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# DETECTING MINORS IN MATROIDS THROUGH TRIANGLES

BORIS ALBAR, DANIEL GONÇALVES, AND JORGE L. RAMÍREZ ALFONSÍN

*Dedicated to the memory of Michel Las Vergnas*

**ABSTRACT.** In this note we investigate some matroid minor structure results. In particular, we present sufficient conditions, in terms of *triangles*, for a matroid to have either  $U_{2,4}$  or  $F_7$  or  $M(K_5)$  as a minor.

## 1. INTRODUCTION

In [5] Mader proved that, for each  $3 \leq r \leq 7$ , if a graph  $G$  on  $n$  vertices has no  $K_r$  minor then it has at most  $n(r-2) - \binom{r-1}{2}$  edges. The latter was used by Nevo [6] to show that, for  $3 \leq r \leq 5$ , if each edge of  $G$  belongs to at least  $r-2$  triangles then  $G$  has a  $K_r$  minor. The latter also holds when  $r = 6, 7$  (see [1]). In the same flavour, we investigate similar conditions for a matroid in order to have certain minors. For general background in matroid theory we refer the reader to [7, 11]. A *triangle* in a matroid is just a circuit of cardinality three. Our main result is the following.

**Theorem 1.** *Let  $M$  be a simple matroid. If every element of  $M$  belongs to at least three triangles then  $M$  has  $U_{2,4}$ ,  $F_7$  or  $M(K_5)$  as a minor.*

We notice that excluding  $U_{2,4}$  as a submatroid (instead of as a minor) would not be sufficient as shown by the matroid  $AG(2, 3)$ . Note that  $AG(2, 3)$  doesn't contain  $M(K_4)$  as a minor. Hence, it doesn't contain  $M(K_5)$  nor  $F_7$  as a minor. Moreover each element of the matroid  $AG(2, 3)$  belongs to 4 triangles but it has no  $U_{2,4}$  submatroid. In the same way, graphic matroids (that are  $U_{2,4}$  and  $F_7$ -minor free) imply that  $M(K_5)$  cannot be simply excluded as a submatroid. We do not know if excluding  $F_7$  as a submatroid in Theorem 1 would be sufficient.

A natural question is whether similar triangle conditions can be used to determine if a matroid admits  $M(K_4)$  as a minor. More precisely,

*is it true that if every element of a matroid  $M$  of rank  $r \geq 3$  belongs to at least two triangles then  $M$  contains  $M(K_4)$  as a minor ?*

The answer to this question is yes if  $M$  is regular (we discuss this at the end of this section see (1)). Moreover, the answer is still yes if  $M$  is binary since the class of binary matroids without a  $M(K_4)$ -minor is the class of series-parallel graphs (a result due to Brylawski [2]). Unfortunately, the answer is no in general, for instance, the reader may take the matroid  $P_7$ , illustrated in Figure 1, as a counterexample.

In the case of ternary matroids, we prove the following.

**Theorem 2.** *Every simple ternary matroid  $M$  such that every element belongs to at least 3 triangles contains a  $P_7$  or a  $M(K_4)$  minor or contains the matroid  $U_{2,4}$  as a submatroid.*

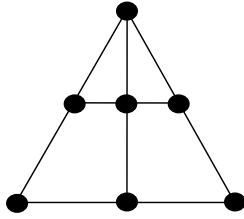
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FIGURE 1. Matroid  $P_7$ .

The above yielded us to consider the following

**Question 1.** Does there exist two finite lists  $\mathcal{L}$  and  $\mathcal{S}$  of matroids such that (a) for each  $M \in \mathcal{L} \cup \mathcal{S}$ , each element  $e \in M$  belongs to at least  $t$  triangles, and (b) for any matroid  $M$  such that each of its elements belong to  $t$  triangles,  $M$  contains one of the matroids in  $\mathcal{L}$  as a minor or  $M$  contains one of the matroids in  $\mathcal{S}$  as a submatroid ?

It is easy to see that  $U_{2,k}$  will belong to either  $\mathcal{L}$  or  $\mathcal{S}$  since it is a matroid with smallest rank such that each element belong to  $t$  triangles for  $k$  big enough (depending on  $t$ ). Moreover, the matroid  $M(K_{t+2})$  will also always be contained in one of these lists since each edge of  $K_{t+2}$  belongs to exactly  $t$  triangles. We finally mention the following generalization of Nevo's result:

- (1) If every element of a simple regular matroid  $M$  belongs to at least  $r - 2$  triangles, with  $3 \leq r \leq 7$ , then  $M$  has  $M(K_r)$  as a minor.

Although this can be proved by applying essentially the same methods as those used in the proof of Theorem 1, we rather prefer to avoid to do this here since the arguments need a more detailed treatment (specially when  $r = 6, 7$ ).

## 2. PROOF OF THEOREM 1

We start by recalling some basic definitions and results needed throughout the paper. We shall denote by  $\mathcal{C}(M)$  the set of circuits of a matroid  $M$ . Let  $k$  be a positive integer. Then, for a matroid  $M$ , a partition  $(X, Y)$  of  $E(M)$  is a  $k$ -separation if  $\min\{|X|, |Y|\} \geq k$  and  $r(X) + r(Y) - r(M) \leq k - 1$ .  $(X, Y)$  is called an *exact  $k$ -separator* if  $r(X) + r(Y) - r(M) = k - 1$ .  $M$  is called  *$k$ -separated* if it has a  $k$ -separation. If  $M$  is  $k$ -separated for some  $k$ , then the connectivity  $\lambda(M)$  of  $M$  is  $\min\{j : M \text{ is } j\text{-separated}\}$ ; otherwise we take  $\lambda(M)$  to be  $\infty$ . We say that a matroid is  *$k$ -connected* if  $\lambda(M) \geq k$ .

Let  $M_1$  and  $M_2$  be two matroid with non-empty ground set  $E_1$  and  $E_2$  respectively. Let

$$\mathcal{C}' = \mathcal{C}(M_1 \setminus (E_1 \cap E_2)) \cup \mathcal{C}(M_2 \setminus (E_1 \cap E_2)) \cup \{C_1 \Delta C_2 : C_i \in \mathcal{C}(M_i) \text{ for } i = 1, 2\}.$$

We denote by  $\mathcal{C}$  the set of minimal elements (by inclusion) of  $\mathcal{C}'$ .

- If  $|E_1 \cap E_2| = 0$ , then  $\mathcal{C}$  is the set of circuits of a matroid with support  $E_1 \Delta E_2$  called the *1-sum* or *direct sum* of  $M_1$  and  $M_2$  and denoted by  $M_1 \oplus_1 M_2$ .
- If  $|E_1 \cap E_2| = 1$ ,  $|E_1|, |E_2| \geq 3$  and  $E_1 \cap E_2$  is not a loop or a coloop of either  $M_1$  or  $M_2$ , then  $\mathcal{C}$  is the set of circuits of a matroid with support  $E_1 \Delta E_2$  called the *2-sum* of  $M_1$  and  $M_2$  and denoted by  $M_1 \oplus_2 M_2$ .
- If  $M_1$  and  $M_2$  are binary matroids with  $|E_1 \cap E_2| = 3$ ,  $|E_1|, |E_2| \geq 7$ , such that  $E_1 \cap E_2$  is a circuit of both  $M_1$  and  $M_2$  and such that  $E_1 \cap E_2$  contains no cocircuit of either  $M_1$  or  $M_2$ , then  $\mathcal{C}$  is the set of circuits of a binary matroid with support  $E_1 \Delta E_2$  called the *3-sum* of  $M_1$  and  $M_2$  and denoted by  $M_1 \oplus_3 M_2$ .

The following structural result is a consequence of Seymour's results in [10] (see also [7, Corollary 11.2.6]):

- (2) [10, (6.5)] Every binary matroid with no  $F_7$  minor can be obtained by a sequence 1- and 2-sums of regular matroids and copies of  $F_7^*$ .

The following results, in relation with  $k$ -separations, are also due to Seymour [9].

- (3) [9, (2.1)] If  $(X, Y)$  is a 1-separator of  $M$  then  $M$  is the 1-sum of  $M|_X$  and  $M|_Y$ ; and conversely, if  $M$  is the 1-sum of  $M_1$  and  $M_2$  then  $(E(M_1), E(M_2))$  is a 1-separation of  $M$ , and  $M_1, M_2$  are isomorphic to proper minors of  $M$ .
- (4) [9, (2.6)] If  $(X, Y)$  is an exact 2-separator of  $M$  then there are matroids  $M_1, M_2$  on  $X \cup \{z\}, Y \cup \{z\}$  respectively (where  $z$  is a new element) such that  $M$  is the 2-sum of  $M_1$  and  $M_2$ . Conversely, if  $M$  is the 2-sum of  $M_1$  and  $M_2$  then  $(E(M_1) - E(M_2), E(M_2) - E(M_1))$  is an exact 2-separation of  $M$ , and  $M_1, M_2$  are isomorphic to proper minors of  $M$ .
- (5) [9, (4.1)] If  $M$  is a 3-connected binary matroid and is the 3-sum of two matroids  $M_1$  and  $M_2$ , then  $M_1$  and  $M_2$  are isomorphic to proper minors of  $M$ .
- (6) [9, (2.10)] A 2-connected matroid  $M$  is not 3-connected if and only if  $M = M_1 \oplus_k M_2$  for some matroids  $M_1$  and  $M_2$ , each of which is isomorphic to a proper minor of  $M$ .

We shall use (2)-(6) and the following three lemmas to prove our main theorem. We will denote by  $si(M)$  the matroid obtained from  $M$  by deleting all its loops and by identifying parallel elements.

**Lemma 1.** *Let  $M_1$  and  $M_2$  be two matroids with ground sets  $E_1$  and  $E_2$  respectively such that  $M = M_1 \oplus_k M_2$ ,  $1 \leq k \leq 3$  and such that  $M$  is a simple matroid. Moreover, we suppose that  $M$  is binary when  $k = 3$ . Let  $e \in E_1 \setminus E_2$  such that  $\{e, x\} \in \mathcal{I}(M_1)$  for any element  $x \in E_1 \cap E_2$  and suppose that  $e$  belongs to  $t$  triangles of  $M$ . Then,  $e$  belongs to at least  $t$  triangles of  $si(M_1)$ .*

*Proof.* Let  $e \in E_1 \setminus E_2$  such that  $\{e, x\} \in \mathcal{I}(M_1)$  for any element  $x \in E_1 \cap E_2$  and suppose that  $e$  belongs to  $t$  triangles of  $M$ . We shall show that  $e$  belongs to at least  $t$  triangles of  $si(M_1)$ .

Let  $T = \{e, f, g\}$  be one of the  $t$  triangles of  $M$  containing  $e$  and note that  $e, g, f \notin E_1 \cap E_2$ . By definition of the  $k$ -sum, either  $T$  is a circuit of  $\mathcal{C}(M_1)$  and we are done, or  $T$  can be written as  $C_1 \Delta C_2$  where  $C_i$  is a circuit of  $M_i, i = 1, 2$ . Since  $M$  is simple and  $E_1 \cap E_2$  contains no loop (by definition of  $k$ -sum) then neither  $M_1$  nor  $M_2$  contain a loop, and thus  $|C_1|, |C_2| \geq 2$ .

If  $|C_1| = 2$ , say  $C_1 = \{e, x\}$ , then  $x \in E_1 \cap E_2$  (otherwise  $e$  and  $x$  would be parallel elements in  $M$ , contradicting the simplicity of  $M$ ). So,  $e$  is parallel to an element  $x$  with  $x \in E_1 \cap E_2$  contradicting the hypothesis of the lemma. We have then that  $|C_1| \geq 3$

If  $|C_2| = 2$ , say  $C_2 = \{g, x\}$ , then  $x \in E_1 \cap E_2$  (otherwise  $g$  and  $x$  would be parallel elements in  $M$ , contradicting the simplicity of  $M$ ). Since  $f \in T = C_1 \Delta C_2$  then  $f \in E_1$  and since  $x \in E_1$  is parallel to  $g$  then  $\{e, f, x\}$  is a triangle of  $M_1$ .

Let us suppose now that  $|C_1|, |C_2| \geq 3$ . Since  $|C_1 \Delta C_2| = |T| = 3$  then  $|C_1 \cap C_2| \geq 2$ . So we are in the case where  $k = 3$  and thus we can suppose that  $M$  is binary. Moreover since  $E_1 \cap E_2$  is a circuit of both  $M_1$  and  $M_2$ , then  $C_1$  and  $C_2$  contain at most two elements of  $E_1 \cap E_2$  or they are equal to  $E_1 \cap E_2$ . In the latter, we have that  $e \in E_1 \cap E_2$  which is a contradiction since  $e \in E_1 \setminus E_2$ . We thus suppose that we are in the former. Hence  $|C_1| + |C_2| = 7$  and we can write  $C_1 \cap C_2 = \{x, y\}$ . Therefore one of  $|C_1|$  or  $|C_2|$  has cardinality at least 4.

We shall use a result due to Fournier [4] stating that a matroid  $M$  is binary if and only if whenever  $C_1$  and  $C_2$  are distinct circuits and  $\{p, q\}$  are elements of  $C_1 \cap C_2$ , then there is a circuit in  $M$  contained in  $C_1 \cup C_2 \setminus \{p, q\}$ .

We have two cases.

Case a)  $|C_2| = 4$  and  $|C_1| = 3$ . We write  $C_1 = \{e, x, y\}$ . By applying Fournier's result to circuits  $E_1 \cap E_2 = \{x, y, z\}$  and  $C_1 = \{e, x, y\}$  we obtain that  $\{e, z\}$  contains a circuit and since by hypothesis neither  $e$  nor  $z$  is a loop, then  $e$  and  $z$  are parallel elements, contradicting the hypothesis because  $z \in E_1 \cap E_2$ .

Case b)  $|C_1| = 4$  and  $|C_2| = 3$ . We write  $C_2 = \{x, y, g\}$ . By Fournier's result applied to circuits  $\{x, y, z\}$  and  $C_2$ , we deduce that  $g$  and  $z$  are parallel elements. Thus  $(T \setminus g) \cup \{z\}$  is a triangle of  $si(M_1)$  and is not a triangle of  $M$ .

It remain to check that two different triangles of  $M$  containing  $e$  induce, by the previous construction, two different triangles in  $si(M_1)$ . Let  $T$  and  $T'$  be two different triangles of  $M$  containing  $e$  that are not triangles of  $M_1$ . Note that  $T$  and  $T'$  have two elements of  $M_1$  because otherwise, as we have previously seen,  $e$  would be parallel to an element of  $E_1 \cap E_2$ , contradicting the hypothesis. We denote by  $w$  (resp.  $w'$ ) the only element of  $T$  (resp.  $T'$ ) that belongs to  $M_2$ . By construction the two triangles of  $si(M_1)$  obtained from  $T$  and  $T'$  respectively contain  $T \setminus \{w\}$  and  $T' \setminus \{w'\}$ . If  $T \setminus \{w\} \neq T' \setminus \{w'\}$ , the resulting triangles of  $si(M_1)$  are different. Suppose now that  $T \setminus \{w\} = T' \setminus \{w'\}$ . In the above construction, the elements  $w$  and  $w'$  are replaced by elements of  $E_1 \cap E_2$  respectively parallel to  $w$  and  $w'$  respectively. Note that  $w$  and  $w'$  cannot be parallel to a common element of  $E_1 \cap E_2$  (indeed if  $w$  and  $w'$  were parallel, it would contradict the simplicity of  $M$ ). So  $w$  and  $w'$  are parallel to two distinct elements of  $E_1 \cap E_2$ , and thus the triangles  $T$  and  $T'$  induces two different triangles in  $si(M_1)$ .  $\square$

**Lemma 2.** *Let  $M$  be a simple connected graphic matroid such that each of its elements belongs to at least three triangles except maybe for one element  $e$  or for some elements of a given triangle  $T$  of  $M$ . If  $M$  is not isomorphic to  $e$  or  $T$ , then  $M$  contains  $M(K_5)$  as a minor.*

*Proof.* Let  $G$  be a graph such that  $M = M(G)$ . We will prove that  $G$  contains a  $K_5$  minor. We will denote by  $X$  the set of vertices corresponding to the extremities of the edge  $e$  or to the vertices of the triangle  $T$  depending on the case. In particular, we have that  $|X| \leq 3$ . Since  $M(G)$  is simple, then  $G$  has at least 4 vertices, so there exists  $u \in V(G) \setminus X$ . Since  $M(G)$  is connected then  $G$  is connected too and so  $\deg(u) \geq 1$ . Moreover, every edge incident to  $u$  belongs to at least 3 triangles, so the graph induced by  $N(u)$  (the set of neighbors of  $u$ ) has minimum degree at least 3. Dirac [3] proved that if  $G$  is a non-null simple graph with no subgraph contractible to  $K_4$ , then  $G$  has a vertex of degree  $\leq 2$ . Therefore, by Dirac's result, the graph induced by the vertices in  $N(u)$  contain a  $K_4$  minor and so the graph induced by  $N(u)$  together with  $u$  contain a  $K_5$  minor.  $\square$

**Lemma 3.** *Let  $M$  be a simple matroid and let  $X$  be a set of element of  $M$  consisting of either an element  $e$  or of the elements of a given triangle  $T$  of  $M$ . If each element of  $M$  belongs to at least three triangles except for the elements of  $X$  and if  $M$  is not isomorphic to  $M|_X$ , then  $M$  is not a cographic matroid.*

*Proof.* We proceed by contradiction. Suppose that there exists a cographic matroid  $M$  contradicting the lemma. Let  $G$  be the graph such that  $E(G)$  is the ground set of  $M$ , and such that the circuits of  $M$  are the edge cuts of  $G$ . We can suppose that  $G$  is connected. Moreover, since  $M$  is simple (i.e. it contains no loop no parallel elements), the graph  $G$  has no edge cut of size one or two and thus  $G$  is 3-edge connected. Let us call an edge cut *trivial* if it corresponds to all the edges incident to a given vertex  $v$ . Note that an edge that belongs to (at least) three 3-edge cuts of  $G$ , belongs to at least one non-trivial 3-edge cut.

In the case where  $M$  has an element  $a$  that does not belong to three triangles, we denote  $v$  one of the endpoints of  $a$  in  $G$ . Now in the case where  $M$  has a triangle  $T = \{a, b, c\}$  which elements do not necessarily belong to three triangles, the edge cut  $\{a, b, c\}$  in  $G$  is either trivial and then we denote  $v$  the degree 3 vertex incident to  $a, b$  and  $c$ , or non-trivial and then every edge of  $G$  (including  $a, b$  and  $c$ ) belongs to a non-trivial 3-edge cut. For every vertex  $v \in V(G)$ , the graph  $G \setminus \{v\}$  is not a stable set. Indeed, suppose that every edge of  $G$  is incident to  $v$  then the graph  $G$  is isomorphic to a star (with eventually multiples edges and loops on  $v$ ), and so, by a result of Whitney [12] the dual matroid of  $M(G)$  (which is isomorphic to  $M$ ) is a graphic matroid associated to the dual graph  $G^*$ . Thus, since  $G$  is a star (with eventually multiple edges and loops on its center), then  $G^*$  is also a star (with eventually multiple edges and loops on its center) of multiples edges. This contradict the fact that each element of  $M$  except at most 3 belongs to at least 3 triangles. which contradicts the simplicity of  $M$ .

We claim that

- (7) there is no 3-edge connected graph  $G$ , with a vertex  $v$ , such that every edge  $e \in E(G \setminus \{v\})$  belongs to some non-trivial 3-edge cut of  $G$  and such that  $G \setminus \{v\}$  is not a stable set.

It is clear that the above claim contradicts the existence of  $G$  and thus implies the lemma. We may now prove (7) by contradiction. So let us consider a graph  $G$  that is 3-edge connected with a distinguished vertex  $v$ , and such that every edge  $e \in E(G \setminus \{v\})$  belongs to at least one non-trivial 3-edge cut of  $G$ . By hypothesis, the graph  $G \setminus \{v\}$  is not a stable set, so there are edges in  $G \setminus \{v\}$ ,  $G$  has some non-trivial 3-edge cuts. Let  $\{e_1, e_2, e_3\} \subset E(G)$  be a non-trivial 3-edge cut of  $G$ , partitioning  $V(G)$  into two sets  $V_1$  and  $V_2$  such that  $v \in V_1$  and such that  $|V_2|$  is minimal (see Figure 2). As this edge cut is non-trivial, there are at least two vertices in  $V_2$ , and as  $G$  is 3-edge connected there is an edge  $f_1$  in  $G[V_2]$ . By hypothesis, let  $\{f_1, f_2, f_3\} \subset E(G)$  be a non-trivial 3-edge cut of  $G$ , partitioning  $V(G)$  into two sets  $X$  and  $Y$  such that  $v \in X$ . Consider now the refined partition defined by the following sets:  $V_1^X = V_1 \cap X$ ,  $V_1^Y = V_1 \cap Y$ ,  $V_2^X = V_2 \cap X$ , and  $V_2^Y = V_2 \cap Y$ . Note that as  $v \in V_1^X$  and as  $f_1$  has both ends in  $V_2$ , the sets  $V_1^X$ ,  $V_2^X$  and  $V_2^Y$  are non-empty. Note also that by definition  $|V_2| \leq |Y|$ , and thus  $|V_2^X| \leq |V_1^Y|$ . This implies that the set  $V_1^Y$  is also non-empty.

By construction, there are at most 6 edges across this partition (if  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$  are disjoint). On the other hand, as  $G$  is 3-edge connected each subset of the partition (as they are non-empty) has at least 3 edges leaving it. This implies that there are exactly 6 edges across the partition and that each set has exactly 3 of them leaving it. Let

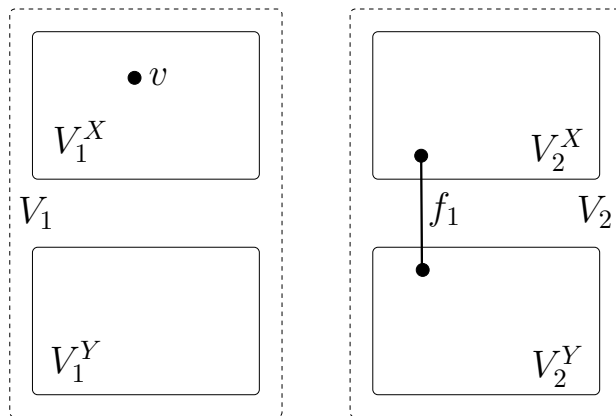


FIGURE 2. The 3-edge connected graph  $G$ , with edges cuts  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3\}$ .

$0 \leq k_e \leq 3$  be the number of edges from  $\{e_1, e_2, e_3\}$  adjacent to  $V_2^X$ , and note that  $V_2^Y$  is adjacent to  $k'_e = 3 - k_e$  of these edges. On the other hand, there is no edge of  $\{f_1, f_2, f_3\}$  going across  $V_1$  and  $V_2$ , thus the number  $k_f$  of edges from this set that are incident to  $V_2^X$  is the same as the number  $k'_f$  of edges from this set that are incident to  $V_2^Y$ . As  $k_e \neq k'_e$  this contradicts the fact that both  $V_2^X$  and  $V_2^Y$  are incident to exactly  $k_e + k_f = k'_e + k'_f = 3$  edges. This concludes the proof of the claim.  $\square$

We may now prove Theorem 1.

*Proof of Theorem 1.* We proceed by contradiction. Let  $M$  be a matroid such that every element belongs to at least three triangles except maybe for one element  $e$  or for some elements of a given triangle  $T$  of  $M$  and assume that  $M$  does not contain  $U_{2,4}$ ,  $F_7$  and  $M(K_5)$  as a minor. We also suppose  $M$  minimal (for the minor relation) with this property.

We first notice that  $M$  must be binary (since it contains no  $U_{2,4}$ -minor). Moreover  $M$  is 2-connected otherwise, by (3),  $M$  can be written as  $M_1 \oplus_1 M_2$ , where  $M_1$  and  $M_2$  are two matroids, but then by Lemma 1, one of  $M_1, M_2$  (say  $M_1$ ) is such that every element belongs to at least 3 triangles, and since both  $M_1$  and  $M_2$  are proper minors of  $M$  by (6), then  $M_1$  contradicts the minimality of  $M$ . Now suppose that  $M$  is 2-connected but not 3-connected, so by (4),  $M$  can be written as a 2-sum of  $M_1$  and  $M_2$  and since  $M$  is such that each element belongs to at least 3 triangles, by Lemma 1, each element of  $si(M_1)$  except the ones of  $E(M_1) \cap E(M_2)$  belongs to at least 3 triangles. But since  $|E(M_1) \cap E(M_2)| \leq 1$  (by definition of 2-sum) and  $si(M_1)$  is a proper minor of  $M$ , then  $si(M_1)$  contradicts the minimality of  $M$ . So we can assume that  $M$  is 3-connected.

Since  $M$  is binary and without  $F_7$ -minor then, by (2), either  $M$  is isomorphic to  $F_7^*$ , either  $M$  is a regular matroid or  $M$  can be written as 2-sum of two smaller matroids. But since  $M$  is 3-connected, by (6), the latter does not hold and for the former, it is easy to check that no element of  $F_7^*$  belongs to at least three triangles, a contradiction. So  $M$  is a 3-connected regular matroid.

By Seymour's regular matroid characterization [9],  $M$  is either graphic, cographic, isomorphic to  $R_{10}$  or is a 3-sum of smaller matroids.

Suppose that  $M$  is isomorphic to  $R_{10}$ . Note that for every element  $e \in E(R_{10})$ , we have that  $R_{10} \setminus e$  is isomorphic to  $M(K_{3,3})$ . Since  $M(K_{3,3})$  is triangle free then every element of  $R_{10}$  should be contained in every triangle of  $R_{10}$  implying that every triangle contains 10

elements, which is a contradiction. Thus  $R_{10}$  is triangle-free, a contradiction. Moreover by Lemmas 2 and 3,  $M$  is neither graphic nor cographic. Thus,  $M$  can be written as a 3-sum of smaller matroids. Suppose that  $M = M_1 \oplus_3 M_2$ . Since the only elements of  $M$  not belonging to three triangles of  $M$  are either a single element or elements that belongs to a triangle of  $M$ , then these elements are contained either in  $M_1$  or  $M_2$ . Without loss of generality we can assume that they are contained in  $M_2$ . But then since  $M$  is 3-connected and binary then, by (6),  $si(M_1)$  is a proper minor of  $M$  and, by Lemma 1, is such that every element except maybe the elements of  $E(si(M_1)) \cap E(M_2)$  belong to at least 3 triangles. This contradicts the minimality of  $M$ .  $\square$

### 3. PROOF OF THEOREM 2

In this section, we will prove Theorem 2 using the following theorem of Oxley [8].

- (8) Any 3-connected ternary matroid with no  $M(K_4)$  minor is either isomorphic to a whirl  $W^r$ , to the matroid  $J$  or to one of the 15 3-connected minors of the Steiner matroid  $S(5, 6, 12)$ .

We will first prove the following lemma about 3-connected matroids.

**Lemma 4.** *Let  $M$  be a 3-connected ternary matroid with no  $M(K_4)$ -minor with at least 2 elements such that every element belongs to at least 2 triangles, except maybe for one element  $e$ , then  $M$  contains  $P_7$  as a minor or is isomorphic to  $U_{2,4}$ .*

*Proof.* By the (8),  $M$  is isomorphic to a whirl  $W^r$ , to  $J$  or is isomorphic to a 3-connected minor of the Steiner matroid  $S(5, 6, 12)$ . Every whirl  $W^r$  for  $r \geq 3$  has at least 2 elements that does not belongs to at least two triangles and the matroid  $J$  has a  $P_7$  minor ([8, (2,9)]). Moreover we checked by computer that all the 3-connected minors of  $S(5, 6, 12)$  has at least 2 elements that does not belongs to at least two triangles or contain a  $P_7$  minor. So either  $M$  contain  $P_7$  as a minor or  $M$  is isomorphic to the whirl  $W^2$ , that is,  $M$  is isomorphic to  $U_{2,4}$ , and the result follows.  $\square$

We may now prove Theorem 2.

*Proof of Theorem 2.* Let  $M$  be a simple ternary matroid with no  $M(K_4)$  minor such that every elements belongs to at least 2 triangles. If  $M$  is 3-connected then, by Lemma 4, the result follows.

Suppose now that  $M$  is not 3-connected. By (3) and (6),  $M$  can be written as  $M_1 \oplus_k M_2$  where  $k \leq 2$  and where  $M_1$  and  $M_2$  are two strict minors of  $M$ . Without loss of generality, we can suppose that  $M_1$  is 3-connected (by taking  $M_1$  and  $M_2$  such that  $|E(M_1)|$  is minimal). Moreover, by Lemma 1, every element of  $M_1$  belongs to at least 2 triangles except maybe for the only element of  $E(M_1) \cap E(M_2)$ . So by the (8),  $M_1$  contains  $P_7$  as a minor or is isomorphic to  $U_{2,4}$ . In the first case, since  $M_1$  is a minor of  $M$ , then  $M$  contain  $P_7$  as a minor and we are done. In the second case, suppose by contradiction that  $M$  does not contain  $U_{2,4}$  as a submatroid. If  $M$  is the direct sum of  $M_1$  and  $M_2$ , then  $M_1$  is a submatroid of  $M$  and thus  $M$  contain  $U_{2,4}$  as a submatroid, contradicting the hypothesis. We thus deduce that  $M$  is the 2-sum of  $M_1$  and  $M_2$ . Let  $p$  be the only element of  $E(M_1) \cap E(M_2)$ . We claim that every element of  $E(M_1) \setminus \{p\}$  belongs to at most one triangle in  $M$ . Suppose that one element of  $M_1 \setminus \{p\}$  belongs to two triangles. As  $|E(M_1) \setminus \{p\}| = 3$ , one of the two triangles denoted by  $T$ , can be written, by the definition of 2-sum, as  $C_1 \Delta C_2$  where  $C_i$  is a circuit of  $M_i$  for  $1 \leq i \leq 2$ . Since  $|T| = |C_1| + |C_2| - 2|C_1 \cap C_2| = 3$  and  $|C_1 \cap C_2| \leq 1$ , we deduce that either  $|C_1| \leq 3$  and  $|C_2| = 2$ , either  $|C_1| = 2$  and  $|C_2| \leq 3$ . The latter cannot happen because otherwise  $C_1$  would be a circuit of  $M_1$  of size 2 which is not possible since  $M_1$  is isomorphic to  $U_{2,4}$ . In



the former case, since  $|C_2| = 2$  and  $p \in C_2$  (by definition of the 2-sum), we may denote  $C_2 = \{p, q\}$ . Since  $p \in M_2$  and  $q$  is parallel to  $p$ ,  $M|_{E(M_1) \setminus \{q\}}$  is isomorphic to  $U_{2,4}$  and thus  $M$  contain  $U_{2,4}$  as a submatroid, which is again a contradiction. Thus every element of  $E(M_1) \setminus \{p\}$  belong to at most one triangle in  $M$ . Therefore all elements of  $M|_{E(M_1) \setminus \{p\}}$  belong to at most one triangle, contradicting the hypothesis, and the result follows.  $\square$

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