$ ,  $l \geq 1$ , of Figures 17 to 19. A dotted half-edge represent the fact that the edge is uni- or bi-directed like the corresponding half edge of  $G'$ .

In each case  $\alpha.k$ ,  $\alpha \in \{a, b, c\}$ , we show that one of the colorings  $\alpha.k.l$ ,  $l \geq 1$ , gives a Schnyder wood of Type 1 of  $G$ . By Lemma 14, we just have to prove that, in each case  $\alpha.k$ , there is one coloring satisfying (T1'), (T2') and (T3'). Properties (T1') and (T3') are satisfied for any colorings  $\alpha.k.l$ ,  $l \geq 1$  but this is not the case for property (T2'). This explains why several possible colorings of  $G$  have to be considered.

(T1') One can easily check that in all the cases  $\alpha.k.l$ , Property (T1') is satisfied for every vertex of  $G$ . To do so one can consider the angle labeling around vertices

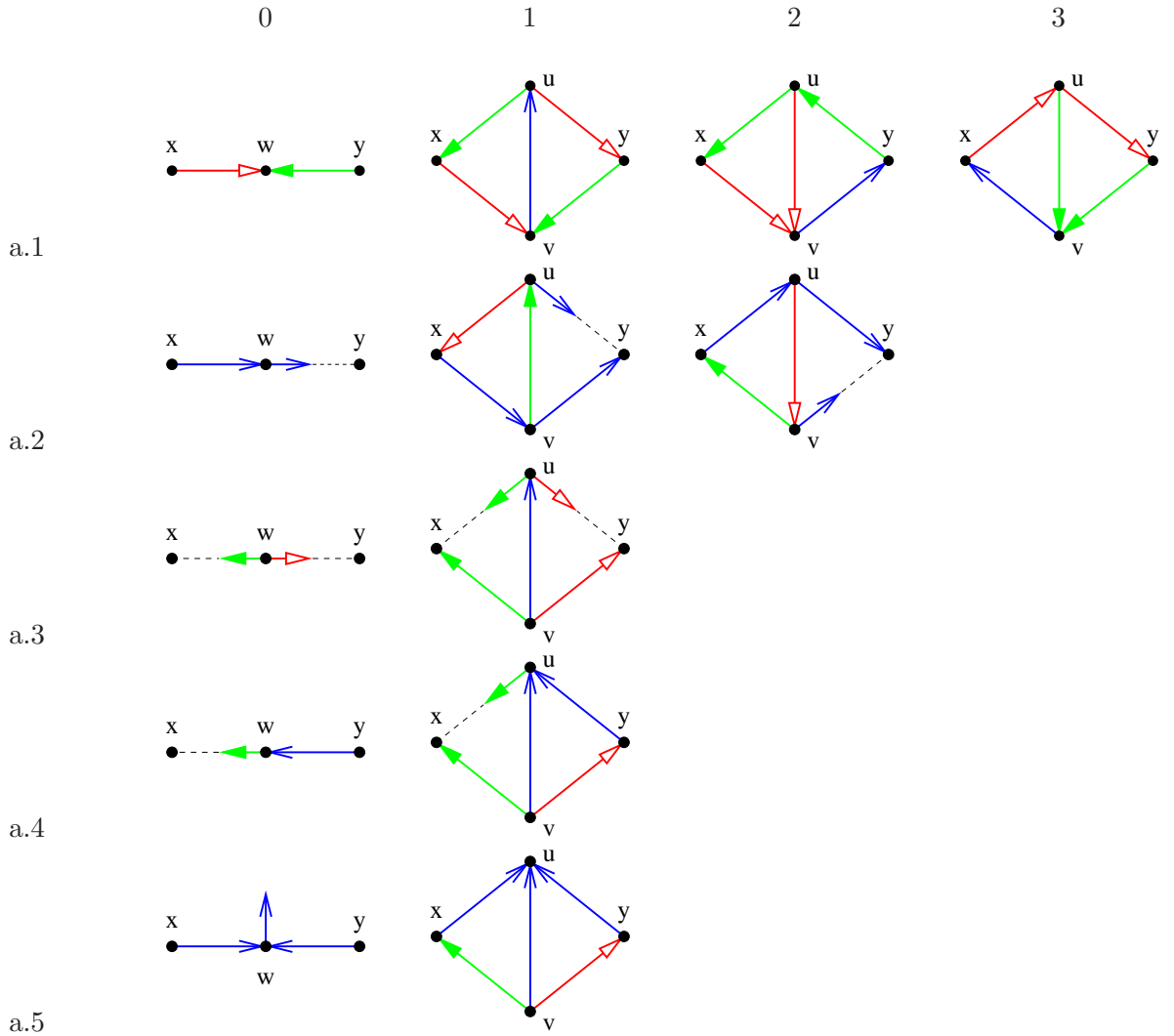


Figure 17: Decontraction rules for case (a).

$w, x, y, (z), (t)$  of  $G'$  in the case  $\alpha.k.0$ . Then one can remark that this angle labeling exports well around vertices  $u, v, x, y, (z), (t)$  of  $G$  and thus the Schnyder property is satisfied for these vertices. On Figure 20, an example is given on how the angle labeling is modified during the decontraction process. It corresponds to case  $c.2.1$  where the dotted half edge is uni-directed. We do not detail more this part that is easy to check.

(T3') One can easily check that in all the cases  $\alpha.k.l$ , Property (T3') is satisfied for  $G$ . If (T3') is not satisfied in  $G$  after applying one of the coloring  $\alpha.k.l$ , then there exists two monochromatic cycles  $C, C'$  of different colors that are reversal. By property (T3') of Lemma 14, there is no reversal cycles in  $G'$ . Thus  $C, C'$  have to use a bi-directed edge  $e'$  of the figures that is newly created and distinct from edge  $e$  and distinct from

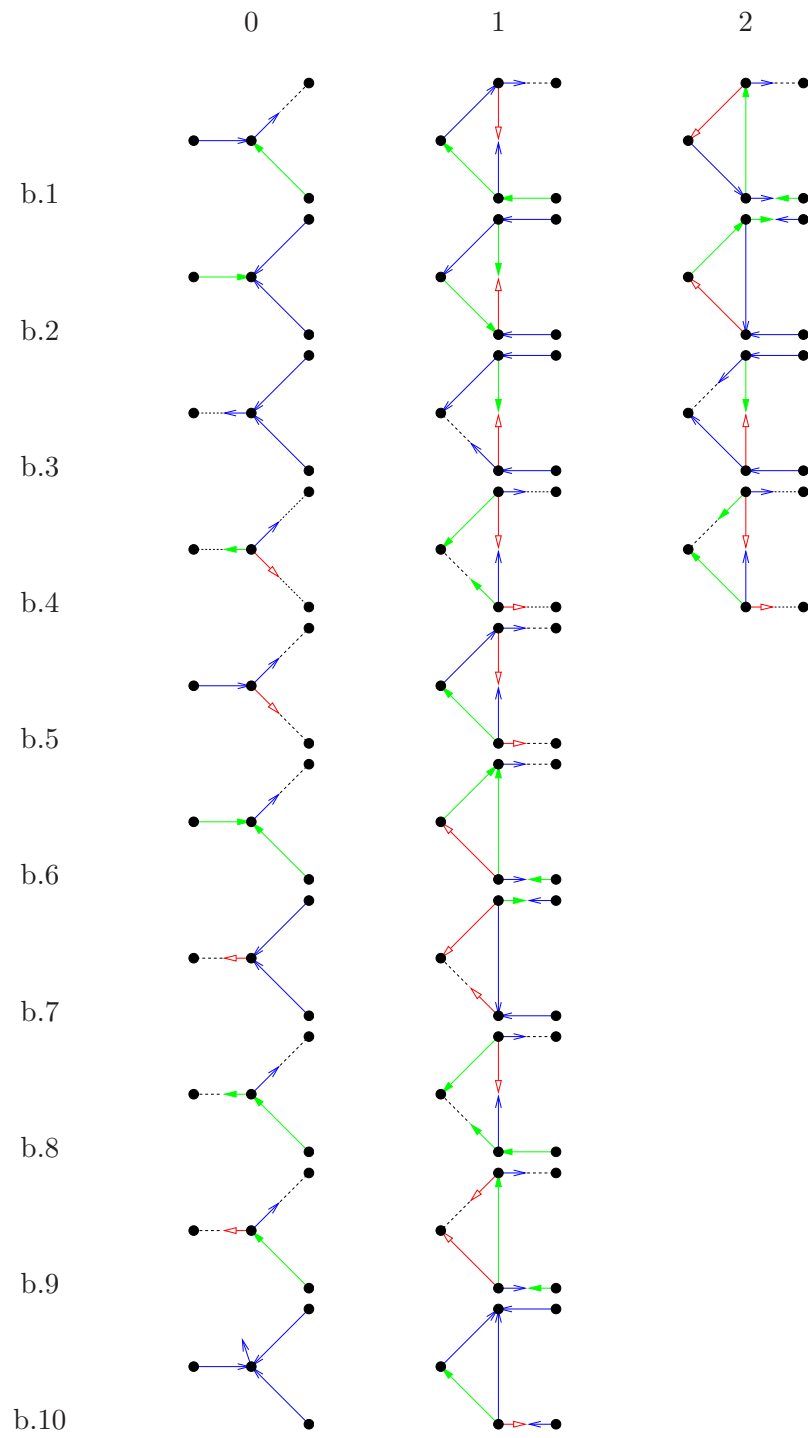


Figure 18: Decontraction rules for case (b)

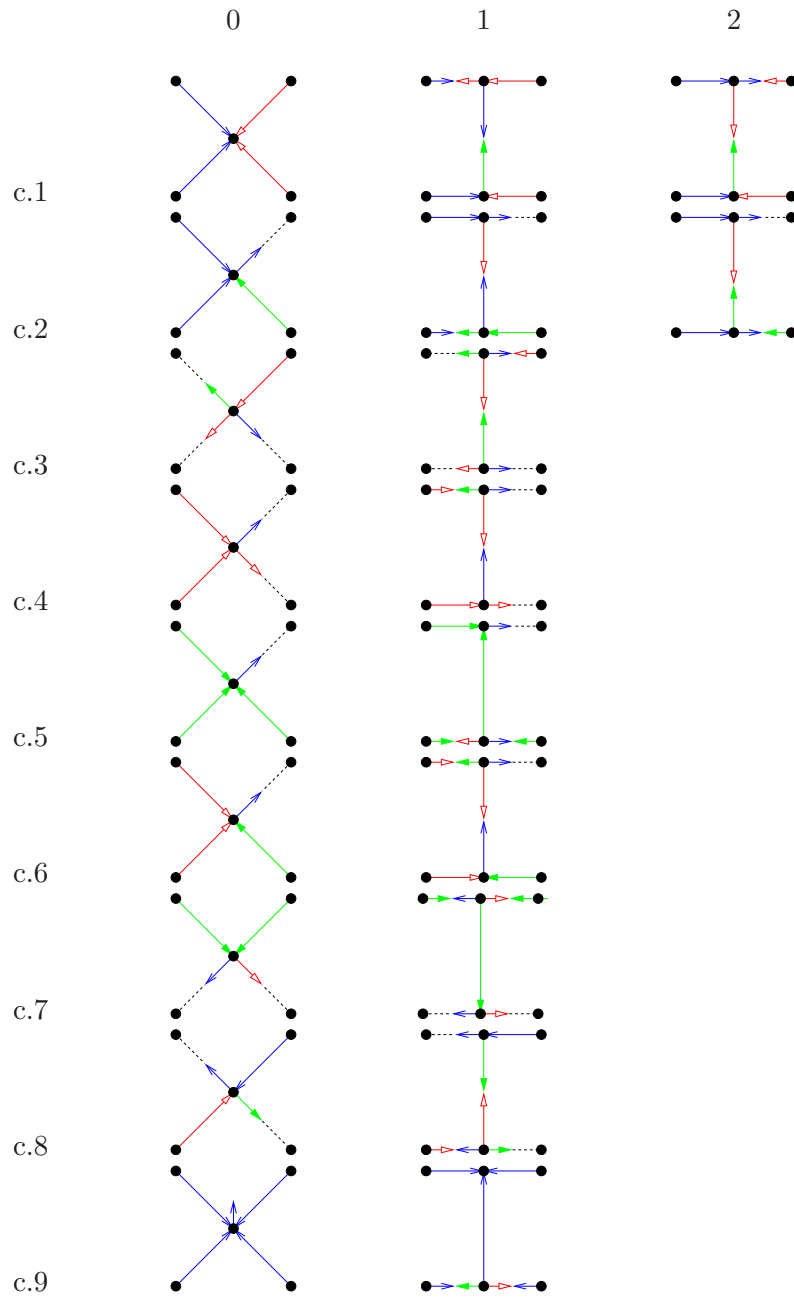


Figure 19: Decontraction rules for case (c)

half-dotted edges (otherwise the cycles are still reversal when  $e$  is contracted). Just some cases have such an edge and one can remark that for all this cases, the cycle, after entering  $u$  or  $v$  by edge  $e'$  have to use the edge  $e$  that is either uni-directed or bi-directed with different colors than  $e'$ , a contradiction. For example in case  $c.2.1$  of Figure 21, two

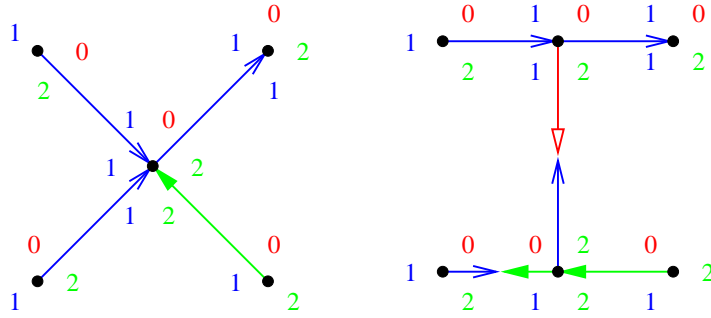


Figure 20: Example on (T1') preservation during decontraction.

reversal cycles of  $G$  that do not corresponds to reversal cycles when  $e$  is contracted, have to use edge  $e_{vt}$ . Then one of the two cycle is entering  $v$  by  $e_{vt}$  in color 1 and thus has to continue by using the only edge leaving  $v$  in color 1, edge  $e$ . As  $e$  and  $e_{vt}$  are colored differently, this is no possible. We do not detail more this part that is easy to check.

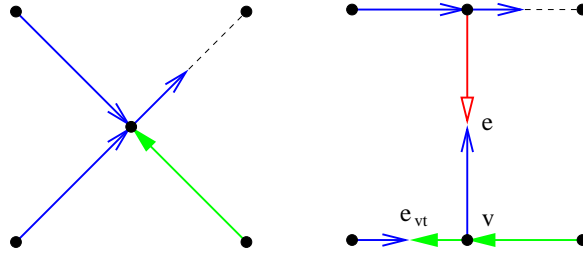


Figure 21: Example on (T3') preservation during decontraction.

(T2') Proving Property (T2') is the main difficulty of the proof. For each case  $\alpha.k$ , there is a case analysis considering the different ways that the monochromatic cycles of  $G'$  go through  $w$  or not. We say that a monochromatic cycle  $C$  of  $G'$  is *safe* if  $C$  does not contain  $w$ . Depending on whether there are safe monochromatic cycles or not of each colors, it may be a different case  $\alpha.k.\ell$  and a different argument that is used to prove that property (T2') is preserved.

- *Case a.1:  $e_{wx}$  and  $e_{wy}$  are entering  $w$  in different color.*

We can assume by symmetry that  $e_{wx} = e_0(x)$  and  $e_{wy} = e_2(y)$  (case a.1.0 of Figure 17). We apply one of the colorings a.1.1, a.1.2 and a.1.3 of Figure 17.

We have a case analysis corresponding to whether there are monochromatic cycles of  $G'$  that are safe.

- ★ *Subcase a.1.{0,1,2}: There are safe monochromatic cycles of colors  $\{0,1,2\}$ .*

Let  $C'_0, C'_1, C'_2$  be safe monochromatic cycles of color 0, 1, 2 in  $G'$ . As the Schnyder wood of  $G'$  is of type 1, they pairwise intersects in  $G'$ . Apply the coloring a.1.1 on  $G$ . As  $C'_0, C'_1, C'_2$  do not contain vertex  $w$ , they are not modified in  $G$ . Thus they still pairwise intersect in  $G$ . So (T2') is satisfied.

★ *Subcase a.1. $\{0, 2\}$ : There are safe monochromatic cycles of colors exactly  $\{0, 2\}$ .*

Let  $C'_0, C'_2$  be safe monochromatic cycles of color 0, 2 in  $G'$ . Let  $C'_1$  be a 1-cycle in  $G'$ . As the Schnyder wood of  $G'$  is of type 1,  $C'_0, C'_1, C'_2$  pairwise intersects in  $G'$ . None of those intersections contain  $w$  as  $C'_0$  and  $C'_2$  do not contain  $w$ . By (T1), the cycle  $C'_1$  enter  $w$  in the sector  $]e_{wx}, e_{wy}[$  and leaves in the sector  $]e_{wy}, e_{wx}[$ . Apply the coloring a.1.1 on  $G$ . The cycle  $C'_1$  is replaced by a new cycle  $C_1 = C'_1 \setminus \{w\} \cup \{u, v\}$ . The cycles  $C'_0, C'_1, C'_2$  were intersecting outside  $w$  in  $G'$  so  $C'_0, C_1, C'_2$  are intersecting in  $G$ . So (T2') is satisfied.

★ *Subcase a.1. $\{1, 2\}$ : There are safe monochromatic cycles of colors exactly  $\{1, 2\}$ .*

Let  $C'_1, C'_2$  be safe monochromatic cycles of color 1, 2 in  $G'$ . Let  $C'_0$  be a 0-cycle in  $G'$ . The cycles  $C'_0, C'_1, C'_2$  pairwise intersects outside  $w$ . The cycle  $C'_0$  enters  $w$  in the sector  $[e_1(w), e_{wx}[$ ,  $[e_{wx}, e_{wx}]$  or  $]e_{wx}, e_2(w)[$ . Apply the coloring a.1.2 on  $G$ . Depending on which of the three sectors  $C'_0$  enters, it is replaced by one of the three following cycle  $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$ ,  $C_0 = C'_0 \setminus \{w\} \cup \{x, v\}$ ,  $C_0 = C'_0 \setminus \{w\} \cup \{v\}$ . In any of the three possibilities,  $C_0, C'_1, C'_2$  are intersecting in  $G$ . So (T2') is satisfied.

★ *Subcase a.1. $\{0, 1\}$ : There are safe monochromatic cycles of colors exactly  $\{0, 1\}$ .*

This case is completely symmetric to the case a.1. $\{1, 2\}$ .

★ *Subcase a.1. $\{2\}$ : There are safe monochromatic cycles of color 2 only.*

Let  $C'_2$  be a safe 2-cycle in  $G'$ . Let  $C'_0, C'_1$  be monochromatic cycles of color 0, 1 in  $G'$ .

Suppose that there exists a path  $Q'_0$  of color 0, from  $y$  to  $w$  such that this path does not intersect  $C'_2$ . Suppose also that there exists a path  $Q'_1$  of color 1, from  $y$  to  $w$  such that this path does not intersect  $C'_2$ . Let  $C''_0 = Q'_0 \cup \{e_{wy}\}$  and  $C''_1 = Q'_1 \cup \{e_{wy}\}$ . By Lemma 1,  $C''_0, C''_1, C'_2$  are not contractible. Both of  $C''_0, C''_1$  does not intersect  $C'_2$  so by Lemma 2, they are both homotopic to  $C'_2$ . Thus by Lemma 3, cycles  $C''_0, C''_1$  are homotopic to each other, contradicting Lemma 16 (with  $i = 2, w, y, Q'_0, Q'_1$ ). So we can assume that one of  $Q'_0$  or  $Q'_1$  as above does not exist.

Suppose that in  $G'$ , there does not exist a path of color 0, from  $y$  to  $w$  such that this path does not intersect  $C'_2$ . Apply the coloring a.1.1 on  $G$ . Cycle  $C'_1$  is replaced by  $C_1 = C'_1 \setminus \{w\} \cup \{u, v\}$ , and intersect  $C'_2$ . Let  $C_0$  be a 0-cycle of  $G$ . Cycle  $C_0$  has to contain  $u$  or  $v$  or both, otherwise it is a safe cycle of  $G'$  of color 0. In any case it intersects  $C_1$ . If  $C_0$  contains  $v$ , then  $C'_0 = C_0 \setminus \{v\} \cup \{w\}$  and so  $C_0$  is intersecting  $C'_2$  and (T2') is satisfied. Suppose now that  $C_0$  does not contain  $v$ . Then  $C_0$  contains  $u$  and  $y$ , the extremity of the edge leaving  $u$  in color 0. Let  $Q_0$  be the part of  $C_0$  consisting of the path from  $y$  to  $u$ . The path  $Q'_0 = Q_0 \setminus \{u\} \cup \{w\}$  is from  $y$  to  $w$ . Thus by assumption  $Q'_0$  intersects  $C'_2$ . So  $C_0$  intersects  $C'_2$  and (T2') is satisfied.

Suppose now that in  $G'$ , there does not exist a path of color 1, from  $y$  to  $w$  such that this path does not intersect  $C'_2$ . Apply the coloring a.1.2 on  $G$ . Depending on which of the three sectors  $C'_0$  enters,  $[e_1(w), e_{wx}[$ ,  $[e_{wx}, e_{wx}]$  or  $]e_{wx}, e_2(w)[$ , it is replaced by one of the three following cycle  $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$ ,  $C_0 = C'_0 \setminus \{w\} \cup \{x, v\}$ ,  $C_0 = C'_0 \setminus \{w\} \cup \{v\}$ . In any of the three possibilities,  $C_0$  contains  $v$  and intersect  $C'_2$ .

Let  $C_1$  be a 1-cycle of  $G$ . Cycle  $C_1$  has to contain  $u$  or  $v$  or both, otherwise it is a safe cycle of  $G'$  of color 1. Vertex  $u$  has no edge entering it in color 1 so  $C_1$  does not contain  $u$  and thus it contains  $v$  and intersects  $C_0$ . Then  $C_1$  contains  $y$ , the extremity of the edge leaving  $v$  in color 1. Let  $Q_1$  be the part of  $C_1$  consisting of the path from  $y$  to  $v$ . The path  $Q'_1 = Q_1 \setminus \{v\} \cup \{w\}$  is from  $y$  to  $w$ . Thus by assumption  $Q'_1$  intersects  $C'_2$ . So  $C_1$  intersects  $C'_2$  and (T2') is satisfied.

★ *Subcase a.1.{0}: There are safe monochromatic cycles of color 0 only.*

This case is completely symmetric to the case a.1.{2}.

★ *Subcase a.1.{1}: There are safe monochromatic cycles of color 1 only.*

Let  $C'_1$  be a safe 1-cycle in  $G'$ . Let  $C'_0$  and  $C'_2$  be monochromatic cycles of color 0 and 2 in  $G'$ .

Suppose  $C'_0$  is entering  $w$  in the sector  $]e_{wx}, e_2(w)[$ . Apply the coloring a.1.3 on  $G$ . The 0-cycle  $C'_0$  is replaced by  $C_0 = C'_0 \setminus \{w\} \cup \{v\}$  and thus contains  $v$  and still intersect  $C'_1$ . Depending on which of the three sectors  $C'_2$  enters,  $[e_0(w), e_{wy}[$ ,  $[e_{wy}, e_{wy}]$  or  $]e_{wy}, e_1(w)[$ , it is replaced by one of the three following cycles  $C_2 = C'_2 \setminus \{w\} \cup \{v\}$ ,  $C_2 = C'_2 \setminus \{w\} \cup \{y, v\}$ ,  $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$ . In any case,  $C_2$  contains  $v$  and still intersect  $C'_1$ . Cycle  $C_0$  and  $C_2$  intersect on  $v$ . So (T2') is satisfied.

The case where  $C'_2$  is entering  $w$  in the sector  $[e_0(w), e_{wy}[$  is completely symmetric and we apply the coloring a.1.2 on  $G$ .

It remains to deal with the case where  $C'_0$  is entering  $w$  in the sector  $[e_1(w), e_{wx}]$  and  $C'_2$  is entering  $w$  in the sector  $[e_{wy}, e_1(w)]$ . Suppose that there exists a path  $Q'_0$  of color 0, from  $y$  to  $w$ , entering  $w$  in the sector  $[e_1(w), e_{wx}]$ , such that this path does not intersect  $C'_1$ . Suppose also that there exists a path  $Q'_2$  of color 2, from  $x$  to  $w$ , entering  $w$  in the sector  $[e_{wy}, e_1(w)]$ , such that this path does not intersect  $C'_1$ . Let  $C''_0 = Q'_0 \cup \{e_{wy}\}$  and  $C''_2 = Q'_2 \cup \{e_{wx}\}$ . By Lemma 1,  $C''_0, C'_1, C''_2$  are not contractible. Cycles  $C''_0, C''_2$  do not intersect  $C'_1$  so by Lemma 2, they are homotopic to  $C'_1$ . Thus by Lemma 3, cycles  $C''_0, C''_2$  are homotopic to each other. Thus by Lemma 17 (with  $i = 1, w, x, y, Q'_0, Q'_2$ ), we have  $C''_0 = (C''_2)^{-1}$ , contradicting (T3') in  $G'$ . So we can assume that one of  $Q'_0$  or  $Q'_2$  as above does not exist. By symmetry, suppose that in  $G'$ , there does not exist a path of color 0, from  $y$  to  $w$ , entering  $w$  in the sector  $[e_1(w), e_{wx}]$ , such that this path does not intersect  $C'_1$ . Apply the coloring a.1.3 on  $G$ . Depending on which of the two sectors  $C'_2$  enters,  $[e_{wy}, e_{wy}]$  or  $]e_{wy}, e_1(w)[$ , it is replaced by one of the two following cycle  $C_2 = C'_2 \setminus \{w\} \cup \{y, v\}$ ,  $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$ . In any case,  $C_2$  still intersects  $C'_1$ . Let  $C_0$  be a 0-cycle of  $G$ . Cycle  $C_0$  has to contain  $u$  or  $v$  or both, otherwise it is a safe cycle of  $G'$  of color 0. Suppose  $C_0$  does not contain  $u$ , then  $C'_0 = C_0 \setminus \{v\} \cup \{w\}$  and  $C'_0$  is not entering  $w$  in the sector  $[e_1(w), e_{wx}]$ , a contradiction. So  $C_0$  contains  $u$ . Thus  $C_0$  contains  $y$ , the extremity of the edge leaving  $u$  in color 0, and it intersects  $C_2$ . Let  $Q_0$  be the part of  $C_0$  consisting of the path from  $y$  to  $u$ . The path  $Q'_0 = Q_0 \setminus \{u\} \cup \{w\}$  is from  $y$  to  $w$  and entering  $w$  in the sector  $[e_1(w), e_{wx}]$ . Thus by assumption  $Q'_0$  intersects  $C'_1$ . So  $C_0$  intersects  $C'_1$  and (T2') is satisfied.

★ *Subcase a.1.{}: There are no safe monochromatic cycle.*



Let  $C'_0, C'_1, C'_2$  be monochromatic cycle of color 0, 1, 2 in  $G'$ . They all pairwise intersect on  $w$ .

Suppose first that  $C'_0$  is entering  $w$  in the sector  $]e_{wx}, e_2(w)[$ . Apply the coloring a.1.3 on  $G$ . The 0-cycle  $C'_0$  is replaced by  $C_0 = C'_0 \setminus \{w\} \cup \{v\}$  and thus contains  $v$ . Depending on which of the three sectors  $C'_2$  enters,  $[e_0(w), e_{wy}[$ ,  $[e_{wy}, e_{wy}]$  or  $]e_{wy}, e_1(w)[$ , it is replaced by one of the three following cycle  $C_2 = C'_2 \setminus \{w\} \cup \{v\}$ ,  $C_2 = C'_2 \setminus \{w\} \cup \{y, v\}$ ,  $C_2 = C'_2 \setminus \{w\} \cup \{u, v\}$ . In any case,  $C_2$  contains  $v$ . Let  $C_1$  be a 1-cycle in  $G$ . Cycle  $C_1$  has to contain  $u$  or  $v$  or both, otherwise it is a safe cycle of  $G'$  of color 1. Vertex  $u$  has no edge entering it in color 1 so  $C_1$  does not contain  $u$  and thus it contains  $v$ . So  $C_0, C_1, C_2$  all intersect on  $v$  and (T2') is satisfied.

The case where  $C'_2$  is entering  $w$  in the sector  $[e_0(w), e_{wy}[$  is completely symmetric and we apply the coloring a.1.2 on  $G$ .

It remains to deal with the case where  $C'_0$  is entering  $w$  in the sector  $[e_1(w), e_{wx}]$  and  $C'_2$  is entering  $w$  in the sector  $[e_{wy}, e_1(w)[$ . Apply the coloring a.1.1 on  $G$ . Cycle  $C'_1$  is replaced by  $C_1 = C'_1 \setminus \{w\} \cup \{u, v\}$ . Let  $C_0$  be a 0-cycle in  $G$ . Cycle  $C_0$  has to contain  $u$  or  $v$  or both, otherwise it is a safe cycle of  $G'$  of color 0. Suppose  $C_0 \cap \{v, x\} = \{v\}$ , then  $C_0 \setminus \{v\} \cup \{w\}$  is a 0-cycle of  $G'$  entering  $w$  in the sector  $]e_{wx}, e_2(w)[$ , contradicting the assumption on  $C'_0$ . Suppose  $C_0$  contains  $u$ , then  $C_0$  contains  $y$ , the extremity of the edge leaving  $u$  in color 0. So  $C_0$  contains  $\{v, x\}$  or  $\{u, y\}$ . Similarly  $C_2$  contains  $\{v, y\}$  or  $\{u, x\}$ . In any case  $C_0, C_1, C_2$  pairwise intersect. So (T2') is satisfied.

• *Case a.2:  $e_{wx}$  and  $e_{wy}$  have the same color, one is entering  $w$ , the other is leaving  $w$ .*

We can assume by symmetry that  $e_{wx} = e_1(x)$  and  $e_{wy} = e_1(w)$  (case a.2.0 of Figure 17). We apply one of the colorings a.2.1 and a.2.2 of Figure 17.

We have a case analysis corresponding to whether there are monochromatic cycles of  $G'$  that are safe.

★ *Subcase a.2. $\{0, 1, 2\}$ : There are safe monochromatic cycles of colors  $\{0, 1, 2\}$ .*

Let  $C'_0, C'_1, C'_2$  be safe monochromatic cycles of color 0, 1, 2 in  $G'$ . They pairwise intersects in  $G'$ . Apply the coloring a.2.1 on  $G$ .  $C'_0, C'_1, C'_2$  still pairwise intersect in  $G$ . So (T2') is satisfied.

★ *Subcase a.2. $\{0, 2\}$ : There are safe monochromatic cycles of colors exactly  $\{0, 2\}$ .*

Let  $C'_0, C'_2$  be safe monochromatic cycles of color 0, 2 in  $G'$ . Let  $C'_1$  be a 1-cycle in  $G'$ . Cycles  $C'_0, C'_2$  still intersects in  $G$ . Apply the coloring a.2.1 on  $G$ . Depending on which of the three sectors  $C'_1$  enters,  $[e_2(w), e_{wx}[$ ,  $[e_{wx}, e_{wx}]$  or  $]e_{wx}, e_0(w)[$ , it is replaced by one of the three following cycles  $C_1 = C'_1 \setminus \{w\} \cup \{u, y\}$ ,  $C_1 = C'_1 \setminus \{w\} \cup \{x, v, y\}$ ,  $C_1 = C'_1 \setminus \{w\} \cup \{v, y\}$ . In any of the three possibilities,  $C_1$  still intersect both  $C'_0, C'_2$ . So (T2') is satisfied.

★ *Subcase a.2. $\{1, 2\}$ : There are safe monochromatic cycles of colors exactly  $\{1, 2\}$ .*

Let  $C'_1, C'_2$  be safe monochromatic cycles of color 1, 2 in  $G'$ . Let  $C'_0$  be a 0-cycle in  $G'$ . Cycles  $C'_1, C'_2$  still intersects in  $G$ . Apply the coloring a.2.2 on  $G$ . Depending on which of the two sectors  $C'_0$  enters,  $[e_{wy}, e_{wy}]$  or  $]e_{wy}, e_2(w)[$ , it is replaced by one of the two following cycles  $C_0 = C'_0 \setminus \{w\} \cup \{y, v\}$ ,  $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$ . In any of the two

possibilities,  $C_0$  still intersect both  $C'_1, C'_2$ . So (T2') is satisfied.

★ *Subcase a.2.{0, 1}*: There are safe monochromatic cycles of colors exactly  $\{0, 1\}$ .

This case is completely symmetric to the case a.2.{1, 2}.

★ *Subcase a.2.{2}*: There are safe monochromatic cycles of color 2 only.

Let  $C'_2$  be a safe 2-cycle in  $G'$ . Let  $C'_0, C'_1$  be monochromatic cycles of color 0, 1 in  $G'$ .

Apply the coloring a.2.2 on  $G$ . Depending on which of the three sectors  $C'_1$  enters,  $[e_2(w), e_{wx}[$ ,  $[e_{wx}, e_{wx}]$  or  $]e_{wx}, e_0(w)]$ , it is replaced by one of the three following cycle  $C_1 = C'_1 \setminus \{w\} \cup \{u, y\}$ ,  $C_1 = C'_1 \setminus \{w\} \cup \{x, u, y\}$ ,  $C_1 = C'_1 \setminus \{w\} \cup \{v, y\}$ . Depending on which of the two sectors  $C'_0$  enters,  $[e_{wy}, e_{wy}]$  or  $]e_{wy}, e_2(w)]$ , it is replaced by one of the two following cycles  $C_0 = C'_0 \setminus \{w\} \cup \{y, v\}$ ,  $C_0 = C'_0 \setminus \{w\} \cup \{u, v\}$ . In any case,  $C_0$  and  $C_1$  intersects each other and intersect  $C'_2$ . So (T2') is satisfied.

★ *Subcase a.2.{0}*: There are safe monochromatic cycles of color 0 only.

This case is completely symmetric to the case a.2.{0}.

★ *Subcase a.2.{1}*: There are safe monochromatic cycles of color 1 only.

Let  $C'_1$  be a safe 1-cycle in  $G'$ . Let  $C'_0, C'_2$  be monochromatic cycles of color 0, 2 in  $G'$ . Suppose that there exists a path  $Q'_0$  of color 0, from  $x$  to  $w$ , that does not intersect  $C'_1$ . Suppose also that there exists a path  $Q'_2$  of color 2, from  $x$  to  $w$ , that does not intersect  $C'_1$ . Let  $C''_0 = Q'_0 \cup \{e_{wx}\}$  and  $C''_2 = Q'_2 \cup \{e_{wx}\}$ . By Lemma 1,  $C''_0, C'_1, C''_2$  are not contractible. Both of  $C''_0, C''_2$  does not intersect  $C'_1$ , so by Lemma 2, they are both homotopic to  $C'_1$ . Thus by Lemma 3, cycles  $C''_0, C''_2$  are homotopic to each other, contradicting Lemma 16 (with  $i = 1, w, x, Q'_0, Q'_2$ ). So we can assume that one of  $Q'_0$  or  $Q'_2$  as above does not exist.

By symmetry, suppose that in  $G'$ , there does not exist a path of color 0, from  $x$  to  $w$ , that does not intersect  $C'_1$ . Apply the coloring a.2.1 on  $G$ . Depending on which of the two sectors  $C'_2$  enters,  $[e_1(w), e_{wy}[$ ,  $[e_{wy}, e_{wy}]$ , it is replaced by one of the two following cycles  $C_2 = C'_2 \setminus \{w\} \cup \{v, u\}$ ,  $C_2 = C'_2 \setminus \{w\} \cup \{y, u\}$ . In any of the two possibilities  $C_2$  intersects  $C'_1$ . Let  $C_0$  be a 0-cycle of  $G$ . Cycle  $C_0$  has to contain  $u$  or  $v$  or both, otherwise it is a safe cycle of  $G'$  of color 0. Vertex  $v$  has no edge entering it in color 0, so  $C_0$  does not contain  $v$  and so it contains  $u$  and  $x$ , the extremity of the edge leaving  $u$  in color 0. Thus  $C_0$  intersect  $C_2$ . Let  $Q_0$  be the part of  $C_0$  consisting of the path from  $x$  to  $u$ . The path  $Q'_0 = Q_0 \setminus \{u\} \cup \{w\}$  of  $G'$  is from  $x$  to  $w$ , thus by assumption  $Q'_0$  intersects  $C'_1$ . So  $C_0$  intersects  $C'_1$  and (T2') is satisfied.

★ *Subcase a.2.{}*: There are no safe monochromatic cycle.

Let  $C'_0, C'_1, C'_2$  be monochromatic cycle of color 0, 1, 2 in  $G'$ .

Suppose  $C'_1$  is entering  $w$  in the sector  $[e_2(w), e_{wx}]$ . Apply the coloring a.2.1 on  $G$ . Cycle  $C'_1$  is replaced by  $C_1 = C'_1 \setminus \{w\} \cup \{u, y\}$  or  $C_1 = C'_1 \setminus \{w\} \cup \{x, v, y\}$ . Cycle  $C'_0$  is replaced by  $C_0 = C'_0 \setminus \{w\} \cup \{u, x\}$  or  $C_0 = C'_0 \setminus \{w\} \cup \{y, u, x\}$ . Cycle  $C'_2$  is replaced by  $C_2 = C'_2 \setminus \{w\} \cup \{v, u\}$  or  $C_2 = C'_2 \setminus \{w\} \cup \{y, u\}$ . So  $C_0, C_1, C_2$  all intersect each other and (T2') is satisfied.

The case where  $C'_1$  is entering  $w$  in the sector  $[e_{wx}, e_0(w)]$  is completely symmetric and we apply the coloring a.2.2 on  $G$ .

- Cases a.3, a.4, a.5:

The proof is simpler for the remaining cases (cases a.3.0, a.4.0, a.5.0 on Figure 17). For each situation, there is only one way to extend the coloring to  $G$  in order to preserve (T1') and this coloring also preserve (T2'). Indeed, in each coloring of  $G$ , case a.3.1, a.4.1, a.5.1 on Figure 17, one can check that every non safe monochromatic cycle  $C'$  is replaced by a cycle  $C$  with  $C' \setminus \{w\} \cup \{u\} \subseteq C$ . Thus all non safe cycles intersects on  $u$  and a non safe cycle of color  $i$  intersect all safe cycles of colors  $i - 1$  and  $i + 1$ . So (T2') is always satisfied.

It remains to analyze the situation for the decontraction case (b) and (c). The colorings that are needed are represented on Figures 18 and 19. The proof is similar to case (a) and we omit it.  $\square$ .

Note that we are not able to prove a lemma analogous to Lemma 15 for Type 2 Schnyder woods. On the example of Figure 22, it is not possible to decontract the graph  $G'$  (Figure 22.(a)), and extend its Schnyder wood to  $G$  (Figure 22.(b)) without modifying the edges that are not incident to the contracted edge  $e$ . Indeed, if we keep the edges non incident to  $e$  unchanged, there are only two possible ways to extend the coloring in order to preserve (T1), but none of them fulfills (T2).

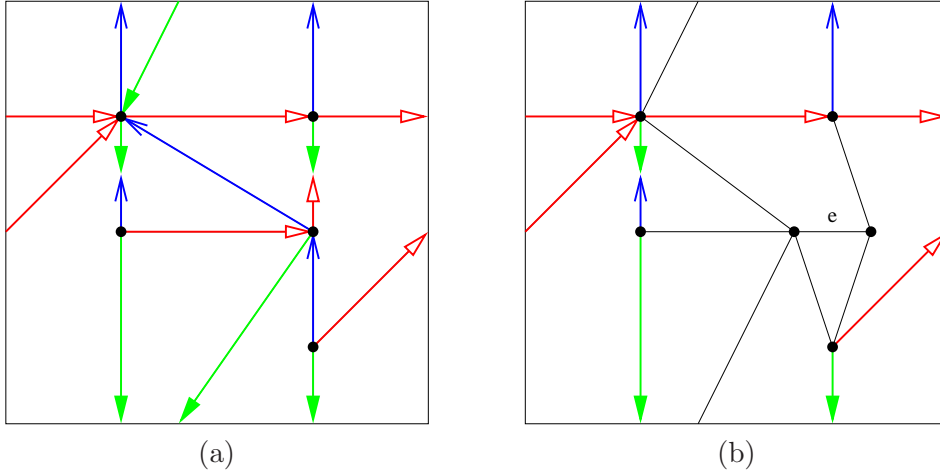


Figure 22: (a) The graph obtained by contracting the edge  $e$  of the graph (b). It is not possible to color and orient the black edges of (b) to obtain a Schnyder wood.

## 9 Existence for essentially 3-connected toroidal maps

Given a map  $G$  embedded on a surface. The *angle map* [22] of  $G$  is a map  $A(G)$  on this surface whose vertices are the vertices of  $G$  plus the vertices of  $G^*$  (i.e. the faces of  $G$ ),

and whose edges are the angles of  $G$ , each angle being incident with the corresponding vertex and face of  $G$ . Note that if  $G$  contains no homotopic multiple edges, then every face of  $G$  has degree at least 3 in  $A(G)$ .

Mohar and Rosenstiehl [23] proved that a map  $G$  is essentially 2-connected if and only if the angle map  $A$  of  $G$  has no pair of (multiple) edges bounding a disk (i.e. no walk of length 2 bounding a disk). As every face in an angle map is a quadrangle, such disk contains some vertices of  $G$ . The following claim naturally extends this characterisation to essentially 3-connected toroidal maps.

**Lemma 18** *A toroidal map  $G$  is essentially 3-connected if and only if the angle map  $A(G)$  has no walk of length at most four bounding a disk which is not a face.*

*Proof.* In  $G^\infty$  any minimal separator of size 1 or 2,  $S = \{v_1\}$  or  $S = \{v_1, v_2\}$ , corresponds to a separating cycle of length 2 or 4 in  $A(G^\infty)$ ,  $C = (v_1, f_1)$  or  $C = (v_1, f_1, v_2, f_2)$ , i.e. a cycle of length at most 4 bounding a disk  $D$  which is not a face.

( $\implies$ ) Any walk in  $A(G)$  of length at most 4 bounding a disk which is not a face lifts to a cycle of length at most 4 bounding a disk which is not a face in  $A(G^\infty)$ . Thus such a walk implies the existence of a small separator in  $G^\infty$ , contradicting its 3-connectedness.

( $\impliedby$ ) According to [23], if  $G$  is essentially 2-connected,  $A(G)$  has no walk of length 2 bounding a disk. Let us now show that if  $G$  is essentially 2-connected but not essentially 3-connected,  $A(G)$  has a walk of length 4 bounding a disk which is not a face.

If  $G$  is essentially 2-connected but not essentially 3-connected, then  $A(G^\infty)$  has a cycle  $C$  of length 4 bounding a disk which is not a face, and this cycle corresponds to a contractible walk  $W$  of length 4 in  $A(G)$ . Since,  $W$  is contractible, it contains a subwalk bounding a disk.  $A(G)$  being bipartite, this subwalk has even length, and since  $A(G)$  is essentially 2-connected, it has no such walk of length 2. Thus  $W$  bounds a disk. Finally, this disk is not a single face since otherwise  $C$  would bound a single face in  $A(G^\infty)$ .  $\square$

A non-loop edge  $e$  of an essentially 3-connected toroidal map is *contractible* if the contraction of  $e$  keeps the map essentially 3-connected. We have the following lemma:

**Lemma 19** *An essentially 3-connected toroidal map that is not reduced to a single vertex has a contractible edge.*

*Proof.* Let  $G$  be an essentially 3-connected toroidal map with at least 2 vertices. Note that for any non-loop  $e$ , the map  $A(G/e)$  has no walk of length 2 bounding a disk which is not a face, otherwise,  $A(G)$  contains a walk of length at most 4 bounding a disk which is not a face and thus, by Lemma 18,  $G$  is not essentially 3-connected.

Suppose by contradiction that contracting any non-loop edge  $e$  of  $G$  yields a non essentially 3-connected map  $G/e$ . By Lemma 18, it means that the angle map  $A(G)$  has no walk of length at most four bounding a disk which is not a face. For any non-loop  $e$ , let  $W_4(e)$  be the 4-walk of  $A(G/e)$  bounding a disk, which is maximal in terms of

the faces it contains. Among all the non-loop edges, let  $e$  be the one such that the number of faces in  $W_4(e)$  is minimum. Let  $W_4(e) = (v_1, f_1, v_2, f_2)$  and assume that the endpoints of  $e$ , say  $a$  and  $b$ , are contracted into  $v_2$  (see Figure 23.(a)). Note that by maximality of  $W_4(e)$ ,  $v_1$  and  $v_2$  do not have any common neighbor  $f$  out of  $W_4(e)$ , such that  $(v_1, f, v_2, f_1)$  bounds a disk.

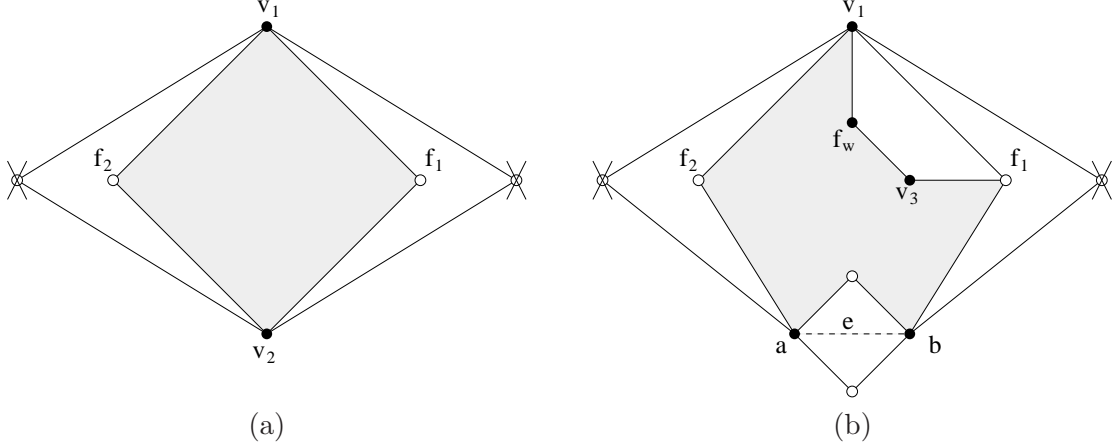


Figure 23: Notations of the proof of Lemma 19

Assume one of  $f_1$  or  $f_2$  has a neighbor inside  $W_4(e)$ . By symmetry, assume  $v_3$  is a vertex inside  $W_4(e)$  such that there is a face  $F = (v_1, f_1, v_3, f_w)$  in  $A(G/e)$ , with eventually  $f_w = f_2$ . Consider now the contraction of the edge  $v_1v_3$ . Let  $P(v_1, v_3) = (v_1, f_x, v_y, f_z, v_3)$  be the path from  $v_1$  to  $v_3$  corresponding to  $W_4(v_1v_3)$  and  $P(a, b) = (a, f_2, v_1, f_1, b)$  the path corresponding to  $W_4(e)$ . Suppose that  $f_z = f_2$ , then  $(v_1, f_2, v_3, f_w)$  bounds a face by Lemma 18 and then  $f_w$  has degree 2 in  $A(G)$ , a contradiction. So  $f_z \neq f_2$ .

Suppose that all the faces of  $W_4(v_1v_3)$  are in  $W_4(e)$ , then with  $F$ ,  $W_4(e)$  contains more faces than  $W_4(v_1v_3)$ , a contradiction to the choice of  $e$ . So in  $A(G)$ ,  $P(v_1, v_3)$  must cross the path  $P(a, b)$ . Then  $v_y$  or  $f_z$  have to intersect  $P(a, b)$ . Suppose  $f_z \neq f_1$ . Then  $v_y = a$  or  $b$ . In this case  $(v_1, f_x, v_2, f_1)$  bounds a disk, a contradiction. Thus  $f_z = f_1$  and the cycle  $(v_1, f_x, v_y, f_1)$  bounds a face by Lemma 18. This implies that  $W_4(v_1v_3)$  bounds a face, a contradiction.

Assume now that none of  $f_1$  or  $f_2$  has a neighbor inside  $W_4(e)$ . Let  $f'_1, f'_2, f_3$  and  $f'_3$  be vertices of  $A(G)$  such that  $(v_1, f_1, b, f'_1)$ ,  $(v_1, f_2, a, f'_2)$  and  $(a, f_3, b, f'_3)$  are faces (see Figure 24). Suppose  $f'_1 = f'_2 = f'_3$ . Then in  $A(G/e)$ , the face  $f'_1$  is deleted (among the 2 homotopic multiple edges between  $v_1, v_2$  that are created, only one is kept in  $G/e$ ). Then  $W_4(e)$  bounds a face, a contradiction. Thus there exists some  $i$  such that  $f'_i \neq f'_{i+1}$ . Assume that  $i = 1$  (resp.  $i = 2$  or  $3$ ), and let  $v_3$  and  $f''$  be such that there is a face  $(v_1, f'_1, v_3, f'')$  in  $A(G)$  (resp.  $(a, f'_2, v_3, f'')$  or  $(b, f'_3, v_3, f'')$ ). As above considering the contraction of the edge  $v_1v_3$  (resp.  $av_3$  or  $bv_3$ ) yields a contradiction.  $\square$

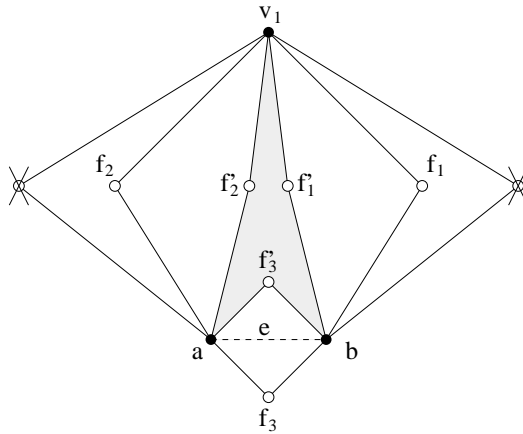


Figure 24: Notations of the proof of Lemma 19

Lemma 19 shows that an essentially 3-connected toroidal map can be contracted step by step by keeping it essentially 3-connected until obtaining a map with just one vertex. The two essentially 3-connected toroidal maps on one vertex are represented on Figure 25 with a Schnyder wood. The graph of Figure 25.(a), the *3-loops*, admits a Schnyder wood of Type 1, and the graph of Figure 25.(b), the *2-loops*, admits a Schnyder wood of Type 2.

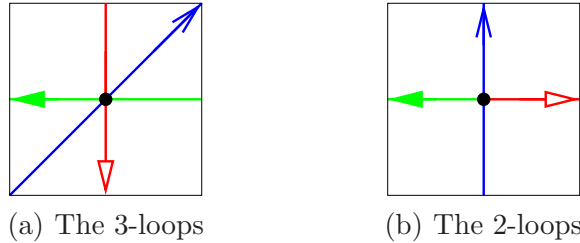


Figure 25: The two essentially 3-connected toroidal maps on one vertex.

It would be convenient if one could contract any essentially 3-connected toroidal map until obtaining one of the two graphs of Figure 25 and then decontract the graph to obtain a Schnyder wood of the original graph. Unfortunately, for Type 2 Schnyder woods we are not able to prove that property (T2) can be preserved during the decontraction process (see Section 8). Fortunately most essentially 3-connected toroidal maps admits Schnyder woods of Type 1. A toroidal map is *basic* if it consists of a non contractible cycle on  $n$  vertices,  $n \geq 1$ , plus  $n$  homotopic loops (see Figure 26). We prove in this section that non-basic essentially 3-connected toroidal maps admits Schnyder woods of Type 1. For this purpose, instead of contracting these maps to one of the two graphs of Figure 25, we contract them to the graph of Figure 25.(a) or to the graph of Figure 27, *the brick*. (One can draw the universal cover of the brick to understand its name.)

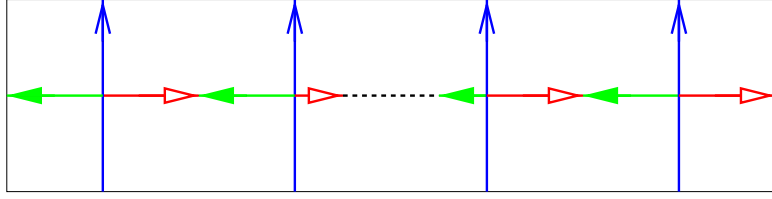


Figure 26: The family of basic toroidal maps, having only Schnyder woods of Type 2

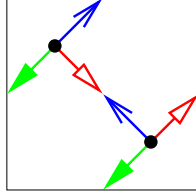


Figure 27: The brick, an essentially 3-connected toroidal map with two vertices.

**Lemma 20** *A non-basic essentially 3-connected toroidal map can be contracted to the 3-loops (Figure 25.(a)) or to the brick (Figure 27).*

*Proof.* Let us prove the lemma by induction on the number of edges of the map. As the 3-loop and the brick are the only non-basic essentially 3-connected toroidal maps with at most 3 edges, the lemma holds for the maps with at most 3 edges. Consider now a non-basic essentially 3-connected toroidal map  $G$  with at least 4 edges. As  $G$  has at least 2 vertices, it has at least one contractible edge by Lemma 19. If  $G$  has a contractible edge  $e$  which contraction yields a non-basic map  $G'$ , then by induction hypothesis on  $G'$  we are done. Let us prove that such an edge always exists. We assume by contradiction, that the contraction of any contractible edge  $e$  yields a basic map  $G'$ . Let us denote  $v_i$ , with  $1 \leq i \leq n$ , the vertices of  $G'$  in such a way that  $(v_1, v_2, \dots, v_n)$  is a cycle of  $G'$ . We can assume that  $v_1$  is the vertex resulting of the contraction of  $e$ . Let  $u$  and  $v$  be the endpoints of  $e$  in  $G$ .

Suppose first that  $u$  or  $v$  is incident to a loop in  $G$ . By symmetry, we can assume that  $v$  is incident to a loop and that  $u$  is in the cylinder between the loops around  $v$  and  $v_n$  (note that if  $n = 1$  then  $v_n = v$ ), and note that  $u$  is the only vertex here. Since  $G$  is non-basic and  $u$  has at least 3 incident edges, two of them go to the same vertex but are non-homothetic. Since after the contraction of  $e$  there is only one edge left in the cylinder, we can deduce that  $u$  has at least two edges in common with  $v$ . On the other side since  $G$  is essentially 3-connected  $u$  has an edge  $e'$  with  $v_n$ . This edge  $e'$  is contractible since its contraction yields a graph containing the basic graph on  $n$  vertices. But since this graph has 2 non-homothetic edges linking  $(uv_n)$  and  $v$ , it is non-basic. So  $G$  has a contractible edge which contraction produces a non-basic graph, contradicting our assumption.

Suppose now that  $u$  and  $v$  do not have incident loop, we thus have that  $G$  contains

a cycle  $C$  of length 2 containing  $e$ . Let  $e'$  be the other edge of  $C$ . Since  $G$  is essentially 3-connected, both  $u$  and  $v$  have at least degree 3, and at least one of them has an incident edge on the left (resp. right) of  $C$ . If  $n = 1$ , since  $G$  has at least 4 edges there are 2 (non-homothetic) edges, say  $f_1$  and  $f_2$  between  $u$  and  $v$  and distinct from  $e$  and  $e'$ . In this case, since the cycles  $(e, f_1)$  and  $(e, f_2)$  were not homothetic, the edges  $f_1$  and  $f_2$  remain non-homothetic in  $G'$ . So in this case  $G$  has one vertex and 3 edges, it is thus non-basic. Assume now that  $n \geq 2$ . In this case  $u$  and  $v$  are contained in a cylinder bordered by the loops at  $v_2$  and at  $v_n$  (with eventually  $n = 2$ ). In this case, we can assume that  $u$  has at least one incident edge  $f_1$  on the left of  $C$  to  $v_n$ , and that  $v$  has at least one incident edge  $f_2$  on the right of  $C$  to  $v_2$ . In this case one can contract  $f_1$  and note that the obtained graph which contain at least 3 non-homothetic edges around  $v$  ( $e$ ,  $e'$  and  $f_2$ ) is essentially 3-connected, and non-basic. So  $G$  has a contractible edge which contraction produces a non-basic graph, contradicting our assumption.  $\square$

**Lemma 21** *A basic toroidal map admits only Schnyder woods of Type 2.*

*Proof.* Basic toroidal map admit Schnyder woods of Type 2 as shown by Figure 26. Suppose that a basic toroidal map  $G$  on  $n$  vertices admits a Schnyder wood of Type 1. Consider one of the vertical loop  $e$  and suppose by symmetry that it is oriented upward in color 1. In a Schnyder wood of type 1, all the monochromatic cycles of different colors are not homotopic, thus all the loops homotopic to  $e$  are also oriented upward in color 1 and they are bi-oriented. It remains just a cycle on  $n$  vertices for edges of color 0 and 2. Thus the Schnyder wood is the one of Figure 26, a contradiction.  $\square$

We are now able to prove the following:

**Theorem 10** *A toroidal graph admits a Schnyder wood of Type 1 if and only if it is an essentially 3-connected non-basic toroidal map.*

*Proof.* ( $\implies$ ) If  $G$  is a toroidal graph given with a Schnyder wood of Type 1. Then, by Lemma 11,  $G$  is essentially 3-connected and by Lemma 21,  $G$  is not basic.

( $\impliedby$ ) Let  $G$  be a non-basic essentially 3-connected toroidal map. By Lemma 20,  $G$  can be contracted to the 3-loops or to the brick. Both of these graphs admit Schnyder woods of type 1 (see Figure 25.(a) and 27). So by Lemma 15 applied successively,  $G$  admits a Schnyder wood of Type 1.  $\square$

Theorem 10 and Lemma 21 imply Theorem 1. One related open problem is to characterize which essentially 3-connected toroidal maps have Schnyder woods of Type 2.

Here is a remark about how to compute a Schnyder wood for an essentially 3-connected toroidal triangulation. Instead of looking carefully at the technical proof of Lemma 15 to know which coloring of the decontracted graph has to be chosen among



the possible choice. One can try the possible cases *a.k.a.*,  $\ell \geq 1$ , and check after which obtained coloring is a Schnyder wood. To do so, one just as to check if (T2') is satisfied. Checking that (T2') is satisfied can be done by the following method: start from any vertex  $v$ , walk along  $P_0(v), P_1(v), P_2(v)$  and mark the three monochromatic cycles  $C_0, C_1, C_2$  reached by the three paths  $P_i$ . Property (T2') is then satisfied if the cycles  $C_0, C_1, C_2$  pairwise intersect.

The existence of Schnyder wood for toroidal triangulations implies the following theorem.

**Theorem 11** *A toroidal triangulation contains three non contractible and non homotopic cycles that are pairwise edge-disjoint.*

*Proof.* One just has to apply Theorem 10 to obtain a Schnyder wood of Type 1 and then, for each color  $i$ , choose arbitrarily a  $i$ -cycle. These cycles are edge-disjoint as, by Euler's formula, there is no bi-oriented edges in Schnyder woods of toroidal triangulations.  $\square$

The conclusion of Theorem 11 is weaker than the one of Theorem 8 but it is not restricted to simple toroidal triangulations. Recall that Theorem 8 is not true for general toroidal triangulations as shown by the graph of Figure 15.

A nonempty family  $\mathcal{R}$  of linear orders on the vertex set  $V$  of a simple graph  $G$  is called a *realizer* of  $G$  if for every edge  $e$ , and every vertex  $x$  not in  $e$ , there is some order  $<_i \in \mathcal{R}$  so that  $y <_i x$  for every  $y \in e$ . The *Dushnik-Miller dimension* [7] of  $G$ , is defined as the least positive integer  $t$  for which  $G$  has a realizer of cardinality  $t$ . Realizers are usually used on finite graphs, but here we allow  $G$  to be an infinite simple graph.

Schnyder woods were originally defined by Schnyder [26] to prove that a finite planar graph  $G$  has Dushnik-Miller dimension at most 3. A consequence of Theorem 1 is an analogous result for the universal cover of a toroidal graph:

**Theorem 12** *The universal cover of a toroidal graph has Dushnik-Miller dimension at most three.*

*Proof.* By eventually adding edges to  $G$  we may assume that  $G$  is a toroidal triangulation. By Theorem 1, it admits a Schnyder wood. For  $i \in \{0, 1, 2\}$ , let  $<_i$  be the order induced by the inclusion of the regions  $R_i$  in  $G^\infty$ . That is  $u <_i v$  if and only if  $R_i(u) \subsetneq R_i(v)$ . Let  $<'_i$  be any linear extension of  $<_i$  and consider  $\mathcal{R} = \{<'_0, <'_1, <'_2\}$ . Let  $e$  be any edge of  $G^\infty$  and  $v$  be any vertex of  $G^\infty$  not in  $e$ . Edge  $e$  is in a region  $R_i(v)$  for some  $i$ , thus  $R_i(u) \subseteq R_i(v)$  for every  $u \in e$  by Lemma 10.(i). As there is no edges oriented in two directions in a Schnyder wood of a toroidal triangulation, we have  $R_i(u) \neq R_i(v)$  and so  $u <_i v$ . Thus  $\mathcal{R}$  is a realizer of  $G^\infty$ .  $\square$

## 10 Orthogonal surfaces

Given two points  $u = (u_0, u_1, u_2)$  and  $v = (v_0, v_1, v_2)$  in  $\mathbb{R}^3$ , we note  $u \vee v = (\max(u_i, v_i))_{i=0,1,2}$  and  $u \wedge v = (\min(u_i, v_i))_{i=0,1,2}$ . We define an order  $\geq$  among the points in  $\mathbb{R}^3$ , in such a way that  $u \geq v$  if  $u_i \geq v_i$  for  $i = 0, 1, 2$ .

Given a set  $\mathcal{V}$  of pairwise incomparable elements in  $\mathbb{R}^3$ , we define the set of vertices that dominates  $\mathcal{V}$  as  $\mathcal{D}_{\mathcal{V}} = \{u \in \mathbb{R}^3 \mid \exists v \in \mathcal{V} \text{ such that } u \geq v\}$ . The *orthogonal surface*  $\mathcal{S}_{\mathcal{V}}$  generated by  $\mathcal{V}$  is the boundary of  $\mathcal{D}_{\mathcal{V}}$ . (Note that orthogonal surfaces are well defined even when  $\mathcal{V}$  is an infinite set.) If  $u, v \in \mathcal{V}$  and  $u \vee v \in \mathcal{S}_{\mathcal{V}}$ , then  $\mathcal{S}_{\mathcal{V}}$  contains the union of the two line segment joining  $u$  and  $v$  to  $u \vee v$ . Such arcs are called *elbow geodesic*. The *orthogonal arc* of  $v \in \mathcal{V}$  in the direction of the standard basis vector  $e_i$  is the intersection of the ray  $v + \lambda e_i$  with  $\mathcal{S}_{\mathcal{V}}$ .

Let  $G$  be a planar map. A *geodesic embedding* of  $G$  on the orthogonal surface  $\mathcal{S}_{\mathcal{V}}$  is a drawing of  $G$  on  $\mathcal{S}_{\mathcal{V}}$  satisfying the following:

- (D1) There is a bijection between the vertices of  $G$  and  $\mathcal{V}$ .
- (D2) Every edge of  $G$  is an elbow geodesic.
- (D3) Every orthogonal arc in  $\mathcal{S}_{\mathcal{V}}$  is part of an edge of  $G$ .
- (D4) There are no crossing edges in the embedding of  $G$  on  $\mathcal{S}_{\mathcal{V}}$ .

Miller [20] (see also [9, 12]) proved that a geodesic embedding of a planar map  $G$  on an orthogonal surface  $\mathcal{S}_{\mathcal{V}}$  induces a Schnyder wood of  $G$ . The edges of  $G$  are colored with the direction of the orthogonal arc contained in the edge. An orthogonal arc intersecting the ray  $v + \lambda e_i$  corresponds to the edge leaving  $v$  in color  $i$ . Edges represented by two orthogonal arcs corresponds to edges oriented in two directions.

Conversely, it has been proved that a Schnyder wood of a planar map  $G$  can be used to obtain a geodesic embedding of  $G$ . Let  $G$  be a planar map given with a Schnyder wood. The method is the following (see [9] for more details): For every vertex  $v$ , one can divide  $G$  into the three regions bounded by the three monochromatic path going out from  $v$ . The *region vector* associated to  $v$  is the vector obtained by counting the number of faces in each of these three regions. The mapping of each vertex on its region vector gives the geodesic embedding. (Note that in this approach, the vertices are all mapped on the same plane as the sum of the coordinates of each region vector is equal to the total number of inner faces of the map.)

Our goal is to generalize geodesic embedding to the torus. More precisely, we want to represent the universal cover of a toroidal map on an infinite and periodic orthogonal surface.

Let  $G$  be a toroidal map. Consider any flat torus representation of  $G$  in a parallelogram  $P$ . The graph  $G^\infty$  is obtained by replicating  $P$  to tile the plane. Given any of these parallelograms  $Q$ , let  $Q^{top}$  (resp.  $Q^{right}$ ) be the copy of  $P$  just above (resp. on the

right of)  $Q$ . Given a vertex  $v$  in  $Q$ , we note  $v^{top}$  (resp.  $v^{right}$ ) its copies in  $Q^{top}$  (resp.  $Q^{right}$ ).

A mapping of the vertices of  $G^\infty$  in  $\mathbb{R}^d$ ,  $d \in \{2, 3\}$ , is *periodic* with respect to vectors  $S$  and  $S'$  of  $\mathbb{R}^d$ , if there exists a flat torus representation  $P$  of  $G$  such that for any vertex  $v$  of  $G^\infty$ , vertex  $v^{top}$  is mapped on  $v + S$  and  $v^{right}$  is mapped on  $v + S'$ . A *geodesic embedding* of a toroidal map  $G$  is a geodesic embedding of  $G^\infty$  on  $\mathcal{S}_{\mathcal{V}^\infty}$ , where  $\mathcal{V}^\infty$  is a periodic mapping of  $G^\infty$  with respect to two non collinear vectors (see example of Figure 28).

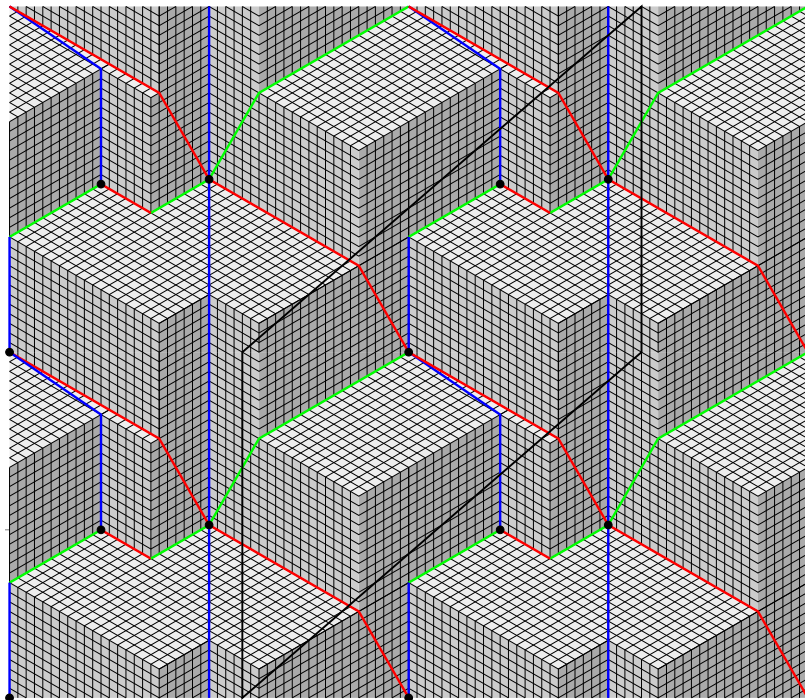


Figure 28: Geodesic embedding of the toroidal map of Figure 2.

Like in the plane Schnyder woods can be used to obtain geodesic embeddings of toroidal maps. For that purpose, we need to generalize the region vector method. The idea is to use the regions  $R_i(v)$  to compute the coordinates of the vertex  $v$  of  $G^\infty$ . The problem is that contrarily to the planar case, these regions are unbounded and contains an infinite number of faces. The method is thus generalized by the following.

Let  $G$  be a toroidal map, given with a Schnyder wood and a flat torus representation in a parallelogram  $P$ .

Recall that  $\mathcal{C}_i = \{C_i^0, \dots, C_i^{k_i-1}\}$  denotes the set of  $i$ -cycles of  $G$  such that there is no  $i$ -cycle in the region  $R(C_i^j, C_i^{j+1})$ . Recall that  $\mathcal{L}_i^j$  denotes the set of  $i$ -lines of  $G^\infty$  corresponding to  $C_i^j$ . The *positive side* of a  $i$ -line is define as the right side while “walking” along the directed path by following the orientation of the edges colored  $i$ .

**Lemma 22** *For any vertex  $v$ , the two monochromatic lines  $L_{i-1}(v)$  and  $L_{i+1}(v)$  intersect. Moreover, if the Schnyder wood is of Type 2.i, then  $L_{i+1}(v) = (L_{i-1}(v))^{-1}$  and  $v$  is situated on the right of  $L_{i+1}(v)$ .*

*Proof.* Let  $j, j'$  be such that  $L_{i-1}(v) \in \mathcal{L}_{i-1}^j$  and  $L_{i+1}(v) \in \mathcal{L}_{i+1}^{j'}$ . If the Schnyder wood is of Type 1 or Type 2.j with  $j \neq i$ , then the two cycles  $C_{i-1}^j$  and  $C_{i+1}^{j'}$  are not homotopic, and so the two lines  $L_{i-1}(v)$  and  $L_{i+1}(v)$  intersect.

If the Schnyder wood is of Type 2.i, we consider the case where  $v \in L_{i-1}(v)$ , and the case where  $v$  does not belong to  $L_{i-1}(v)$  nor  $L_{i+1}(v)$ . Then  $v$  lies between two consecutive  $(i+1)$ -lines (which are also  $(i-1)$ -lines). Let us denote those two lines  $L_{i+1}$  and  $L'_{i+1}$ , such that  $L'_{i+1}$  is situated on the right of  $L_{i+1}$  and  $v \notin L'_{i+1}$ . By property (T1),  $P_{i+1}(v)$  and  $P_{i-1}(v)$  cannot reach  $L'_{i+1}$ . Thus  $L_{i+1} = L_{i+1}(v) = (L_{i-1}(v))^{-1}$ .  $\square$

The *size* of the region  $R(C_i^j, C_i^{j+1})$  of  $G$ , denoted  $f_i^j = |R(C_i^j, C_i^{j+1})|$ , is equal to the number of faces in  $R(C_i^j, C_i^{j+1})$ . Remark that for each color, we have  $\sum_{j=0}^{k_i-1} f_i^j$  equals the total number of faces  $f$  of  $G$ . If  $L$  and  $L'$  are consecutive  $i$ -lines of  $G^\infty$  with  $L \in \mathcal{L}_i^j$  and  $L' \in \mathcal{L}_i^{j+1}$ , then the *size* of the (unbounded) region  $R(L, L')$ , denoted  $|R(L, L')|$ , is equal to  $f_i^j$ . If  $L$  and  $L'$  are any  $i$ -lines, the *size* of the (unbounded) region  $R(L, L')$ , denoted  $|R(L, L')|$ , is equal to the sum of the size of all the regions delimited by consecutive  $i$ -lines inside  $R(L, L')$ . For each color  $i$ , choose arbitrarily a  $i$ -line  $L_i^*$  in  $\mathcal{L}_i^0$  that is used as an origin for  $i$ -lines. Given a  $i$ -line  $L$ , we define the value  $f_i(L)$  of  $L$  as follows:  $f_i(L) = |R(L, L_i^*)|$  if  $L$  is on the positive side of  $L_i^*$  and  $f_i(L) = -|R(L, L_i^*)|$  otherwise.

Consider two vertices  $u, v$  such that  $L_{i-1}(u) = L_{i-1}(v)$  and  $L_{i+1}(u) = L_{i+1}(v)$ . Even if the two regions  $R_i(u)$  and  $R_i(v)$  are unbounded, their *difference* is bounded. Let  $d_i(u, v)$  be the number of faces in  $R_i(u) \setminus R_i(v)$  minus the number of faces in  $R_i(v) \setminus R_i(u)$ . For any vertex, by Lemma 22, there exists  $z_i(v)$  a vertex on the intersection of the two lines  $L_{i-1}(v)$  and  $L_{i+1}(v)$ . Let  $N$  be a constant  $\geq n$  (in this section we can have  $N = n$  but in Section 12 we need to choose  $N$  bigger). We are now able to define the region vector of a vertex of  $G^\infty$ , that is a mapping of this vertex in  $\mathbb{R}^3$ .

**Definition 4 (Region vector)** *The  $i$ -th coordinate of the region vector of a vertex  $v$  of  $G^\infty$  is equal to  $v_i = d_i(v, z_i(v)) + N \times (f_{i+1}(L_{i+1}(v)) - f_{i-1}(L_{i-1}(v)))$  (see Figure 29).*

**Lemma 23** *The sum of the coordinates of a vertex  $v$  equals the number of faces in the bounded region delimited by the lines  $L_0(v)$ ,  $L_1(v)$  and  $L_2(v)$  if the Schnyder wood is of Type 1 and this sum equals zero if the Schnyder wood is of Type 2.*

*Proof.* We have  $v_0 + v_1 + v_2 = d_0(v, z_0(v)) + d_1(v, z_1(v)) + d_2(v, z_2(v)) = \sum_i (|R_i(v) \setminus R_i(z_i(v))| - |R_i(z_i(v)) \setminus R_i(v)|)$ . We use the characteristic function  $\mathbf{1}$  to deal with infinite regions. We note  $\mathbf{1}(R)$ , the function defined on the faces of  $G^\infty$  that has value 1 on

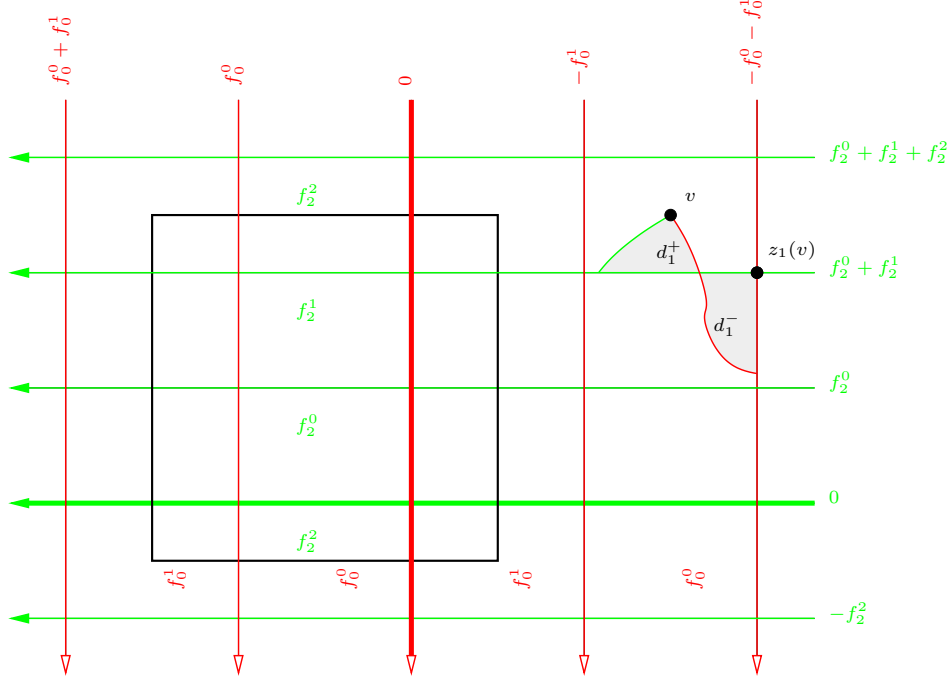


Figure 29: Coordinate 1 of vertex  $v$ , is equal to the number of faces in the region  $d_1^+$ , minus the number of faces in the region  $d_1^-$ , plus  $N$  times  $(f_2^0 + f_2^1) - (-f_0^0 - f_0^1)$ .

each face of region  $R$  and 0 elsewhere. Given a function  $g : F(G^\infty) \rightarrow \mathbb{Z}$ , we note  $|g| = \sum_{F \in F(G^\infty)} g(F)$  (when the sum is finite). Thus  $\sum_i v_i = \sum_i (|\mathbf{1}(R_i(v) \setminus R_i(z_i(v)))| - |\mathbf{1}(R_i(z_i(v)) \setminus R_i(v))|) = |\sum_i (\mathbf{1}(R_i(v) \setminus R_i(z_i(v))) - \mathbf{1}(R_i(z_i(v)) \setminus R_i(v)))|$ . Now we compute  $g = \sum_i (\mathbf{1}(R_i(v) \setminus R_i(z_i(v))) - \mathbf{1}(R_i(z_i(v)) \setminus R_i(v)))$ . We have:

$$g = \sum_i (\mathbf{1}(R_i(v) \setminus R_i(z_i(v))) + \mathbf{1}(R_i(v) \cap R_i(z_i(v))) - \mathbf{1}(R_i(z_i(v)) \setminus R_i(v)) - \mathbf{1}(R_i(v) \cap R_i(z_i(v))))$$

As  $R_i(v) \setminus R_i(z_i(v))$  and  $R_i(z_i(v)) \setminus R_i(v)$  are disjoint from  $R_i(v) \cap R_i(z_i(v))$ , we have

$$g = \sum_i (\mathbf{1}(R_i(v)) - \mathbf{1}(R_i(z_i(v)))) = \sum_i \mathbf{1}(R_i(v)) - \sum_i \mathbf{1}(R_i(z_i(v)))$$

The interior of the three regions  $R_i(v)$ , for  $i = 0, 1, 2$ , being disjoint and spanning the whole plane  $\mathbb{P}$  (by definition), we have  $\sum_i \mathbf{1}(R_i(v)) = \mathbf{1}(\cup_i R_i(v)) = \mathbf{1}(\mathbb{P})$ . Moreover the regions  $R_i(z_i(v))$ , for  $i = 0, 1, 2$ , are also disjoint and  $\sum_i \mathbf{1}(R_i(z_i(v))) = \mathbf{1}(\cup_i R_i(z_i(v))) = \mathbf{1}(\mathbb{P} \setminus T)$  where  $T$  is the bounded region delimited by the lines  $L_0(v)$ ,  $L_1(v)$  and  $L_2(v)$ . So  $g = \mathbf{1}(\mathbb{P}) - \mathbf{1}(\mathbb{P} \setminus T) = \mathbf{1}(T)$ . And thus  $\sum_i v_i = |g| = |\mathbf{1}(T)|$ .  $\square$

Lemma 23 shows that if the Schnyder wood is of Type 1, then the set of points are not necessarily coplanar like in the planar case [13], but all the copies of a vertex lies on the same plane (the bounded region delimited by the lines  $L_0(v)$ ,  $L_1(v)$  and  $L_2(v)$ ) has

the same number of faces for any copies of a vertex  $v$ ). Surprisingly, for Schnyder woods of Type 2, all the points are coplanar.

For each color  $i$ , let  $c_i$  (resp.  $c'_i$ ), be the algebraic number of times a  $i$ -cycle is traversing the vertical (resp. horizontal) side of the parallelogram  $P$  (that was the parallelogram containing the flat torus representation of  $G$ ) from right to left (resp. from top to bottom). This number increases by one each time a monochromatic cycle traverses the side in the given direction and decreases by one when it traverses in the other way. Let  $S$  and  $S'$  be the two vectors of  $\mathbb{R}^3$  with coordinates  $S_i = N(c_{i+1} - c_{i-1})f$  and  $S'_i = N(c'_{i+1} - c'_{i-1})f$ . Note that  $S_0 + S_1 + S_2 = 0$  and  $S'_0 + S'_1 + S'_2 = 0$

**Lemma 24** *The mapping is periodic with respect to  $S$  and  $S'$ .*

*Proof.* Let  $v$  be any vertex of  $G^\infty$ . Then  $v_i^{top} - v_i = N(f_{i+1}(L_{i+1}(v^{top})) - f_{i+1}(L_{i+1}(v))) - N(f_{i-1}(L_{i-1}(v^{top})) - f_{i-1}(L_{i-1}(v))) = N(c_{i+1} - c_{i-1})f$ . So  $v^{top} = v + S$ . Similarly  $v^{right} = v + S'$ .  $\square$

For each color  $i$ , let  $\gamma_i$  be the integer such that two monochromatic cycles of  $G$  of respective colors  $i - 1$  and  $i + 1$  intersect exactly  $\gamma_i$  times, with the convention that  $\gamma_i = 0$  if the Schnyder wood is of Type 2.i. By Lemma 5,  $\gamma_i$  is properly defined and do not depend on the choice of the monochromatic cycles. Note that if the Schnyder wood is of Type 2.i, then  $\gamma_{i-1} = \gamma_{i+1}$  and if the Schnyder wood is not of Type 2.i, then  $\gamma_i \neq 0$ . Let  $\gamma = \max(\gamma_0, \gamma_1, \gamma_2)$ . Let  $Z_0 = ((\gamma_1 + \gamma_2)Nf, -\gamma_1Nf, -\gamma_2Nf)$  and  $Z_1 = (-\gamma_0Nf, (\gamma_0 + \gamma_2)Nf, -\gamma_2Nf)$  and  $Z_2 = (-\gamma_0Nf, -\gamma_1Nf, (\gamma_0 + \gamma_1)Nf)$ .

**Lemma 25** *For any vertex  $u$ , we have  $\{u + k_0Z_0 + k_1Z_1 + k_2Z_2 \mid k_0, k_1, k_2 \in \mathbb{Z}\} \subseteq \{u + kS + k'S' \mid k, k' \in \mathbb{Z}\}$ .*

*Proof.* Let  $u, v$  be two copies of the same vertex, such that  $v$  is the first copy of  $u$  in the direction of  $L_0(u)$ . (That is  $L_0(u) = L_0(v)$  and on the path  $P_0(u) \setminus P_0(v)$  there is no two copies of the same vertex.) Then  $v_i - u_i = N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) - N(f_{i-1}(L_{i-1}(v)) - f_{i-1}(L_{i-1}(u)))$ . We have  $|R(L_0(v), L_0(u))| = 0$ ,  $|R(L_1(v), L_1(u))| = \gamma_2f$  and  $|R(L_2(v), L_2(u))| = \gamma_1f$ . So  $v_0 - u_0 = N(\gamma_1 + \gamma_2)f$  and  $v_1 - u_1 = -N\gamma_1f$  and  $v_2 - u_2 = -N\gamma_2f$ . So  $v = u + Z_0$ . Similarly for the other colors. So the first copy of  $u$  in the direction of  $L_i(u)$  is equal to  $u + Z_i$ . By Lemma 24, all the copies of  $u$  are mapped on  $\{u + kS + k'S' \mid k, k' \in \mathbb{Z}\}$ , so we have the result.  $\square$

**Lemma 26** *We have  $\dim(Z_0, Z_1, Z_2) = 2$  and if the Schnyder wood is not of Type 2.i, then  $\dim(Z_{i-1}, Z_{i+1}) = 2$ .*

*Proof.* We have  $\gamma_0Z_0 + \gamma_1Z_1 + \gamma_2Z_2 = 0$  and so  $\dim(Z_0, Z_1, Z_2) \leq 2$ . We can assume by symmetry that the Schnyder wood is not of Type 2.1 and so  $\gamma_1 \neq 0$ . Thus  $Z_0 \neq 0$  and  $Z_2 \neq 0$ . Suppose by contradiction that  $\dim(Z_0, Z_2) = 1$ . Then there exist  $\alpha \neq 0$ ,

$\beta \neq 0$ , such that  $\alpha Z_0 + \beta Z_2 = 0$ . The sum of this equation for the coordinates 0 and 2 gives  $(\alpha + \beta)\gamma_1 = 0$  and thus  $\alpha = -\beta$ . Then the equation for coordinate 0 gives  $\gamma_0 + \gamma_1 + \gamma_2 = 0$  contradicting the fact that  $\gamma_1 > 1$  and  $\gamma_0, \gamma_2 \geq 0$ .  $\square$

**Lemma 27** *The vectors  $S, S'$  are not collinear.*

*Proof.* By Lemma 25, the set  $\{u + k_0 Z_0 + k_1 Z_1 + k_2 Z_2 \mid k_0, k_1, k_2 \in \mathbb{Z}\}$  is a subset of  $\{u + kS + k'S' \mid k, k' \in \mathbb{Z}\}$ . By Lemma 26, we have  $\dim(Z_0, Z_1, Z_2) = 2$ , thus  $\dim(S, S') = 2$ .  $\square$

**Lemma 28** *If  $u, v$  are two distinct vertices such that  $v$  is in  $L_{i-1}(v)$ ,  $u$  is in  $P_{i-1}(v)$ , both  $u$  and  $v$  are in the region  $R(L_{i+1}(u), L_{i+1}(v))$  and  $L_{i+1}(u)$  and  $L_{i+1}(v)$  are two consecutive  $(i+1)$ -lines with  $L_{i+1}(u) \in \mathcal{L}_{i+1}^j$  (see Figure 30). Then  $d_i(z_i(v), v) + d_i(u, z_i(u)) < (n-1) \times f_{i+1}^j$ .*

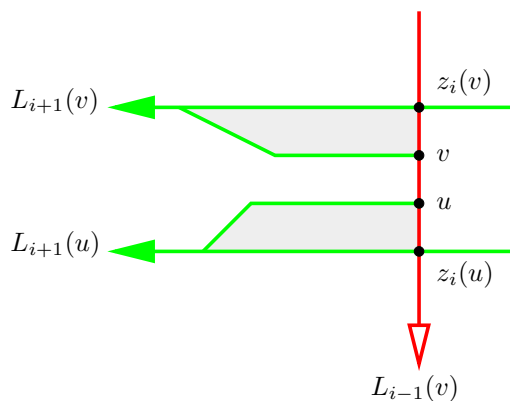


Figure 30: The gray area, corresponding to the quantity  $d_i(z_i(v), v) + d_i(u, z_i(u))$ , has size bounded by  $(n-1) \times f_{i+1}^j$ .

*Proof.* Let  $Q_{i+1}(v)$  the subpath of  $P_{i+1}(v)$  between  $v$  and  $L_{i+1}(v)$  (maybe  $Q_{i+1}(v)$  has length 0 if  $v = z_i(v)$ ). Let  $Q_{i+1}(u)$  the subpath of  $P_{i+1}(u)$  between  $u$  and  $L_{i+1}(u)$  (maybe  $Q_{i+1}(u)$  has length 0 if  $u = z_i(u)$ ). The path  $Q_{i+1}(v)$  cannot contain two different copies of a vertex of  $G$ , otherwise  $Q_{i+1}(v)$  will correspond to a non contractible cycle of  $G$  and thus will contain an edge of  $L_{i+1}(v)$ . So the length of  $Q_{i+1}(v)$  is  $\leq n-1$ .

The total number of times a copy of a given face of  $G$  can appear in the region  $R = R_i(z_i(v)) \setminus R_i(v)$ , corresponding to  $d_i(z_i(v), v)$ , can be bounded as follow. Region  $R$  is between two consecutive copies of  $L_{i+1}(u)$ . So in  $R$ , all the copies of a given face are separated by a copy of  $L_{i-1}(v)$ . Each copy of  $L_{i-1}(v)$  intersecting  $R$  have to intersect  $Q_{i+1}(v)$  on a specific vertex. As  $Q_{i+1}(v)$  has at most  $n$  vertices. A given face can appear at most  $n-1$  times in  $R$ . Similarly, the total number of times that a copy of a given

face of  $G$  can appear in the region  $R_i(u) \setminus R_i(z_i(u))$ , corresponding to  $d_i(u, z_i(u))$ , is  $\leq (n-1)$ .

A given face of  $G$  can appear in only one of the two gray regions of Figure 30. So a face is counted  $\leq n-1$  times in the quantity  $d_i(z_i(v), v) + d_i(u, z_i(u))$ . Only the faces of the region  $R(C_{i+1}^j, C_{i+1}^{j+1})$  can be counted. And there is at least one face of  $R(C_{i+1}^j, C_{i+1}^{j+1})$  (for example one incident to  $v$ ) that is not counted. So in total  $d_i(z_i(v), v) + d_i(u, z_i(u)) \leq (n-1) \times (f_{i+1}^j - 1) < (n-1) \times f_{i+1}^j$ .  $\square$

Clearly, the symmetric of Lemma 28, where the role of  $i+1$  and  $i-1$  are exchanged, is also true.

The bound of Lemma 28 is somehow sharp. In the example of Figure 31, the rectangle represent a toroidal map  $G$  and the universal cover is partially represented. If the map  $G$  has  $n$  vertices and  $f$  faces ( $n = 5$  and  $f = 5$  in the example), then the gray region, representing the quantity  $d_1(z_1(v), v) + d_1(u, z_1(u))$ , has size  $\frac{n(n-1)}{2} = \Omega(n \times f)$ .

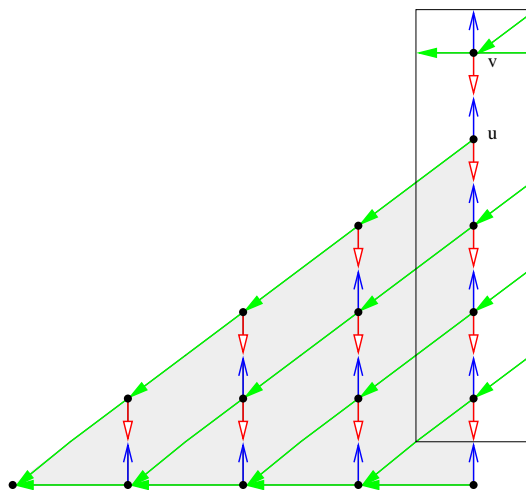


Figure 31: Example of a toroidal map where  $d_1(u, z_1(u))$  has size  $\Omega(n \times f)$ .

**Lemma 29** *Let  $u, v$  be vertices of  $G^\infty$  such that  $R_i(u) \subseteq R_i(v)$ , then  $u_i \leq v_i$ . Moreover if  $R_i(u) \subsetneq R_i(v)$ , then  $v_i - u_i > (N-n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|) \geq 0$ .*

*Proof.* We distinguish two cases depending of the fact that the Schnyder wood is of type 2.i or not.

- *Case 1: The Schnyder wood is not of Type 2.i.*

Suppose first that  $u$  and  $v$  are both in a region delimited by two consecutive lines of color  $i-1$  and two consecutive lines of color  $i+1$ . Let  $L_{i-1}^j, L_{i-1}^{j+1}, L_{i+1}^{j'}, L_{i+1}^{j'+1}$  be these lines such that  $L_{i-1}^{j+1}$  is on the positive side of  $L_{i-1}^j$ ,  $L_{i+1}^{j'+1}$  is on the positive side



of  $L_{i+1}^{j'}$ , and  $L_k^\ell \in \mathcal{L}_k^\ell$  (see Figure 32). We distinguish cases corresponding to equality or not between lines  $L_{i-1}(u)$ ,  $L_{i-1}(v)$  and  $L_{i+1}(u)$ ,  $L_{i+1}(v)$ .

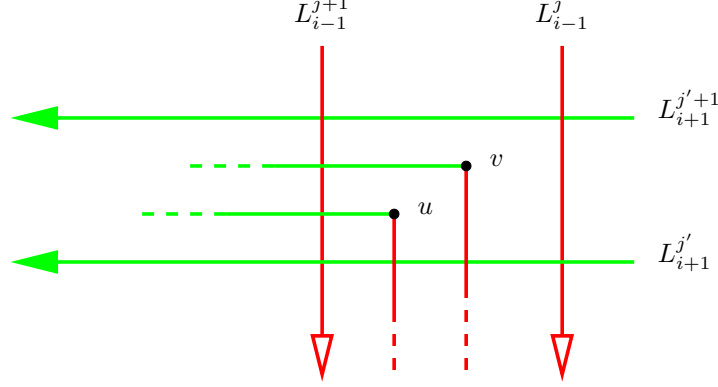


Figure 32: Position of  $u$  and  $v$  in the proof of Lemma 29

★ *Case 1.1:*  $L_{i-1}(u) = L_{i-1}(v)$  and  $L_{i+1}(u) = L_{i+1}(v)$ . Then  $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) = d_i(v, u)$ . Thus clearly, if  $R_i(u) \subseteq R_i(v)$ , then  $u_i \leq v_i$  and if  $R_i(u) \subsetneq R_i(v)$ ,  $v_i - u_i > 0 = (N - n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|)$ .

★ *Case 1.2:*  $L_{i-1}(u) = L_{i-1}(v)$  and  $L_{i+1}(u) \neq L_{i+1}(v)$ . As  $u \in R_i(v)$ , we have  $L_{i+1}(u) = L_{i+1}^{j'}$  and  $L_{i+1}(v) = L_{i+1}^{j'+1}$ . Then  $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'}$ . Let  $u'$  be the intersection of  $P_{i+1}(u)$  with  $L_{i-1}^{j+1}$  (maybe  $u = u'$ ). Let  $v'$  be the intersection of  $P_{i+1}(v)$  with  $L_{i-1}^{j+1}$  (maybe  $v = v'$ ). Since  $L_{i+1}(u) \neq L_{i+1}(v)$ , we have  $u' \neq v'$ . Since  $u \in R_i(v)$ , we have  $u' \in R_i(v')$  and so  $u' \in P_{i-1}(v')$ . Then, by Lemma 28,  $d_i(z_i(v'), v') + d_i(u', z_i(u')) < (n - 1)f_{i+1}^{j'}$ . If  $L_{i-1}(u) = L_{i-1}^{j+1}$  then, one can see that  $d_i(v, z_i(v)) - d_i(u, z_i(u)) \geq d_i(v', z_i(v')) - d_i(u', z_i(u'))$ . If  $L_{i-1}(u) = L_{i-1}^j$ , one can see that  $d_i(v, z_i(v)) - d_i(u, z_i(u)) \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) - f_{i+1}^{j'}$ . So finally,  $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'} \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) + (N - 1)f_{i+1}^{j'} > (N - n)f_{i+1}^{j'} = (N - n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|) \geq 0$ .

★ *Case 1.3:*  $L_{i-1}(u) \neq L_{i-1}(v)$  and  $L_{i+1}(u) = L_{i+1}(v)$ . This case is completely symmetric to the previous case.

★ *Case 1.4:*  $L_{i-1}(u) \neq L_{i-1}(v)$  and  $L_{i+1}(u) \neq L_{i+1}(v)$ . As  $u \in R_i(v)$ , we have  $L_{i+1}(u) = L_{i+1}^{j'}$ ,  $L_{i+1}(v) = L_{i+1}^{j'+1}$ ,  $L_{i-1}(u) = L_{i-1}^{j+1}$ , and  $L_{i-1}(v) = L_{i-1}^j$ . Then  $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) - N(f_{i-1}(L_{i-1}(v)) - f_{i-1}(L_{i-1}(u))) = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'} + Nf_{i-1}^j$ . Let  $u'$  be the intersection of  $P_{i+1}(u)$  with  $L_{i-1}^{j+1}$  (maybe  $u = u'$ ). Let  $u''$  be the intersection of  $P_{i-1}(u)$  with  $L_{i+1}^{j'}$  (maybe  $u = u''$ ). Let  $v'$  be the intersection of  $P_{i+1}(v)$  with  $L_{i-1}^{j+1}$  (maybe  $v = v'$ ). Let  $v''$  be the intersection of  $P_{i-1}(v)$  with  $L_{i+1}^{j'+1}$  (maybe  $v = v''$ ). Since  $L_{i+1}(u) \neq L_{i+1}(v)$ , we have  $u' \neq v'$ . Since  $u \in R_i(v)$ , we have  $u' \in R_i(v')$  and so  $u' \in P_{i-1}(v')$ . Then, by Lemma 28,  $d_i(z_i(v'), v') +$

$d_i(u', z_i(u')) < (n-1)f_{i+1}^{j'}$ . Symmetrically,  $d_i(z_i(v''), v'') + d_i(u'', z_i(u'')) < (n-1)f_{i-1}^j$ . Moreover, we have  $d_i(v, z_i(v)) - d_i(u, z_i(u)) \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) + d_i(v'', z_i(v'')) - d_i(u'', z_i(u'')) - f_{i+1}^{j'} - f_{i-1}^j$ . So finally,  $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + Nf_{i+1}^{j'} + Nf_{i-1}^j \geq d_i(v', z_i(v')) - d_i(u', z_i(u')) + d_i(v'', z_i(v'')) - d_i(u'', z_i(u'')) + (N-1)f_{i+1}^{j'} + (N-1)f_{i-1}^j > (N-n)f_{i+1}^{j'} + (N-n)f_{i-1}^j = (N-n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|) \geq 0$ .

Suppose now that  $u$  and  $v$  do not lie in a region delimited by two consecutive lines of color  $i-1$  and/or in a region delimited by two consecutive lines of color  $i+1$ . One can easily find distinct vertices  $w_0, \dots, w_r$  ( $w_i$ ,  $1 \leq i < r$  chosen at intersections of monochromatic lines of colors  $i-1$  and  $i+1$ ) such that  $w_0 = u$ ,  $w_r = v$ , and for  $0 \leq \ell \leq r-1$ , we have  $R_i(w_\ell) \subsetneq R_i(w_{\ell+1})$  and  $w_\ell, w_{\ell+1}$  are both in a region delimited by two consecutive lines of color  $i-1$  and in a region delimited by two consecutive lines of color  $i+1$ . Thus by the first part of the proof,  $(w_\ell)_i - (w_{\ell+1})_i > (N-n)(|R(L_{i-1}(w_{\ell+1}), L_{i-1}(w_\ell))| + |R(L_{i+1}(w_{\ell+1}), L_{i+1}(w_\ell))|)$ . Thus  $v_i - u_i > (N-n) \sum_{\ell} (|R(L_{i-1}(w_{\ell+1}), L_{i-1}(w_\ell))| + |R(L_{i+1}(w_{\ell+1}), L_{i+1}(w_\ell))|)$ . For any  $a, b, c$  such  $R_i(a) \subseteq R_i(b) \subseteq R_i(c)$ , we have  $|R(L_j(a), L_j(b))| + |R(L_j(b), L_j(c))| = |R(L_j(a), L_j(c))|$ . Thus we obtain the result by summing the size of the regions.

• *Case 2: The Schnyder wood is of Type 2.i.*

Suppose first that  $u$  and  $v$  are both in a region delimited by two consecutive lines of color  $i+1$ .

Let  $L_{i+1}^j, L_{i+1}^{j+1}$  be these lines such that  $L_{i+1}^{j+1}$  is on the positive side of  $L_{i+1}^j$ , and  $L_{i+1}^\ell \in \mathcal{L}_{i+1}^\ell$ . We can assume that we do not have both  $u$  and  $v$  in  $L_{i+1}^{j+1}$  (by eventually choosing other consecutive lines of color  $i+1$ ). We consider two cases :

★ *Case 2.1:  $v \notin L_{i+1}^{j+1}$ .* Then by Lemma 22,  $L_{i+1}^j = L_{i+1}(u) = (L_{i-1}(u))^{-1} = L_{i+1}(v) = (L_{i-1}(v))^{-1}$ . Then  $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) = d_i(v, u)$ . Thus clearly, if  $R_i(u) \subseteq R_i(v)$ , then  $u_i \leq v_i$  and if  $R_i(u) \subsetneq R_i(v)$ , then  $v_i - u_i > 0 = (N-n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|)$ .

★ *Case 2.2:  $v \in L_{i+1}^{j+1}$ .* Then  $L_{i+1}^{j+1} = L_{i+1}(v) = (L_{i-1}(v))^{-1}$  and  $d_i(v, z_i(v)) = 0$ . By assumption  $u \notin L_{i+1}^{j+1}$  and by Lemma 22,  $L_{i+1}^j = L_{i+1}(u) = (L_{i-1}(u))^{-1}$ . Then  $v_i - u_i = d_i(v, z_i(v)) - d_i(u, z_i(u)) + N(f_{i+1}(L_{i+1}(v)) - f_{i+1}(L_{i+1}(u))) - N(f_{i-1}(L_{i-1}(v)) - f_{i-1}(L_{i-1}(u))) = -d_i(u, z_i(u)) + 2Nf_{i+1}^j$ . Let  $L_i$  and  $L'_i$  be two consecutive  $i$ -lines such that  $u$  lies in the region between them and  $L'_i$  is on the right of  $L_i$ . Let  $u'$  be the intersection of  $P_{i+1}(u)$  with  $L_i$  (maybe  $u = u'$ ). Let  $u''$  be the intersection of  $P_{i-1}(u)$  with  $L'_i$  (maybe  $u = u''$ ). Then, by Lemma 28,  $d_i(u', z_i(u')) < (n-1)f_{i+1}^j$  and  $d_i(u'', z_i(u'')) < (n-1)f_{i+1}^j$ . Thus we have  $d_i(u, z_i(u)) \leq d_i(u', z_i(u')) + d_i(u'', z_i(u'')) + f_{i+1}^j < (2(n-1)+1)f_{i+1}^j$ . So finally,  $v_i - u_i > -(2n-1)f_{i+1}^j + 2Nf_{i+1}^j > 2(N-n)f_{i+1}^j = (N-n)(|R(L_{i-1}(u), L_{i-1}(v))| + |R(L_{i+1}(u), L_{i+1}(v))|) \geq 0$ .

If  $u$  and  $v$  do not lie in a region delimited by two consecutive lines of color  $i+1$ , then as in case 1, one can find intermediate vertices to obtain the result.  $\square$

**Lemma 30** *If two vertices  $u, v$  are adjacent, then for each color  $i$ , we have  $|v_i - u_i| \leq$*

$2Nf$ .

*Proof.* Since  $u, v$  are adjacent, they are both in a region delimited by two consecutive lines of color  $i - 1$  and in a region delimited by two consecutive lines of color  $i + 1$ . Let  $L_{i-1}^j, L_{i-1}^{j+1}$  be these two consecutive lines of color  $i - 1$  and  $L_{i+1}^{j'}, L_{i+1}^{j'+1}$  these two consecutive lines of color  $i + 1$  with  $L_k^\ell \in \mathcal{L}_k^\ell$ ,  $L_{i-1}^{j+1}$  is on the positive side of  $L_{i-1}^j$  and  $L_{i+1}^{j'+1}$  is on the positive side of  $L_{i+1}^{j'}$  (see Figure 32 when the Schnyder wood is not of Type 2.i). If the Schnyder wood is of Type 2.i we assume that  $L_{i-1}^{j+1} = (L_{i+1}^{j'})^{-1}$  and  $L_{i-1}^j = (L_{i+1}^{j'+1})^{-1}$ . Let  $z$  be a vertex on the intersection of  $L_{i-1}^{j+1}$  and  $L_{i+1}^{j'}$ . Let  $z'$  be a vertex on the intersection of  $L_{i-1}^j$  and  $L_{i+1}^{j'+1}$ . Thus we have  $R_i(z) \subseteq R_i(u) \subseteq R_i(z')$  and  $R_i(z) \subseteq R_i(v) \subseteq R_i(z')$ . So by Lemma 29,  $z_i \leq u_i \leq z'_i$  and  $z_i \leq v_i \leq z'_i$ . So  $|v_i - u_i| \leq z'_i - z_i = N(f_{i+1}(L_{i+1}^{j'+1}) - f_{i+1}(L_{i+1}^{j'})) - N(f_{i-1}(L_{i-1}^j) - f_{i-1}(L_{i-1}^{j+1})) = Nf_{i+1}^{j'} + Nf_{i-1}^j \leq 2Nf$ .  $\square$

We are now able to prove the following :

**Theorem 13** *If  $G$  is a toroidal map given with a Schnyder wood, then the mapping of each vertex of  $G^\infty$  on its region vector gives a geodesic embedding of  $G$ .*

*Proof.* By Lemmas 24 and 27, the mapping of  $G^\infty$  on its region vector is periodic with respect to  $S, S'$  that are not collinear. For any pair  $u, v$  of distinct vertices of  $G^\infty$ , by Lemma 10.(iii), there exists  $i, j$  with  $R_i(u) \subsetneq R_i(v)$  and  $R_j(v) \subsetneq R_j(u)$  thus, by Lemma 29,  $u_i < v_i$  and  $v_j < u_j$ . So  $\mathcal{V}^\infty$  is a set of pairwise incomparable elements of  $\mathbb{R}^3$ .

(D1)  $\mathcal{V}^\infty$  is a set of pairwise incomparable elements so the mapping between vertices of  $G^\infty$  and  $\mathcal{V}^\infty$  is a bijection.

(D2) Let  $e = uv$  be an edge of  $G^\infty$ . We show that  $w = u \vee v$  is on the surface  $S_{\mathcal{V}^\infty}$ . By definition  $u \vee v$  is in  $\mathcal{D}_{\mathcal{V}^\infty}$ . Suppose, by contradiction that  $w \notin S_{\mathcal{V}^\infty}$ . Then there exist  $x \in \mathcal{V}^\infty$  with  $x < w$ . Let  $x$  also denote the corresponding vertex of  $G^\infty$ . Edge  $e$  is in a region  $R_i(x)$  for some  $i$ . So  $u, v \in R_i(x)$  and thus by Lemma 10.(i),  $R_i(u) \subseteq R_i(x)$  and  $R_i(v) \subseteq R_i(x)$ . Then by Lemma 29,  $w_i = \max(u_i, v_i) \leq x_i$ , a contradiction. Thus the elbow geodesic between  $u$  and  $v$  is on the surface.

(D3) Consider a vertex  $v \in \mathcal{V}$  and a color  $i$ . Let  $u$  be the extremity of the arc  $e_i(v)$ . We have  $u \in R_{i-1}(v)$  and  $u \in R_{i+1}(v)$ , so by Lemma 10.(i),  $R_{i-1}(u) \subseteq R_{i-1}(v)$  and  $R_{i+1}(u) \subseteq R_{i+1}(v)$ . Thus by Lemma 10.(iii),  $R_i(v) \subsetneq R_i(u)$ . So, by Lemma 29,  $v_i < u_i$ ,  $u_{i-1} \leq v_{i-1}$  and  $u_{i+1} \leq v_{i+1}$ . So the orthogonal arc of vertex  $v$  in direction of the basis vector  $e_i$  is part of the elbow geodesic of the edge  $e_i(v)$ .

(D4) Suppose there exists a pair of crossing edges  $e = uv$  and  $e' = u'v'$  on the surface  $S_{\mathcal{V}^\infty}$ . The two edges  $e, e'$  cannot intersect on orthogonal arcs so they intersect on a plane orthogonal to one of the coordinate axis. Up to symmetry we may assume that we are in the situation of Figure 33 with  $u_1 = u'_1$ ,  $u_2 > u'_2$  and  $v_2 < v'_2$ . Between

$u$  and  $u'$ , there is a path consisting of orthogonal arcs only. With (D3), this implies that there is a bi-directed path  $P^*$  colored 0 from  $u$  to  $u'$  and colored 2 from  $u'$  to  $u$ . We have  $u \in R_2(v)$ , so by Lemma 10.(i),  $R_2(u) \subseteq R_2(v)$ . We have  $u' \in R_2(u)$ , so  $u' \in R_2(v)$ . If  $P_0(v)$  contains  $u'$ , then there is a contractible cycle containing  $v, u, u'$  in  $G_1 \cup G_0^{-1} \cup G_2^{-1}$ , contradicting Lemma 1, so  $P_0(v)$  does not contain  $u'$ . If  $P_1(v)$  contains  $u'$ , then  $u' \in P_1(u) \cap P_0(u)$ , contradicting Lemma 9. So  $u' \in R_2^o(v)$ . Thus the edge  $u'v'$  implies that  $v' \in R_2(v)$ . So by Lemma 29,  $v'_2 \leq v_2$ , a contradiction.  $\square$

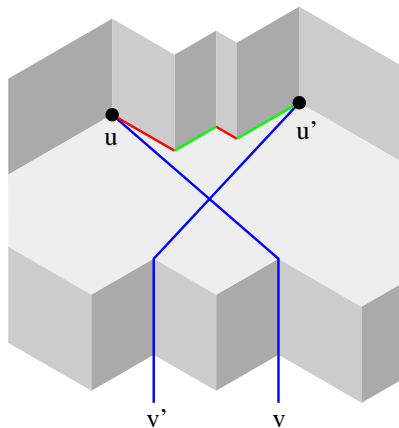


Figure 33: A pair of crossing elbow geodesic

Theorem 1 and 13 implies Theorem 2.

One can ask what is the “size” of the obtained geodesic embedding of Theorem 13? Of course this mapping is infinite so there is no real size, but as the object is periodic one can consider the smallest size of the vectors such that the mapping is periodic with respect to them. There are several such pairs of vectors, one is  $S, S'$ . Recall that  $S_i = N(c_{i+1} - c_{i-1})f$  and  $S'_i = N(c'_{i+1} - c'_{i-1})f$ . Unfortunately the size of  $S, S'$  can be arbitrarily large. Indeed, the values of  $c_{i+1} - c_{i-1}$  and  $c'_{i+1} - c'_{i-1}$  are unbounded as a toroidal map can be artificially “very twisted” in the considered flat torus representation (independently from the number of vertices or faces). Nevertheless we can prove existence of bounded size vectors for which the mapping is periodic with respect to them.

**Lemma 31** *If  $G$  is a toroidal map given with a Schnyder wood, then the mapping of each vertex of  $G^\infty$  on its region vector gives a periodic mapping of  $G^\infty$  with respect to non collinear vectors  $Y$  and  $Y'$  where the size of  $Y$  and  $Y'$  is in  $\mathcal{O}(\gamma Nf)$ . In general we have  $\gamma \leq n$  and in the case where  $G$  is a simple toroidal triangulation given with a Schnyder wood obtained by Theorem 9, we have  $\gamma = 1$ .*

*Proof.* By Lemma 25, the vectors  $Z_{i-1}, Z_{i+1}$  (when the Schnyder wood is not of Type 2.i) span a subset of  $S, S'$  (it can happen that this subset is strict). Thus in the parallelogram delimited by the vectors  $Z_{i-1}, Z_{i+1}$  (that is a parallelogram by Lemma 26), there is a

parallelogram with sides  $Y, Y'$  containing a copy of  $V$ . The size of the vectors  $Z_i$  is in  $\mathcal{O}(\gamma N f)$  and so  $Y$  and  $Y'$  also.

In general we have  $\gamma_i \leq n$  as each intersection between two monochromatic cycles of  $G$  of color  $i - 1$  and  $i + 1$  corresponds to a different vertex of  $G$  and thus  $\gamma \leq n$ . In the case of simple toroidal triangulation given with a Schnyder wood obtained by Theorem 9, we have, for each color  $i$ ,  $\gamma_i = 1$ , and thus  $\gamma = 1$ .  $\square$

We use the example of the toroidal map  $G$  of Figure 2 to illustrate the region vector method. This toroidal map has 3 vertices, 4 faces and 7 edges. There are two edges that are oriented in two directions. The Schnyder wood is of Type 1, with two 1-cycles. We choose as origin the three bold monochromatic lines of Figure 34. Then the region vectors of the vertices of  $G^\infty$  are  $\{u \in \mathbb{R}^3 \mid \exists v \in \mathcal{V}, k_1, k_2 \in \mathbb{Z} \text{ such that } u = v + kS + k'S'\}$ , with  $N = n = 3$ ,  $\mathcal{V} = \{(0, 0, 0), (0, 12, -11), (6, 12, -18)\}$ ,  $S = (-12, 24, -12)$ ,  $S' = (12, 24, -36)$ ,  $c_0 = -1$ ,  $c_1 = 0$ ,  $c_2 = 1$ ,  $c'_0 = -2$ ,  $c'_1 = 1$ ,  $c'_2 = 0$ . The points are not coplanar. They are on the two different planes of equation  $x + y + z = 0$  and  $x + y + z = 1$ . The geodesic embedding that is obtained by mapping each vertex to its region vector is the geodesic embedding of Figure 28. The parallelogram has sides the vectors  $S, S'$ .

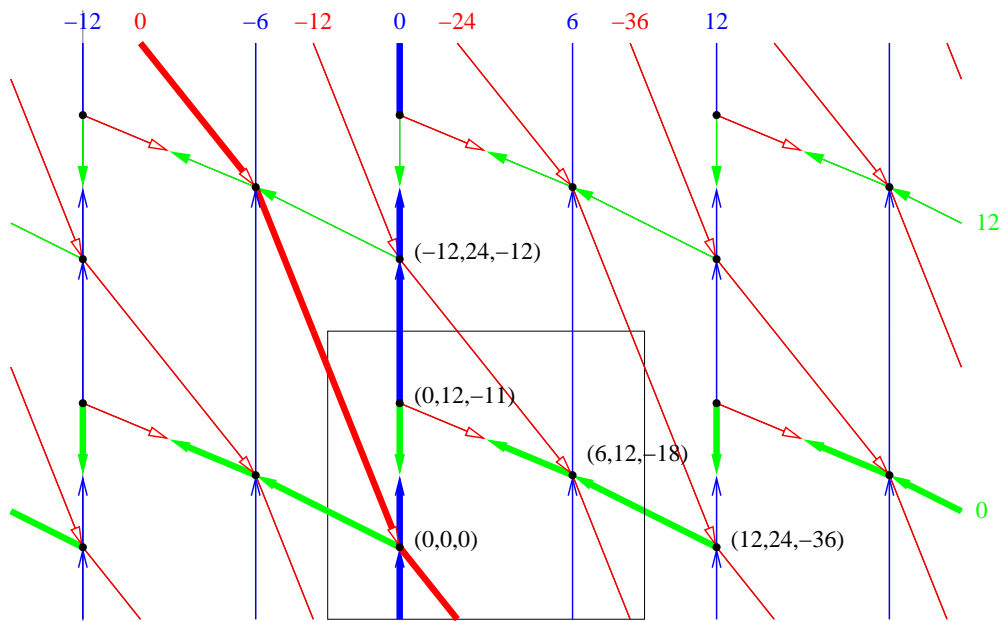


Figure 34: Coordinates of the vertices.

Like in the plane, one can give weights to faces of  $G$ . Then all their copies in  $G^\infty$  have the same weight and instead of counting the number of faces in each region one can compute the weighted sum.

Note that the geodesic embeddings of Theorem 13 are not necessarily rigid. A geodesic embedding is *rigid* [12, 20] if for every pair  $u, v \in \mathcal{V}$  such that  $u \vee v$  is in

$\mathcal{S}_\mathcal{V}$ , then  $u$  and  $v$  are the only elements of  $\mathcal{V}$  that are dominated by  $u \vee v$ . The geodesic embedding of Figure 28 is not rigid has the bend corresponding to the loop of color 1 is dominated by three vertices of  $G^\infty$ . We do not know if it is possible to build a rigid geodesic embedding from the Schnyder wood of a toroidal map. Maybe a technique similar to the one presented in [12] can be generalized to the torus.

It has been already mentioned that in the geodesic embeddings of Theorem 13 the points corresponding to vertices are not coplanar. The problem to build a coplanar geodesic embedding from the Schnyder wood of a toroidal map is open. In the plane, there are some examples of maps  $G$  [12] for which it is not possible to require both rigidity and co-planarity. Thus the same is true in the torus for the graph  $G^+$ .

Another question related to co-planarity is whether one can require that the points of the orthogonal surface corresponding to edges of the graph (i.e. bends) are coplanar. This property is related to contact representation by homothetic triangles [12]. It is known that in the plane, not all Schnyder woods are supported by such surfaces. Kratochvil's conjecture [19], recently proved [17], states that every 4-connected planar triangulation admits a contact representation by homothetic triangles. Can this be extended to the torus ?

When considering non necessarily homothetic triangles, it has been proved [11] that there is a bijection between Schnyder woods of planar triangulations and contact representations by triangles. This results has been generalized to internally 3-connected planar map [16] by exhibiting a bijection between Schnyder woods of internally 3-connected planar maps and primal-dual contact representations by triangles (i.e. representations where both the primal and the dual are represented). It would be interesting to generalize these results to the torus.

## 11 Duality of orthogonal surfaces

Given an orthogonal surface generated by  $\mathcal{V}$ , let  $\mathcal{F}_\mathcal{V}$  be the maximal points of  $\mathcal{S}_\mathcal{V}$ , i.e. the points of  $\mathcal{S}_\mathcal{V}$  that are not dominated by any vertex of  $\mathcal{S}_\mathcal{V}$ . If  $A, B \in \mathcal{F}_\mathcal{V}$  and  $A \wedge B \in \mathcal{S}_\mathcal{V}$ , then  $\mathcal{S}_\mathcal{V}$  contains the union of the two line segments joining  $A$  and  $B$  to  $A \wedge B$ . Such arcs are called *dual elbow geodesic*. The *dual orthogonal arc* of  $A \in \mathcal{F}_\mathcal{V}$  in the direction of the standard basis vector  $e_i$  is the intersection of the ray  $A + \lambda e_i$  with  $\mathcal{S}_\mathcal{V}$ .

Given a toroidal map  $G$ , let  $G^{\infty*}$  be the dual of  $G^\infty$ . A *dual geodesic embedding* of  $G$  is a drawing of  $G^{\infty*}$  on the orthogonal surface  $\mathcal{S}_{\mathcal{V}^\infty}$ , where  $\mathcal{V}^\infty$  is a periodic mapping of  $G^\infty$  with respect to two non collinear vectors, satisfying the following (see example of Figure 35):

- (D1\*) There is a bijection between the vertices of  $G^{\infty*}$  and  $\mathcal{F}_{\mathcal{V}^\infty}$ .
- (D2\*) Every edge of  $G^{\infty*}$  is a dual elbow geodesic.
- (D3\*) Every dual orthogonal arc in  $\mathcal{S}_{\mathcal{V}^\infty}$  is part of an edge of  $G^{\infty*}$ .

(D4\*) There are no crossing edges in the embedding of  $G^{\infty*}$  on  $\mathcal{S}_{\mathcal{V}^\infty}$ .

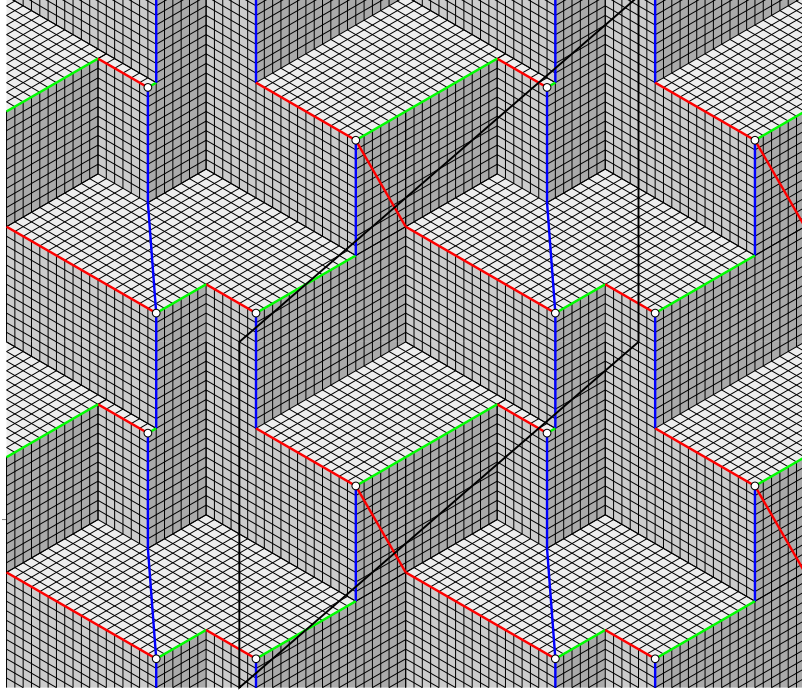


Figure 35: Dual geodesic embedding of the toroidal map of Figure 2.

Let  $G$  be a toroidal map given with a Schnyder wood. Consider the mapping of each vertex on its region vector. We consider the dual of the Schnyder wood of  $G$ . By Lemma 13, it is a Schnyder wood of  $G^*$ . A face  $F$  of  $G^\infty$  is mapped on the point  $\bigvee_{v \in F} v$ . Let  $\widehat{G}^\infty$  be a simultaneous drawing of  $G^\infty$  and  $G^{\infty*}$  such that only dual edges intersect. To avoid confusion, we note  $R_i$  the regions of the primal Schnyder wood and  $R_i^*$  the regions of the dual Schnyder wood.

**Lemma 32** *For any face  $F$  of  $G^\infty$ , we have that  $\bigvee_{v \in F} v$  is a maximal point of  $\mathcal{S}_{\mathcal{V}^\infty}$ .*

*Proof.* Let  $F$  be a face of  $G^\infty$ . For any vertex  $u$  of  $\mathcal{V}^\infty$ , there exists a color  $i$ , such that the face  $F$  is in the region  $R_i(u)$ . Thus for  $v \in F$ , we have  $v \in R_i(u)$ . By Lemma 29, we have  $v_i \leq u_i$  and so  $F_i \leq u_i$ . So  $F = \bigvee_{v \in F} v$  is a point of  $\mathcal{S}_{\mathcal{V}^\infty}$ .

Suppose, by contradiction, that  $F$  is not a maximal point of  $\mathcal{S}_{\mathcal{V}^\infty}$ . Then there is a point  $\alpha \in \mathcal{S}_{\mathcal{V}^\infty}$  that dominates  $F$  and for at least one coordinate  $j$ , we have  $F_j < \alpha_j$ . By Lemma 12, the angles at  $F$  form, in counterclockwise order, nonempty intervals of 0's, 1's and 2's. For each color, let  $z^i$  be a vertex of  $F$  with angle  $i$ . We have  $F$  is in the region  $R_i(z^i)$ . So  $z^{i-1} \in R_i(z^i)$  and by Lemma 10.(i), we have  $R_i(z^{i-1}) \subseteq R_i(z^i)$ . Since  $F$  is in  $R_{i-1}(z^{i-1})$ , it is not in  $R_i(z^{i-1})$  and thus  $R_i(z^{i-1}) \subsetneq R_i(z^i)$ . Then by Lemma 29, we

have  $(z^{i-1})_i < (z^i)_i$  and symmetrically  $(z^{i+1})_i < (z^i)_i$ . So  $F_{j-1} = (z^{j-1})_{j-1} > (z^j)_{j-1}$  and  $F_{j+1} > (z^j)_{j+1}$ . Thus  $\alpha$  strictly dominates  $z^j$ , a contradiction to  $\alpha \in \mathcal{S}_{\mathcal{V}^\infty}$ . Thus  $F$  is a maximal point of  $\mathcal{S}_{\mathcal{V}^\infty}$   $\square$

**Lemma 33** *If two faces  $A, B$  are such that  $R_i^*(B) \subseteq R_i^*(A)$ , then  $A_i \leq B_i$ .*

*Proof.* Let  $v \in B$  be a vertex whose angle at  $B$  is labeled  $i$ . We have  $v \in R_i^*(B)$  and so  $v \in R_i^*(A)$ . In  $\widetilde{G^\infty}$ , the path  $P_i(v)$  cannot leave  $R_i^*(A)$ , the path  $P_{i+1}(v)$  cannot intersect  $P_{i+1}(A)$  and the path  $P_{i-1}(v)$  cannot intersect  $P_{i-1}(A)$ . Thus  $P_{i+1}(v)$  intersect  $P_{i-1}(A)$  and the path  $P_{i-1}(v)$  cannot intersect  $P_{i+1}(A)$ . So  $A \in R_i(v)$ . Thus for all  $u \in A$ , we have  $u \in R_i(v)$ , so  $R_i(u) \subseteq R_i(v)$ , and so  $u_i \leq v_i$ . Then  $A_i = \max_{u \in A} u_i \leq v_i \leq \max_{w \in B} w_i = B_i$ .  $\square$

**Theorem 14** *If  $G$  is a toroidal map given with a Schnyder wood and each vertex of  $G^\infty$  is mapped on its region vector, then the mapping of each face of  $G^{\infty*}$  on the point  $\bigvee_{v \in F} v$  gives a dual geodesic embedding of  $G$ .*

*Proof.* By Lemmas 24 and 27, the mapping is periodic with respect to non collinear vectors.

(D1\*) Consider a counting of elements on the orthogonal surface, where we count two copies of the same object just once (note that we are on an infinite and periodic object). We have that the sum of primal orthogonal arcs plus dual ones is exactly  $3m$ . There are  $3n$  primal orthogonal arcs and thus there are  $3m - 3n = 3f$  dual orthogonal arcs. Each maximal point of  $\mathcal{S}_{\mathcal{V}^\infty}$  is incident to 3 dual orthogonal arcs and there is no dual orthogonal arc incident to two distinct maximal points. So there is  $f$  maximal points. Thus by Lemma 32, we have a bijection between faces of  $G^\infty$  and maximal points of  $\mathcal{S}_{\mathcal{V}^\infty}$ .

Let  $\mathcal{V}^{\infty*}$  be the maximal points of  $\mathcal{S}_{\mathcal{V}^\infty}$ . Let  $\mathcal{D}_{\mathcal{V}^\infty}^* = \{A \in \mathbb{R}^3 \mid \exists B \in \mathcal{V}^{\infty*} \text{ such that } A \leq B\}$ . Note that the boundary of  $\mathcal{D}_{\mathcal{V}^\infty}^*$  is  $\mathcal{S}_{\mathcal{V}^\infty}$ .

(D2\*) Let  $e = AB$  be an edge of  $G^{\infty*}$ . We show that  $w = A \wedge B$  is on the surface  $\mathcal{S}_{\mathcal{V}^\infty}$ . By definition  $w$  is in  $\mathcal{D}_{\mathcal{V}^\infty}^*$ . Suppose, by contradiction that  $w \notin \mathcal{S}_{\mathcal{V}^\infty}$ . Then there exist  $C$  a maximal point of  $\mathcal{S}_{\mathcal{V}^\infty}$  with  $w < C$ . By the bijection (D1\*) between maximal point and vertices of  $G^{\infty*}$ , the point  $C$  corresponds to a vertex of  $G^{\infty*}$ , also denoted  $C$ . Edge  $e$  is in a region  $R_i^*(C)$  for some  $i$ . So  $A, B \in R_i^*(C)$  and thus, by Lemma 10.(i),  $R_i^*(A) \subseteq R_i^*(C)$  and  $R_i^*(B) \subseteq R_i^*(C)$ . Then by Lemma 33, we have  $C_i \leq \min(A_i, B_i) = w_i$ , a contradiction. Thus the dual elbow geodesic between  $A$  and  $B$  is also on the surface.

(D3\*) Consider a vertex  $A$  of  $G^{\infty*}$  and a color  $i$ . Let  $B$  be the extremity of the arc  $e_i(A)$ . We have  $B \in R_{i-1}^*(A)$  and  $B \in R_{i+1}^*(A)$ , so by Lemma 10.(i),  $R_{i-1}^*(B) \subseteq R_{i-1}^*(A)$  and  $R_{i+1}^*(B) \subseteq R_{i+1}^*(A)$ . So by Lemma 33,  $A_{i-1} \leq B_{i-1}$  and  $A_{i+1} \leq B_{i+1}$ . As  $A$  and



$B$  are distinct maximal point of  $S_{\mathcal{V}^\infty}$ , they are incomparable, thus  $B_i < A_i$ . So the dual orthogonal arc of vertex  $A$  in direction of the basis vector  $e_i$  is part of edge  $e_i(A)$ .

(D4\*) Suppose there exists a pair of crossing edges  $e = AB$  and  $e' = A'B'$  of  $G^{\infty*}$  on the surface  $S_{\mathcal{V}^\infty}$ . The two edges  $e, e'$  cannot intersect on orthogonal arcs so they intersects on a plane orthogonal to one of the coordinate axis. Up to symmetry we may assume that we are in the situation  $A_1 = A'_1$ ,  $A'_0 > A_0$  and  $B'_0 < B_0$ . Between  $A$  and  $A'$ , there is a path consisting of orthogonal arcs only. With (D3\*), this implies that there is a bi-directed path  $P^*$  colored 2 from  $A$  to  $A'$  and colored 0 from  $A'$  to  $A$ . We have  $A \in R_0(B)$ , so by Lemma 10.(i),  $R_0(A) \subseteq R_0(B)$ . We have  $A' \in R_0(A)$ , so  $A' \in R_0(B)$ . If  $P_2(B)$  contains  $A'$ , then there is a contractible cycle containing  $A, A', B$  in  $G_1^* \cup G_0^{*-1} \cup G_2^{*-1}$ , contradicting Lemma 1, so  $P_2(B)$  does not contain  $A'$ . If  $P_1(B)$  contains  $A'$ , then  $A' \in P_1(A) \cap P_2(A)$ , contradicting Lemma 9. So  $A' \in R_0^o(B)$ . Thus the edge  $A'B'$  implies that  $B' \in R_0(B)$ . So by Lemma 33,  $B'_0 \geq B_0$ , a contradiction.  $\square$

Theorems 13 and 14 can be combined to obtain a simultaneous representation of a Schnyder wood and its dual on an orthogonal surface. The projection of this 3-dimensional object on the plane of equation  $x + y + z = 0$  gives a representation of the primal and the dual where edges are allowed to have one bend and two dual edges have to cross on their bends (see example of Figure 36).

**Theorem 15** *An essentially 3-connected toroidal map admits a simultaneous flat torus representation of the primal and the dual where edges are allowed to have one bend and two dual edges have to cross on their bends. Such a representation is contained in a (triangular) grid of size  $\mathcal{O}(n^2f) \times \mathcal{O}(n^2f)$  in general and  $\mathcal{O}(nf) \times \mathcal{O}(nf)$  if the map is a simple triangulation. Furthermore the length of the edges are in  $\mathcal{O}(nf)$ .*

*Proof.* Let  $G$  be an essentially 3-connected toroidal map. By Theorems 1 (or Theorem 9 if  $G$  is a simple triangulation),  $G$  admits a Schnyder wood (where monochromatic cycles of different colors intersect just once if  $G$  is simple). By Theorems 13 and 14, the mapping of each vertex of  $G^\infty$  on its region vector gives a primal and dual geodesic embedding. Thus the projection of this embedding on the plane of equation  $x + y + z = 0$  gives a representation of the primal and the dual of  $G^\infty$  where edges are allowed to have one bend and two dual edges have to cross on their bends.

By Lemma 31, the obtained mapping is a periodic mapping of  $G^\infty$  with respect to non collinear vectors  $Y$  and  $Y'$  where the size of  $Y$  and  $Y'$  is in  $\mathcal{O}(\gamma Nf)$ , with  $\gamma \leq n$  in general and  $\gamma = 1$  in case of a simple triangulation. Let  $N = n$ . The embedding gives a representation in the flat torus of sides  $Y, Y'$  where the size of the vectors  $Y$  and  $Y'$  is in  $\mathcal{O}(n^2f)$  in general and in  $\mathcal{O}(nf)$  if the graph is simple and the Schnyder wood is obtained by Theorem 9. By Lemma 30 the length of the edges in this representation are in  $\mathcal{O}(nf)$ .  $\square$

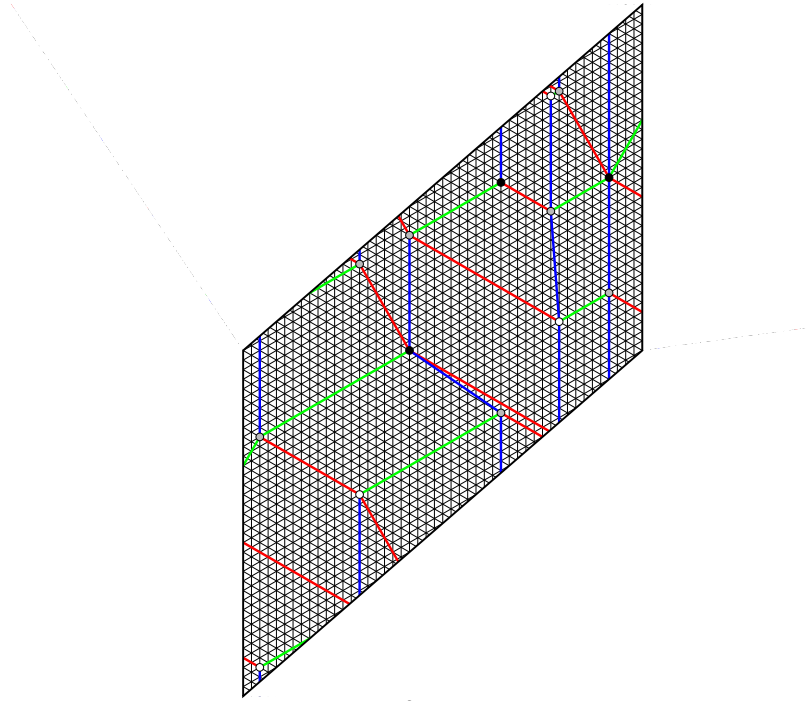


Figure 36: Simultaneous representation of the primal and the dual of the toroidal map of Figure 2 with edges having one bend (in grey).

## 12 Straight-line representation of toroidal maps

The geodesic embedding obtained by the region vector method can be used to obtain a straight-line representation of a toroidal map (see Figure 37). For this purpose, we have to choose  $N$  bigger than previously. Note that Figure 37 is the projection of the geodesic embedding of Figure 28 obtained with the value of  $N = n$ . In this particular case this gives a straight-line representation but in this section we only prove that such a technique works for triangulations and for  $N$  sufficiently large. To obtain a straight-line representation of a general toroidal map, one first has to triangulate it.

Let  $G$  be a toroidal triangulation given with a Schnyder wood and  $V^\infty$  the set of region vectors of vertices of  $G^\infty$ . The Schnyder wood is of Type 1 by Theorem 6. Recall that  $\gamma_i$  is the integer such that two monochromatic cycles of  $G$  of colors  $i - 1$  and  $i + 1$  intersect exactly  $\gamma_i$  times.

**Lemma 34** *For any vertex  $v$ , the number of faces in the bounded region delimited by the three lines  $L_i(v)$  is strictly less than  $(5 \min(\gamma_i) + \max(\gamma_i))f$ .*

*Proof.* Suppose by symmetry that  $\min(\gamma_i) = \gamma_1$ . Let  $L_i = L_i(v)$  and  $z_i = z_i(v)$ . Let  $T$  be the bounded region delimited by the three monochromatic lines  $L_i$ . The boundary

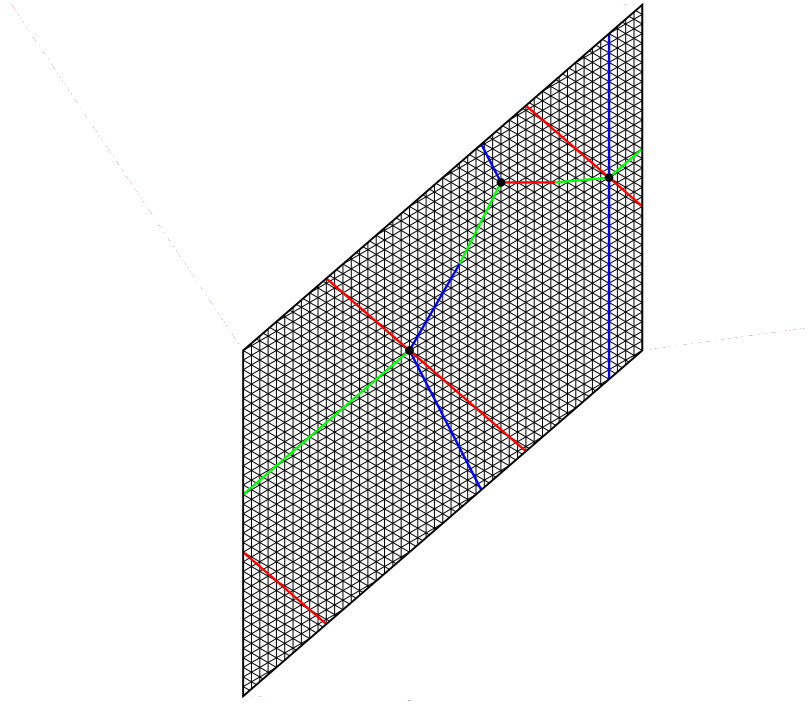


Figure 37: Straight-line representation of the graph of Figure 2 obtained by projecting the geodesic embedding of Figure 28

of  $T$  is a cycle  $C$  oriented clockwise or counterclockwise. Assume that  $C$  is oriented counterclockwise (the proof is similar if oriented clockwise). The region  $T$  is on the left sides of the lines  $L_i$ . We have  $z_{i-1} \in P_i(z_{i+1})$ .

We define, for  $j, k \in \mathbb{N}$ , monochromatic lines  $L_2(j)$ ,  $L_0(k)$  and vertices  $z(j, k)$  as follows (see Figure 38). Let  $L_2(1)$  be the first 2-line intersecting  $L_0 \setminus \{z_1\}$  while walking from  $z_1$ , along  $L_0$  in the direction of  $L_0$ . Let  $L_0(1)$  be the first 0-line of color 0 intersecting  $L_2 \setminus \{z_1\}$  while walking from  $z_1$ , along  $L_2$  in the reverse direction of  $L_2$ . Let  $z(1, 1)$  be the intersection between  $L_2(1)$  and  $L_0(1)$ . Let  $z(j, 1)$ ,  $j \geq 0$ , be the consecutive copies of  $z(1, 1)$  along  $L_0(1)$  such that  $z(j+1, 1)$  is after  $z(j, 1)$  in the direction of  $L_0(1)$ . Let  $L_2(j)$ ,  $j \geq 0$ , be the 2-line of color 2 containing  $z(j, 1)$ . Note that we may have  $L_2 = L_2(0)$  but in any case  $L_2$  is between  $L_2(0)$  and  $L_2(1)$ . Let  $z(j, k)$ ,  $k \geq 0$ , be the consecutive copies of  $z(j, 1)$  along  $L_2(j)$  such that  $z(j, k+1)$  is after  $z(j, k)$  in the reverse direction of  $L_2(j)$ . Let  $L_0(k)$ ,  $k \geq 0$ , be the 0-line containing  $z(1, k)$ . Note that we may have  $L_0 = L_0(0)$  but in any case  $L_0$  is between  $L_0(0)$  and  $L_0(1)$ . Let  $S(j, k)$  be the region delimited by  $L_2(j)$ ,  $L_2(j+1)$ ,  $L_0(k)$ ,  $L_0(k+1)$ . All the region  $S(j, k)$  are copies of  $S(0, 0)$ . The region  $S(0, 0)$  may contain several copies of a face of  $G$  but the number of copies of a face in  $S(0, 0)$  is equal to  $\gamma_1$ . Let  $R$  be the unbounded region situated on the right of  $L_0(1)$  and on the right of  $L_2(1)$ . As  $P_0(v)$  cannot intersect  $L_0(1)$  and  $P_2(v)$  cannot intersect  $L_2(1)$ , vertex  $v$  is in  $R$ . Let  $P(j, k)$  be the subpath of  $L_0(k)$  between  $z(j, k)$  and  $z(j+1, k)$ .

All the lines  $L_0(k)$  are composed only of copies of  $P(0,0)$ . The interior vertices of the path  $P(0,0)$  cannot contain two copies of the same vertex, otherwise there will be a vertex  $z(j,k)$  between  $z(0,0)$  and  $z(1,0)$ . Thus all interior vertices of a path  $P(j,k)$  corresponds to distinct vertices of  $G$ .

The Schnyder wood is of Type 1, thus 1-lines are crossing 0-lines. As a line  $L_0(k)$  is composed only of copies of  $P(0,0)$ , any path  $P(j,k)$  is crossed by a 1-line. Let  $L'_1$  be the first 1-line crossing  $P(1,1)$  on a vertex  $x$  while walking from  $z(1,1)$  along  $L_0(1)$ . By (T1), line  $L'_1$  is not intersecting  $R \setminus \{z(1,1)\}$ . As  $v \in R$  we have  $L_1$  is on the left of  $L'_1$  (maybe  $L_1 = L'_1$ ). Thus the region  $T$  is included in the region  $T'$  delimited by  $L_0, L'_1, L_2$ .

Let  $y$  be the vertex where  $L'_1$  is leaving  $S(1,1)$ . We claim that  $y \in L_2(1)$ . Note that by (T1), we have  $y \in L_2(1) \cup P(1,2)$ . Suppose by contradiction that  $y$  is an interior vertex of  $P(1,2)$ . Let  $d_x$  be the length of the subpath of  $P(1,1)$  between  $z(1,1)$  and  $x$ . Let  $d_y$  be the length of the subpath of  $P(1,2)$  between  $z(1,2)$  and  $y$ . Suppose  $d_y < d_x$ , then there should be a distinct copy of  $L'_1$  intersecting  $P(1,1)$  between  $z(1,1)$  and  $x$  on a copy of  $y$ , a contradiction to the choice of  $L'_1$ . So  $d_x \leq d_y$ . Let  $A$  be the subpath of  $L'_1$  between  $x$  and  $y$ . Let  $B$  be the subpath of  $P(1,1)$  between  $x$  and the copy of  $y$  (if  $d_x = d_y$ , then  $B$  is just a single vertex). Consider all the copies of  $A$  and  $B$  between lines  $L_2(1)$  and  $L_2(2)$ , they form an infinite line  $L$  situated on the right of  $L_2(1)$  that prevents  $L'_1$  from crossing  $L_2(1)$ , a contradiction.

By the position of  $x$  and  $y$ . We have  $L'_1$  intersects  $S(0,1)$  and  $S(1,0)$ . We claim that  $L'_1$  cannot intersect both  $S(0,3)$  and  $S(3,0)$ . Suppose by contradiction that  $L'_1$  intersects both  $S(0,3)$  and  $S(3,0)$ . Then  $L'_1$  is crossing  $S(0,2)$  without crossing  $L_2(0)$  or  $L_2(1)$ . Similarly  $L'_1$  is crossing  $S(2,0)$  without crossing  $L_0(0)$  or  $L_0(1)$ . Thus by superposing what happen in  $S(0,2)$  and  $S(2,0)$  in a square  $S(j,k)$ , we have that there are two crossing 1-lines, a contradiction. Thus  $L'_1$  intersects at most one of  $S(0,3)$  and  $S(3,0)$ .

Suppose that  $L'_1$  does not intersect  $S(3,0)$ . Then the part of  $T'$  situated right of  $L_0(2)$  (left part on Figure 38) is strictly included in  $(S(0,0) \cup S(1,0) \cup S(2,0) \cup S(0,1) \cup S(1,1))$ . Thus this part of  $T'$  contains at most  $5\gamma_1 f$  faces. Now consider the part of  $T'$  situated on the left of  $L_0(2)$  (right part on Figure 38). Let  $y'$  be the intersection of  $L'_1$  with  $L_2$ . Let  $Q$  be the subpath of  $L'_1$  between  $y$  and  $y'$ . By definition of  $L_2(1)$ , there are no 2-lines between  $L_2$  and  $L_2(1)$ . So  $Q$  cannot intersect a 2-line on one of its interior vertices. Thus  $Q$  is crossing at most  $\gamma_2$  consecutive 0-lines (that are not necessarily lines of type  $L_0(k)$ ). Let  $L'_0$  be the  $\gamma_2 + 1$ -th consecutive 0-line that is on the left of  $L_0(2)$  (counting  $L_0(2)$ ). Then the part of  $T'$  situated on the left of  $L_0(2)$  is strictly included in the region delimited by  $L_0(2), L'_0, L_2, L_2(1)$ , and thus contains at most  $\gamma_2$  copies of a face of  $G$ . Thus  $T'$  contains at most  $(\gamma_2 + 5\gamma_1)f$  faces.

Symmetrically if  $L'_1$  does not intersect  $S(0,3)$  we have that  $T'$  contains at most  $(\gamma_0 + 5\gamma_1)f$  faces. Then in any case,  $T'$  contains at most  $(\max(\gamma_0, \gamma_2) + 5\gamma_1)f$  faces and the lemma is true.  $\square$

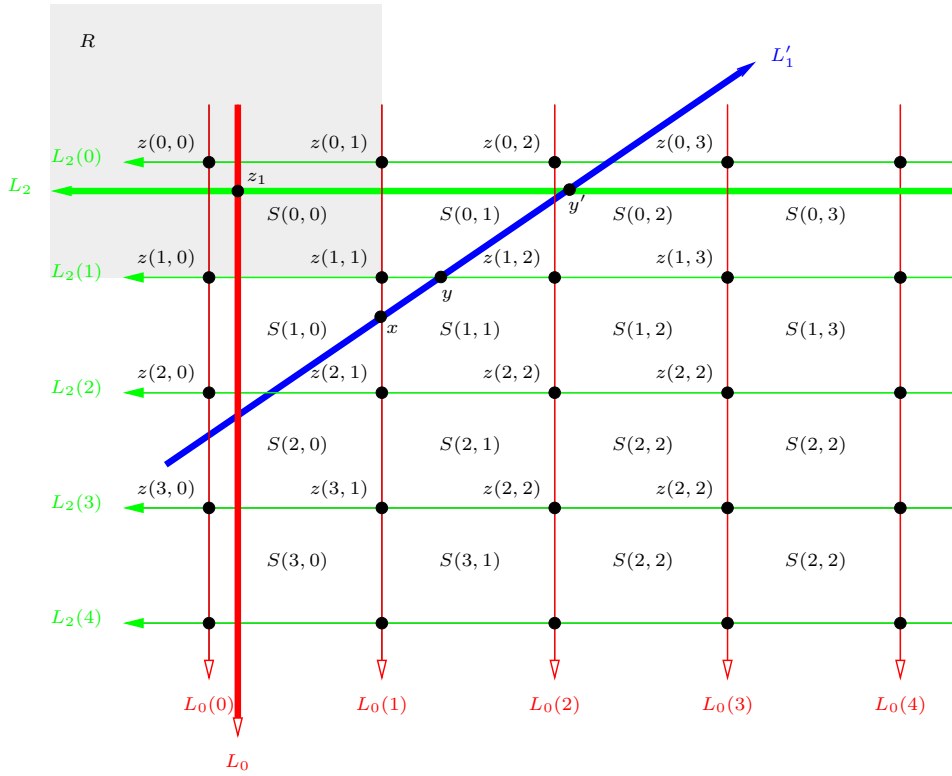


Figure 38: Notations of the proof of Lemma 34.

The bound of Lemma 34 is somehow sharp. In the example of Figure 39, the rectangle represent a toroidal triangulation  $G$  and the universal cover is partially represented. For each value of  $k \geq 0$ , there is a toroidal triangulation  $G$  with  $n = 4(k + 1)$  vertices, where the gray region, representing the region delimited by the three monochromatic lines  $L_i(v)$  contains  $4 \sum_{j=1}^{2k+1} + 3(2k + 2) = \Omega(n \times f)$  faces. Figure 39 represent such a triangulation for  $k = 2$ .

For planar graphs the region vector method gives vertices that all lie on the same plane. This property is very helpful in proving that the position of the points on  $P$  gives straight-line representations. In the torus, things are more complicated as our generalization of the region vector method does not give coplanar points. But Lemma 23 and 34 show that all the points lie in the region situated between the two planes of equation  $x + y + z = 0$  and  $x + y + z = t$ , with  $t = (5 \min(\gamma_i) + \max(\gamma_i))f$ . Note that  $t$  is bounded by  $6nf$  by Lemma 31 and this is independent from  $N$ . Thus from “far away” it looks like the points are coplanar and by taking  $N$  sufficiently large, non coplanar points are “far enough” from each other to enable the region vector method to give straight-line representations.

Let  $N = t + n$ .

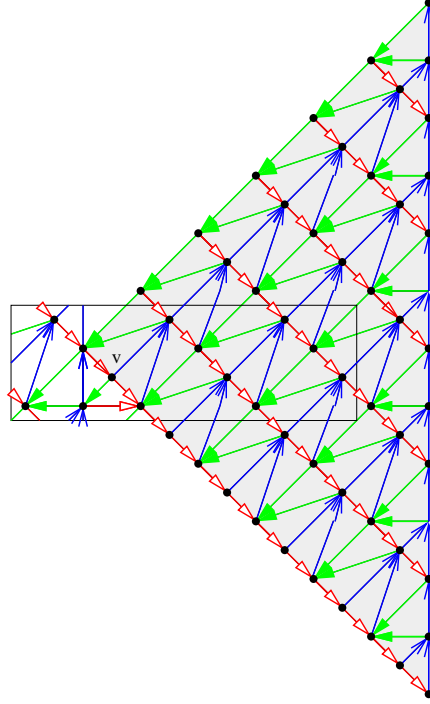


Figure 39: Example of a toroidal triangulation where the number of faces in the region delimited by the three monochromatic lines  $L_i(v)$  contains  $\Omega(n \times f)$  faces.

**Lemma 35** *Let  $u, v$  be two vertices such that  $e_{i-1}(v) = uv$ ,  $L_i = L_i(u) = L_i(v)$ , and such that both  $u, v$  are in the region  $R(L_i, L'_i)$  for  $L'_i$  a  $i$ -line consecutive to  $L_i$ . Then  $v_{i+1} - u_{i+1} < |R(L_i, L'_i)|$  and  $e_{i-1}(v)$  is going counterclockwise around the closed disk bounded by  $\{e_{i-1}(v)\} \cup P_i(u) \cup P_i(v)$ .*

*Proof.* Let  $y$  be the first vertex of  $P_i(v)$  that is also in  $P_i(u)$ . Let  $Q_u$  (resp.  $Q_v$ ) the part of  $P_i(u)$  (resp.  $P_i(v)$ ) between  $u$  (resp.  $v$ ) and  $y$ .

Let  $D$  be the closed disk bounded by the cycle  $C = (Q_v)^{-1} \cup \{e_{i-1}(v)\} \cup Q_u$ . If  $C$  is going clockwise around  $D$ , then  $P_{i+1}(v)$  is leaving  $v$  in  $D$  and thus has to intersect  $Q_u$  or  $Q_v$ . In both cases, there is a cycle in  $G_{i+1} \cup (G_i)^{-1} \cup (G_{i-1})^{-1}$ , a contradiction to Lemma 8. So  $C$  is going clockwise around  $D$ .

As  $L_i(u) = L_i(v)$  and  $L_{i-1}(u) = L_{i-1}(v)$ , we have  $v_{i+1} - u_{i+1} = d_{i+1}(v, u)$  and this is equal to the number of faces in  $D$ . We have  $D \subsetneq R(L_i, L'_i)$ . Suppose  $D$  contains two copies of a given face. Then, these two copies are on different sides of a 1-line. By property (T1), it is not possible to have a 1-line entering  $D$ . So  $D$  contains at most one copy of each face of  $R(L_i, L'_i)$ .  $\square$

**Lemma 36** *For any face  $F$  of  $G^\infty$ , incident to vertices  $u, v, w$  (given in counterclockwise order around  $F$ ), the cross product  $\vec{vw} \wedge \vec{vu}$  has strictly positive coordinates.*

*Proof.* Consider the angle labeling corresponding to the Schnyder wood. By Lemma 12, the angles at  $F$  are labeled in counterclockwise order 0, 1, 2. As  $\vec{uv} \wedge \vec{vw} = \vec{vw} \wedge \vec{vu} = \vec{wu} \wedge \vec{wv}$ , we may assume that  $u, v, w$  are such that  $u$  is in the angle labeled 0, vertex  $v$  in the angle labeled 1 and vertex  $w$  in the angle labeled 2. The face  $F$  is either a cycle completely directed into one direction or it has two edges oriented in one direction and one edge oriented in the other. Let

$$\vec{X} = \vec{vw} \wedge \vec{vu} = \begin{pmatrix} (w_1 - v_1)(u_2 - v_2) - (w_2 - v_2)(u_1 - v_1) \\ -(w_0 - v_0)(u_2 - v_2) + (w_2 - v_2)(u_0 - v_0) \\ (w_0 - v_0)(u_1 - v_1) - (w_1 - v_1)(u_0 - v_0) \end{pmatrix}$$

By symmetry, we consider the following two cases:

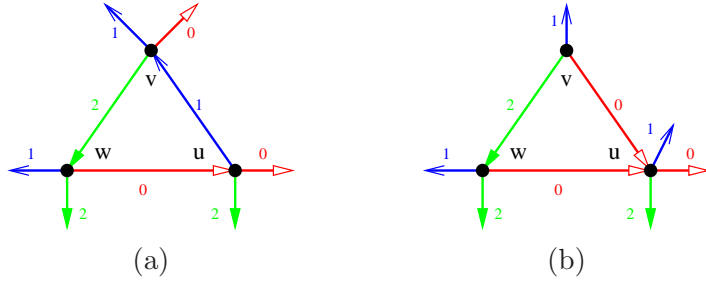


Figure 40: (a) case 1 and (b) case 2 of the proof of Lemma 36

- *Case 1: the edges of the face  $F$  are in counterclockwise order  $e_1(u)$ ,  $e_2(v)$ ,  $e_0(w)$  (see Figure 40.(a)).*

We have  $v \in P_1(u)$ , so  $v \in R_0(u) \cap R_2(u)$  and  $u \in R_1^\circ(v)$  (as there is no edges oriented in two direction). By Lemma 10, we have  $R_0(v) \subseteq R_0(u)$  and  $R_2(v) \subseteq R_2(u)$  and  $R_1(u) \subsetneq R_1(v)$ . In fact the first two inclusions are strict as  $u \notin R_0(v) \cup R_2(v)$ . So by Lemma 29, we have  $v_0 < u_0$ ,  $v_2 < u_2$ ,  $u_1 < v_1$ . We can prove similar inequalities for the other pairs of vertices and we obtain  $w_0 < v_0 < u_0$ ,  $u_1 < w_1 < v_1$ ,  $v_2 < u_2 < w_2$ . By just studying the signs of the different terms occurring in the value of the coordinates of  $\vec{X}$ , it is clear that  $\vec{X}$  as strictly positive coordinates. (For the first coordinates, it is easier if written in the following form  $X_0 = (u_1 - w_1)(v_2 - w_2) - (u_2 - w_2)(v_1 - w_1)$ .)

- *Case 2: the edges of the face  $F$  are in counterclockwise order  $e_0(v)$ ,  $e_2(v)$ ,  $e_0(w)$ . (see Figure 40.(b)).*

As in the previous case, one can easily obtain the following inequalities:  $w_0 < v_0 < u_0$ ,  $u_1 < w_1 < v_1$ ,  $u_2 < v_2 < w_2$  (the only difference with case 1 is between  $u_2$  and  $v_2$ ). Exactly like in the previous case, it is clear to see that  $X_0$  and  $X_2$  are strictly positive. But there is no way to reformulate  $X_1$  to have a similar proof. Let  $A = w_2 - v_2$ ,  $B = u_0 - v_0$ ,  $C = v_0 - w_0$  and  $D = v_2 - u_2$ , so  $X_1 = AB - CD$  and  $A, B, C, D$  are all strictly positive.

Vertices  $u, v, w$  are in the region  $R(L_1, L'_1)$  for  $L'_1$  a 1-line consecutive to  $L_1$ . We consider two cases depending on equality or not between  $L_1(u)$  and  $L_1(v)$ .

★ *Subcase 2.1:*  $L_1(u) = L_1(v)$ .

We have  $X_1 = A(B - D) + D(A - C)$ .

We have  $B - D = (u_0 + u_2) - (v_0 + v_2) = (v_1 - u_1) + (\sum u_i - \sum v_i)$ . Since  $u \in P_0(v)$ , we have  $L_0(u) = L_0(v)$ . Suppose that  $L_2(u) = L_2(v)$ , then by Lemma 23, we have  $\sum u_i = \sum v_i$ , and thus  $B - D = v_1 - u_1 > 0$ . Suppose now that  $L_2(u) \neq L_2(v)$ . By Lemmas 23 and 34,  $\sum u_i - \sum v_i > -t$ . By Lemma 29,  $v_1 - u_1 > (N - n)|R(L_2(u), L_2(v))| \geq N - n$ . So  $B - D > N - n - t \geq 0$ .

We have  $A - C = (w_0 + w_2) - (v_0 + v_2) = (v_1 - w_1) + (\sum w_i - \sum v_i) > \sum w_i - \sum v_i$ . Suppose that  $L_1(v) = L_1(w)$ , then by Lemma 23, we have  $\sum v_i = \sum w_i$  and thus  $A - C = v_1 - w_1 > 0$ . Then  $X_1 > 0$ . Suppose now that  $L_1(v) \neq L_1(w)$ . By Lemma 35,  $D = v_2 - u_2 < |R(L_1, L'_1)|$ . By Lemma 29,  $A = w_2 - v_2 > (N - n)|R(L_1, L'_1)|$ . By Lemma 23 and 34,  $\sum w_i - \sum v_i > -t$ , so  $A - C > -t$ . Then  $X_1 > (N - n - t)|R(L_1, L'_1)| > 0$ .

★ *Subcase 2.2:*  $L_1(u) \neq L_1(v)$ .

We have  $X_1 = B(A - C) + C(B - D)$ .

Suppose that  $L_1(w) \neq L_1(v)$ . Then  $L_1(w) = L_1(u)$ . By Lemma 35  $e_0(w)$  is going counterclockwise around the closed disk  $D$  bounded by  $\{e_0(w)\} \cup P_1(w) \cup P_1(u)$ . Then  $v$  is inside  $D$  and  $P_1(v)$  has to intersect  $P_1(w) \cup P_1(u)$ , so  $L_1(v) = L_1(u)$ , contradiction our assumption. So  $L_1(v) = L_1(w)$ .

By Lemma 29,  $B = u_0 - v_0 > (N - n)|R(L_1, L'_1)|$ . We have  $A - C = (w_0 + w_2) - (v_0 + v_2) = (v_1 - w_1) + (\sum w_i - \sum v_i)$ . By Lemma 23, we have  $\sum v_i = \sum w_i$  and thus  $A - C = v_1 - w_1 > 0$ . By (the symmetric of) Lemma 35,  $C = v_0 - w_0 < |R(L_1, L'_1)|$ . By Lemma 23 34,  $B - D = (u_0 + u_2) - (v_0 + v_2) = (v_1 - u_1) + (\sum u_i - \sum v_i) > -t$ . So  $X_1 > (N - n - t)|R(L_1, L'_1)| > 0$ .  $\square$

Let  $G$  be an essentially 3-connected toroidal map. Consider a periodic mapping of  $G^\infty$  embedded graph  $H$  (finite or infinite) and a face  $F$  of  $H$ . Denote  $(f_1, f_2, \dots, f_t)$  the counterclockwise facial walk around  $F$ . Given a mapping of the vertices of  $H$  in  $\mathbb{R}^2$ , we say that  $F$  is *correctly oriented* if for any triplet  $1 \leq i_1 < i_2 < i_3 \leq t$ , the points  $f_{i_1}$ ,  $f_{i_2}$ , and  $f_{i_3}$  form a counterclockwise triangle. Note that a correctly oriented face is drawn as a convex polygon.

**Lemma 37** *Let  $G$  be an essentially 3-connected toroidal map given with a periodic mapping of  $G^\infty$  such that every face of  $G^\infty$  is correctly oriented. This mapping gives a straight-line representation of  $G^\infty$ .*

*Proof.* We proceed by induction on the number of vertices  $n$  of  $G$ . Note that the theorem holds for  $n = 1$ , so we assume that  $n > 1$ . Given any vertex  $v$  of  $G$ , let  $(u_0, u_1, \dots, u_{d-1})$  be the sequence of its neighbors in counterclockwise order (subscript understood modulo  $d$ ). Every face being correctly oriented, for every  $i \in [0, d - 1]$  the oriented angle (oriented counterclockwise)  $(\overrightarrow{vu_i}, \overrightarrow{vu_{i+1}}) < \pi$ . Let the winding number  $k_v$  of  $v$  be the integer such that  $2k_v\pi = \sum_{i \in [0, d-1]} (\overrightarrow{vu_i}, \overrightarrow{vu_{i+1}})$ . It is clear that  $k_v \geq 1$ . Let



us prove that  $k_v = 1$  for every vertex  $v$ .

(4) For any vertex  $v$ , its winding number  $k_v = 1$ .

In a flat torus representation of  $G$ , we can sum up all the angles by grouping them around the vertices or around the faces.

$$\sum_{v \in V(G)} \sum_{u_i \in N(v)} (\overrightarrow{vu_i}, \overrightarrow{vu_{i+1}}) = \sum_{F \in F(G)} \sum_{f_i \in F} (\overrightarrow{f_i f_{i-1}}, \overrightarrow{f_i f_{i+1}})$$

The face being correctly oriented, they form convex polygons. Thus the angles of a face  $F$  sum at  $(|F| - 2)\pi$ .

$$\begin{aligned} \sum_{v \in V(G)} 2k_v \pi &= \sum_{F \in F(G)} (|F| - 2)\pi \\ \sum_{v \in V(G)} k_v &= \frac{1}{2} \sum_{F \in F(G)} |F| - f \\ \sum_{v \in V(G)} k_v &= m - f \end{aligned}$$

So by Euler's formula  $\sum_{v \in V(G)} k_v = n$ , and thus  $k_v = 1$  for every vertex  $v$ . This proves claim (4).  $\diamond$

Let  $v$  be a vertex of  $G$  that minimizes the number of loops whose ends are on  $v$ . Thus either  $v$  has no incident loop, or every vertex is incident to at least one loop.

Assume that  $v$  has no incident loop. Let  $v'$  be any copy of  $v$  in  $G^\infty$  and denote its neighbors  $(u_0, u_1, \dots, u_{d-1})$  in counterclockwise order. As  $k_v = 1$ , the points  $u_0, u_1, \dots, u_{d-1}$  form a polygon  $P$  containing the point  $v'$  and the segments  $[v', u_i]$  for any  $i \in [0, d-1]$ . It is well known that any polygon, admits a partition into triangles by adding some of the chords. Let us call  $O$  the outerplanar graph with outer boundary  $(u_0, u_1, \dots, u_{d-1})$ , obtained by this "triangulation" of  $P$ . Let us now consider the toroidal map  $G' = (G \setminus \{v\}) \cup O$  and its periodic embedding obtained from the mapping of  $G^\infty$  by removing the copies of  $v$ . It is easy to see that in this embedding every face of  $G'$  is correctly oriented (including the inner faces of  $O$ , or the faces of  $G$  that have been shortened by an edge  $u_i u_{i+1}$ ). Thus by induction hypothesis, the mapping gives a straight-line representation of  $G'^\infty$ . It is also a straight-line representation of  $G^\infty$  minus the copies of  $v$  where the interior of each copy of the polygons  $P$  are pairwise disjoint and do not intersect any vertex or edge. Thus one can add the copies of  $v$  on their initial positions and add the edges with their neighbors without intersecting any edge. The obtained drawing is thus a straight-line representation of  $G^\infty$ .

Assume now that every vertex is incident to at least one loop. Since these loops are non-contractible and do not cross each other, they form homothetic cycles. Thus  $G$  is as depicted in Figure 41, where the dotted segments stand for edges that may be in  $G$  but not necessarily. Since the mapping is periodic the edges corresponding to

loops of  $G$  form several parallel lines, cutting the plane into infinite strips. Since for any  $1 \leq i \leq n$ ,  $k_{v_i} = 1$ , a line of copies of  $v_i$  divides the plane, in such a way that their neighbors which are copies of  $v_{i-1}$  and their neighbors which are copies of  $v_{i+1}$  are in distinct half-planes. Thus adjacent copies of  $v_i$  and  $v_{i+1}$  are on two lines bounding a strip. Then one can see that the edges between copies of  $v_i$  and  $v_{i+1}$  are contained in this strip without intersecting each other. Thus the obtained mapping of  $G^\infty$  is a straight-line representation.  $\square$

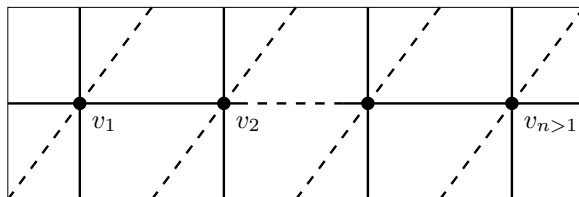


Figure 41: The graph  $G$  if every vertex is incident to a loop.

A plane is *positive* if it has equation  $\alpha x + \beta y + \gamma z = 0$  with  $\alpha, \beta, \gamma \geq 0$ .

**Theorem 16** *If  $G$  is a toroidal triangulation given with a Schnyder wood, and  $V^\infty$  the set of region vectors of vertices of  $G^\infty$ . Then the projection of  $V^\infty$  on a positive plane gives a straight-line representation of  $G^\infty$ .*

*Proof.* Let  $\alpha, \beta, \gamma \geq 0$  and consider the projection of  $V^\infty$  on the plane  $P$  of equation  $\alpha x + \beta y + \gamma z = 0$ . A normal vector of the plane is given by the vector  $\vec{n} = (\alpha, \beta, \gamma)$ . Consider a face  $F$  of  $G^\infty$ . Suppose that  $F$  is incident to vertices  $u, v, w$  (given in counterclockwise order around  $F$ ). By Lemma 36,  $(\vec{uv} \wedge \vec{vw}) \cdot \vec{n}$  is positive. Thus the projection of the face  $F$  on  $P$  is correctly oriented. So by Lemma 37, the projection of  $V^\infty$  on  $P$  gives a straight-line representation of  $G^\infty$ .  $\square$

Theorems 1 and 16 implies Theorem 3. Indeed, any toroidal graph  $G$  can be transformed into a toroidal triangulation  $G'$  by adding a linear number of vertices and edges and such that  $G'$  is simple if and only if  $G$  is simple (see for example the proof of Lemma 2.3 of [21]). Then by Theorem 1,  $G'$  admits a Schnyder wood. By Theorem 16, the projection of the set of region vectors of vertices of  $G'^\infty$  on a positive plane gives a straight-line representation of  $G'^\infty$ . The grid where the representation is obtained can be the triangular grid, if the projection is done on the plane of equation  $x + y + z = 0$ , or the square grid, if the projection is done on one of the plane of equation  $x = 0$ ,  $y = 0$  or  $z = 0$ . By Lemma 31, and the choice of  $N$ , the obtained mapping is a periodic mapping of  $G^\infty$  with respect to non collinear vectors  $Y$  and  $Y'$  where the size of these vectors is in  $\mathcal{O}(\gamma^2 n^2)$  with  $\gamma \leq n$  in general and  $\gamma = 1$  if the graph is simple and the Schnyder wood obtained by Theorem 9. By Lemma 30, the length of the edges in this representation are in  $\mathcal{O}(n^3)$  in general and in  $\mathcal{O}(n^2)$  if the graph is simple. When the

graph is not simple, there is a non contractible cycle of length 1 or 2 and thus the size of one of the two vectors  $Y, Y'$  is in  $\mathcal{O}(n^3)$ . Thus the grid obtained in Theorem 3 has size in  $\mathcal{O}(n^3) \times \mathcal{O}(n^4)$  in general and  $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$  if the graph is simple.

The method presented here gives a polynomial algorithm to obtain flat torus straight-line representation of any toroidal maps in polynomial size grids. Indeed, all the proofs leads to polynomial algorithms, even the proof of Theorem 8 [14] which uses results from Robertson and Seymour [25] on disjoint paths problems.

It would be nice to extend Theorem 16 to obtain convex straight-line representation for essentially 3-connected toroidal maps.

## 13 Conclusion

We have proposed a generalization of Schnyder woods to toroidal maps with application to graph drawing. Along these lines, several questions were raised. We recall some briefly:

- Does the set of Schnyder woods of a given toroidal map has a kind of lattice structure ?
- Does any simple toroidal triangulation admits a Schnyder wood where the set of edges of each color induces a connected subgraph ?
- Is it possible to use Schnyder woods to embed the universal cover of a toroidal map on rigid or coplanar orthogonal surfaces ?
- Which toroidal maps admits (primal-dual) contact representation by (homothetic) triangles in a flat torus ?
- Can geodesic embeddings be used to obtain convex straight-line representation for essentially 3-connected toroidal maps ?

The guideline of Castelli Aleardi et al. [3] to generalize Schnyder wood to higher genus was to preserve the tree structure of planar Schnyder woods and to use this structure for efficient encoding. For that purpose they introduce several special rules (even in the case of genus 1). Our main guideline while working on this paper was that the surface of genus 1, the torus, seems to be the perfect surface to define Schnyder woods. Euler's formula gives exactly  $m = 3n$  for toroidal triangulations. Thus a simple and symmetric object can be defined by relaxing the tree constraint. For genus 0, the plane, there are not enough edges in planar triangulations to have outdegree three for every vertex. For higher genus (the double torus, ...) there are too many edges in triangulations. An open problem is to find what would be the natural generalization of our definition of toroidal Schnyder woods to higher genus.

The results presented here motivated Castelli Aleardi and Fusy [4] to developed direct methods to obtain straight line representations for toroidal maps. They manage to

generalize planar canonical ordering to the cylinder to obtain straight-line representation of simple toroidal triangulations in grids of size  $\mathcal{O}(n) \times \mathcal{O}(n^2)$  thus improving the size of our grid that is  $\mathcal{O}(n^2) \times \mathcal{O}(n^2)$  in the case of a simple toroidal map. It should be interesting to investigate further the links between the two methods as canonical ordering are strongly related to Schnyder woods.

Planar Schnyder woods appear to have many applications in various areas like enumeration [1], compact coding [24], representation by geometric objects [11, 16], graph spanners [2], graph drawing [8, 18], etc. In this paper we use a new definition of Schnyder wood for graph drawing purpose, it would also be interesting to see if it can be used in other computer science domains.

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