Detecting minors in matroids throughout triangles

Boris Albar, Daniel Gonçalves, Jorge Luis Ramírez Alfonsín

DETECTING MINORS IN MATROIDS THROUGH TRIANGLES

BORIS ALBAR, DANIEL GONÇALVES, AND JORGE L. RAMÍREZ ALFONSÍN

Dedicated to the memory of Michel Las Vergnas

Abstract. In this note we investigate some matroid minor structure results. In particular, we present sufficient conditions, in terms of *triangles*, for a matroid to have either $U_{2,4}$ or F_7 or $M(K_5)$ as a minor.

1. Introduction

In [5] Mader proved that, for each $3 \le r \le 7$, if a graph G on n vertices has no K_r minor then it has at most $n(r-2) - \binom{r-1}{2}$ edges. The latter was used by Nevo [6] to show that, for $3 \le r \le 5$, if each edge of G belongs to at least $r-2$ triangles then G has a K_r minor. The latter also holds when $r = 6, 7$ (see [1]). In the same flavour, we investigate similar conditions for a matroid in order to have certain minors. For general background in matroid theory we refer the reader to [7, 11]. A *triangle* in a matroid is just a circuit of cardinality three. Our main result is the following.

Theorem 1. *Let M be a simple matroid. If every element of M belongs to at least three triangles then M has $U_{2,4}$, F_7 or $M(K_5)$ as a minor.*

We notice that excluding $U_{2,4}$ as a submatroid (instead of as a minor) would not be sufficient as shown by the matroid $AG(2,3)$. Note that $AG(2,3)$ doesn't contain $M(K_4)$ as a minor. Hence, it doesn't contain $M(K_5)$ nor F_7 as a minor. Moreover each element of the matroid $AG(2,3)$ belongs to 4 triangles but it has no $U_{2,4}$ submatroid. In the same way, graphic matroids (that are $U_{2,4}$ and F_7-minor free) imply that $M(K_5)$ cannot be simply excluded as a submatroid. We do not know if excluding F_7 as a submatroid in Theorem 1 would be sufficient.
A natural question is whether similar triangle conditions can be used to determine if a matroid admits $M(K_4)$ as a minor. More precisely,

> *is it true that if every element of a matroid M of rank $r \ge 3$ belongs to at least two triangles then M contains $M(K_4)$ as a minor ?*

The answer to this question is yes if M is regular (we discuss this at the end of this section see (1)). Moreover, the answer is still yes if M is binary since the class of binary matroids without a $M(K_4)$-minor is the class of series-parallel graphs (a result due to Brylawski [2]). Unfortunately, the answer is no in general, for instance, the reader may take the matroid P_7, illustrated in Figure 1, as a counterexample.
In the case of ternary matroids, we prove the following.

Theorem 2. *Every simple ternary matroid M such that every element belongs to at least 3 triangles contains a P_7 or a $M(K_4)$ minor or contains the matroid $U_{2,4}$ as a submatroid.*

Date: November 9, 2018.
2010 *Mathematics Subject Classification.* 05B35.
Key words and phrases. Matroids, Minors.
This work was supported by the ANR grants TEOMATRO ANR-10-BLAN 0207 and EGOS 12 JS02 002 01.

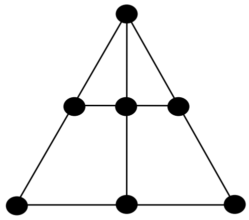

FIGURE 1. Matroid P_7.

The above yielded us to consider the following

Question 1. Does there exist two finite lists $\mathcal{L}$ and $\mathcal{S}$ of matroids such that (a) for each $M \in \mathcal{L} \cup \mathcal{S}$, each element $e \in M$ belongs to at least t triangles, and (b) for any matroid M such that each of its elements belong to t triangles, M contains one of the matroids in $\mathcal{L}$ as a minor or M contains one of the matroids in $\mathcal{S}$ as a submatroid ?

It is easy to see that $U_{2,k}$ will belong to either $\mathcal{L}$ or $\mathcal{S}$ since it is a matroid with smallest rank such that each element belong to t triangles for k big enough (depending on t). Moreover, the matroid $M(K_{t+2})$ will also always be contained in one of these lists since each edge of K_{t+2} belongs to exactly t triangles. We finally mention the following generalization of Nevo's result:

(1) If every element of a simple regular matroid M belongs to at least $r-2$ triangles, with $3 \leq r \leq 7$, then M has $M(K_r)$ as a minor.

Although this can be proved by applying essentially the same methods as those used in the proof of Theorem 1, we rather prefer to avoid to do this here since the arguments need a more detailed treatment (specially when $r = 6, 7$).

2. PROOF OF THEOREM 1

We start by recalling some basic definitions and results needed throughout the paper. We shall denote by $\mathcal{C}(M)$ the set of circuits of a matroid M. Let k be a positive integer. Then, for a matroid M, a partition (X, Y) of $E(M)$ is a *k-separation* if $\min\{|X|, |Y|\} \geq k$ and $r(X)+r(Y)-r(M) \leq k-1$. (X, Y) is called an *exact k-separator* if $r(X)+r(Y)-r(M) = k-1$. M is called *k-separated* if it has a k-separation. If M is k-separated for some k, then the connectivity $\lambda(M)$ of M is $\min\{j : M \text{ is } j\text{-separated }\}$; otherwise we take $\lambda(M)$ to be ∞. We say that a matroid is *k-connected* if $\lambda(M) \geq k$.

Let M_1 and M_2 be two matroid with non-empty ground set E_1 and E_2 respectively. Let

$$\mathcal{C}' = \mathcal{C}(M_1 \setminus (E_1 \cap E_2)) \cup \mathcal{C}(M_2 \setminus (E_1 \cap E_2)) \cup \{C_1 \Delta C_2 \ : \ C_i \in \mathcal{C}(M_i) \text{ for } i = 1, 2\}.$$

We denote by $\mathcal{C}$ the set of minimal elements (by inclusion) of $\mathcal{C}'$.

- If $|E_1 \cap E_2| = 0$, then $\mathcal{C}$ is the set of circuits of a matroid with support $E_1 \Delta E_2$ called the 1-*sum* or *direct sum* of M_1 and M_2 and denoted by $M_1 \oplus_1 M_2$.
- If $|E_1 \cap E_2| = 1$, $|E_1|, |E_2| \geq 3$ and $E_1 \cap E_2$ is not a loop or a coloop of either M_1 or M_2, then $\mathcal{C}$ is the set of circuits of a matroid with support $E_1 \Delta E_2$ called the 2-*sum* of M_1 and M_2 and denoted by $M_1 \oplus_2 M_2$.
- If M_1 and M_2 are binary matroids with $|E_1 \cap E_2| = 3$, $|E_1|, |E_2| \geq 7$, such that $E_1 \cap E_2$ is a circuit of both M_1 and M_2 and such that $E_1 \cap E_2$ contains no cocircuit of either M_1 or M_2, then $\mathcal{C}$ is the set of circuits of a binary matroid with support $E_1 \Delta E_2$ called the 3-*sum* of M_1 and M_2 and denoted by $M_1 \oplus_3 M_2$.

The following structural result is a consequence of Seymour's results in [10] (see also [7, Corollary 11.2.6]):

(2) [10, (6.5)] Every binary matroid with no F_7 minor can be obtained by a sequence 1- and 2-sums of regular matroids and copies of F_7^*.

The following results, in relation with k-separations, are also due to Seymour [9].

(3) [9, (2.1)] If (X, Y) is a 1-separator of M then M is the 1-sum of $M|_X$ and $M|_Y$; and conversely, if M is the 1-sum of M_1 and M_2 then $(E(M_1), E(M_2))$ is a 1-separation of M, and M_1, M_2 are isomorphic to proper minors of M.

(4) [9, (2.6)] If (X, Y) is an exact 2-separator of M then there are matroids M_1 M_2 on $X \cup \{z\}, Y \cup \{z\}$ respectively (where z is a new element) such that M is the 2-sum of M_1 and M_2. Conversely, if M is the 2-sum of M_1 and M_2 then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is an exact 2-separation of M, and M_1, M_2 are isomorphic to proper minors of M.

(5) [9, (4.1)] If M is a 3-connected binary matroid and is the 3-sum of two matroids M_1 and M_2, then M_1 and M_2 are isomorphic to proper minors of M.

(6) [9, (2.10)] A 2-connected matroid M is not 3-connected if and only if $M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, each of which is isomorphic to a proper minor of M.

We shall use (2)-(6) and the following three lemmas to prove our main theorem. We will denote by $si(M)$ the matroid obtained from M by deleting all its loops and by identifying parallel elements.

Lemma 1. *Let M_1 and M_2 be two matroids with ground sets E_1 and E_2 respectively such that $M = M_1 \oplus_k M_2$, $1 \le k \le 3$ and such that M is a simple matroid. Moreover, we suppose that M is binary when $k = 3$. Let $e \in E_1 \setminus E_2$ such that $\{e, x\} \in \mathcal{I}(M_1)$ for any element $x \in E_1 \cap E_2$ and suppose that e belongs to t triangles of M. Then, e belongs to at least t triangles of $si(M_1)$.*

Proof. Let $e \in E_1 \setminus E_2$ such that $\{e, x\} \in \mathcal{I}(M_1)$ for any element $x \in E_1 \cap E_2$ and suppose that e belongs to t triangles of M. We shall show that e belongs to at least t triangles of $si(M_1)$.

Let $T = \{e, f, g\}$ be one of the t triangles of M containing e and note that $e, g, f \notin E_1 \cap E_2$. By definition of the k-sum, either T is a circuit of $\mathcal{C}(M_1)$ and we are done, or T can be written as $C_1 \Delta C_2$ where C_i is a circuit of $M_i, i = 1, 2$. Since M is simple and $E_1 \cap E_2$ contains no loop (by definition of k-sum) then neither M_1 nor M_2 contain a loop, and thus $|C_1|, |C_2| \ge 2$.

If $|C_1| = 2$, say $C_1 = \{e, x\}$, then $x \in E_1 \cap E_2$ (otherwise e and x would be parallel elements in M, contradicting the simplicity of M). So, e is parallel to an element x with $x \in E_1 \cap E_2$ contradicting the hypothesis of the lemma. We have then that $|C_1| \ge 3$

If $|C_2| = 2$, say $C_2 = \{g, x\}$, then $x \in E_1 \cap E_2$ (otherwise g and x would be parallel elements in M, contradicting the simplicity of M). Since $f \in T = C_1 \Delta C_2$ then $f \in E_1$ and since $x \in E_1$ is parallel to g then $\{e, f, x\}$ is a triangle of M_1.

Let us suppose now that $|C_1|, |C_2| \geq 3$. Since $|C_1 \Delta C_2| = |T| = 3$ then $|C_1 \cap C_2| \geq 2$. So we are in the case where $k = 3$ and thus we can suppose that M is binary. Moreover since $E_1 \cap E_2$ is a circuit of both M_1 and M_2, then C_1 and C_2 contain at most two elements of $E_1 \cap E_2$ or they are equal to $E_1 \cap E_2$. In the latter, we have that $e \in E_1 \cap E_2$ which is a contradiction since $e \in E_1 \setminus E_2$. We thus suppose that we are in the former. Hence $|C_1| + |C_2| = 7$ and we can write $C_1 \cap C_2 = \{x, y\}$. Therefore one of $|C_1|$ or $|C_2|$ has cardinality at least 4.

We shall use a result due to Fournier [4] stating that a matroid M is binary if and only if whenever C_1 and C_2 are distinct circuits and $\{p, q\}$ are elements of $C_1 \cap C_2$, then there is a circuit in M contained in $C_1 \cup C_2 \setminus \{p, q\}$.

We have two cases.
Case a) $|C_2| = 4$ and $|C_1| = 3$. We write $C_1 = \{e, x, y\}$. By applying Fournier's result to circuits $E_1 \cap E_2 = \{x, y, z\}$ and $C_1 = \{e, x, y\}$ we obtain that $\{e, z\}$ contains a circuit and since by hypothesis neither e nor z is a loop, then e and z are parallel elements, contradicting the hypothesis because $z \in E_1 \cap E_2$.

Case b) $|C_1| = 4$ and $|C_2| = 3$. We write $C_2 = \{x, y, g\}$. By Fournier's result applied to circuits $\{x, y, z\}$ and C_2, we deduce that g and z are parallel elements. Thus $(T \setminus g) \cup \{z\}$ is a triangle of $si(M_1)$ and is not a triangle of M.

It remain to check that two different triangles of M containing e induce, by the previous construction, two different triangles in $si(M_1)$. Let T and T' be two different triangles of M containing e that are not triangles of M_1. Note that T and T' have two elements of M_1 because otherwise, as we have previously seen, e would be parallel to an element of $E_1 \cap E_2$, contradicting the hypothesis. We denote by w (resp. w') the only element of T (resp. T') that belongs to M_2. By construction the two triangles of $si(M_1)$ obtained from T and T' respectively contain $T \setminus \{w\}$ and $T' \setminus \{w'\}$. If $T \setminus \{w\} \neq T' \setminus \{w'\}$, the resulting triangles of $si(M_1)$ are different. Suppose now that $T \setminus \{w\} = T' \setminus \{w\}$. In the above construction, the elements w and w' are replaced by elements of $E_1 \cap E_2$ repectively parallel to w and w' respectively. Note that w and w' cannot be parallel to a common element of $E_1 \cap E_2$ (indeed if w and w' were parallel, it would contradict the simplicity of M). So w and w' are parallel to two distinct elements of $E_1 \cap E_2$, and thus the triangles T and T' induces two different triangles in $si(M_1)$. □

Lemma 2. *Let M be a simple connected graphic matroid such that each of its elements belongs to at least three triangles except maybe for one element e or for some elements of a given triangle T of M. If M is not isomorphic to e or T, then M contains $M(K_5)$ as a minor.*

Proof. Let G be a graph such that $M = M(G)$. We will prove that G contains a K_5 minor. We will denote by X the set of vertices corresponding to the extremities of the edge e or to the vertices of the triangle T depending on the case. In particular, we have that $|X| \leq 3$. Since $M(G)$ is simple, then G has at least 4 vertices, so there exists $u \in V(G) \setminus X$. Since $M(G)$ is connected then G is connected too and so $\deg(u) \geq 1$. Moreover, every edge incident to u belongs to at least 3 triangles, so the graph induced by $N(u)$ (the set of neighbors of u) has minimum degree at least 3. Dirac [3] proved that if G is a non-null simple graph with no subgraph contractible to K_4, then G has a vertex of degree ≤ 2. Therefore, by Dirac's result, the graph induced by the vertices in $N(u)$ contain a K_4 minor and so the graph induced by $N(u)$ together with u contain a K_5 minor. □

Lemma 3. *Let M be a simple matroid and let X be a set of element of M consisting of either an element e or of the elements of a given triangle T of M. If each element of M belongs to at least three triangles except for the elements of X and if M is not isomorphic to $M|_X$, then M is not a cographic matroid.*

Proof. We proceed by contradiction. Suppose that there exists a cographic matroid M contradicting the lemma. Let G be the graph such that $E(G)$ is the ground set of M, and such that the circuits of M are the edge cuts of G. We can suppose that G is connected. Moreover, since M is simple (i.e. it contains no loop no parallel elements), the graph G has no edge cut of size one or two and thus G is 3-edge connected. Let us call an edge cut *trivial* if it corresponds to all the edges incident to a given vertex v. Note that an edge that belongs to (at least) three 3-edge cuts of G, belongs to at least one non-trivial 3-edge cut.

In the case where M has an element a that does not belong to three triangles, we denote v one of the endpoints of a in G. Now in the case where M has a triangle $T = \{a, b, c\}$ which elements do not necessarily belong to three triangles, the edge cut $\{a, b, c\}$ in G is either trivial and then we denote v the degre 3 vertex incident to a, b and c, or non-trivial and then every edge of G (including a, b and c) belongs to a non-trivial 3-edge cut. For every vertex $v \in V(G)$, the graph $G \setminus \{v\}$ is not a stable set. Indeed, suppose that every edge of G is incident to v then the graph G is isomorphic to a star (with eventually multiples edges and loops on v), and so, by a result of Whitney [12] the dual matroid of $M(G)$ (which is isomorphic to M) is a graphic matroid associated to the dual graph G^*. Thus, since G is a star (with eventually multiple edges and loops on its center), then G^* is also a star (with eventually multiple edges and loops on its center) of multiples edges. This contradict the fact that each element of M except at most 3 belongs to at least 3 triangles. which contradicts the simplicity of M.
We claim that

(7) there is no 3-edge connected graph G, with a vertex v, such that every edge $e \in E(G \setminus \{v\})$ belongs to some non-trivial 3-edge cut of G and such that $G \setminus \{v\}$ is not a stable set.

It is clear that the above claim contradicts the existence of G and thus implies the lemma. We may now prove (7) by contradiction. So let us consider a graph G that is 3-edge connected with a distinguished vertex v, and such that every edge $e \in E(G \setminus \{v\})$ belongs to at least one non-trivial 3-edge cut of G. By hypothesis, the graph $G \setminus \{v\}$ is not a stable set, so there are edges in $G \setminus \{v\}$, G has some non-trivial 3-edge cuts. Let $\{e_1, e_2, e_3\} \subset E(G)$ be a non-trivial 3-edge cut of G, partitioning $V(G)$ into two sets V_1 and V_2 such that $v \in V_1$ and such that $|V_2|$ is minimal (see Figure 2). As this edge cut is non-trivial, there are at least two vertices in V_2, and as G is 3-edge connected there is an edge f_1 in $G[V_2]$. By hypothesis, let $\{f_1, f_2, f_3\} \subset E(G)$ be a non-trivial 3-edge cut of G, partitioning $V(G)$ into two sets X and Y such that $v \in X$. Consider now the refined partition defined by the following sets: $V_1^X = V_1 \cap X$, $V_1^Y = V_1 \cap Y$, $V_2^X = V_2 \cap X$, and $V_2^Y = V_2 \cap Y$. Note that as $v \in V_1^X$ and as f_1 has both ends in V_2, the sets V_1^X V_2^X and V_2^Y are non-empty. Note also that by definition $|V_2| \le |Y|$, and thus $|V_2^X| \le |V_1^Y|$. This implies that the set V_1^Y is also non-empty.
By construction, there are at most 6 edges across this partition (if $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are disjoint). On the other hand, as G is 3-edge connected each subset of the partition (as they are non-empty) has at least 3 edges leaving it. This implies that there are exactly 6 edges across the partition and that each set has exactly 3 of them leaving it. Let

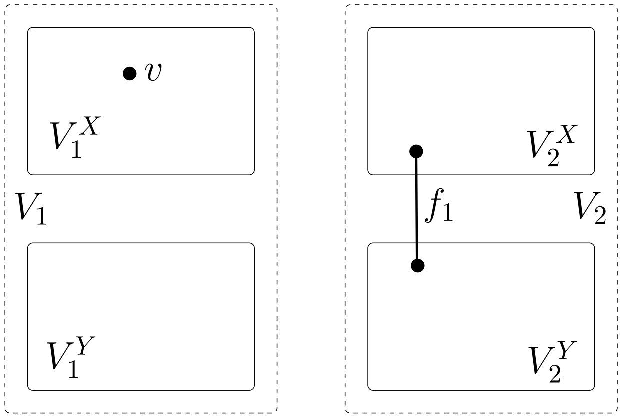

FIGURE 2. The 3-edge connected graph G, with edges cuts $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$.

$0 \leq k_e \leq 3$ be the number of edges from $\{e_1, e_2, e_3\}$ adjacent to V_2^X, and note that V_2^Y is adjacent to $k'_e = 3 - k_e$ of these edges. On the other hand, there is no edge of $\{f_1, f_2, f_3\}$ going across V_1 and V_2, thus the number k_f of edges from this set that are incident to V_2^X is the same as the number k'_f of edges from this set that are incident to V_2^Y. As $k_e \neq k'_e$ this contradicts the fact that both V_2^X and V_2^Y are incident to exactly $k_e + k_f = k'_e + k'_f = 3$ edges. This concludes the proof of the claim. $\square$

We may now prove Theorem 1.

Proof of Theorem 1. We proceed by contradiction. Let M be a matroid such that every element belongs to at least three triangles except maybe for one element e or for some elements of a given triangle T of M and assume that M does not contain $U_{2,4}$, F_7 and $M(K_5)$ as a minor. We also suppose M minimal (for the minor relation) with this property.

We first notice that M must be binary (since it contains no $U_{2,4}$-minor). Moreover M is 2-connected otherwise, by (3), M can be written as $M_1 \oplus_1 M_2$, where M_1 and M_2 are two matroids, but then by Lemma 1, one of M_1, M_2 (say M_1) is such that every element belongs to at least 3 triangles, and since both M_1 and M_2 are proper minors of M by (6), then M_1 contradicts the minimality of M. Now suppose that M is 2-connected but not 3-connected, so by (4), M can be written as a 2-sum of M_1 and M_2 and since M is such that each element belongs to at least 3 triangles, by Lemma 1, each element of $si(M_1)$ except the ones of $E(M_1) \cap E(M_2)$ belongs to at least 3 triangles. But since $|E(M_1) \cap E(M_2)| \leq 1$ (by definition of 2-sum) and $si(M_1)$ is a proper minor of M, then $si(M_1)$ contradicts the minimality of M. So we can assume that M is 3-connected.

Since M is binary and without F_7-minor then, by (2), either M is isomorphic to F_7^*, either M is a regular matroid or M can be written as 2-sum of two smaller matroids. But since M is 3-connected, by (6), the latter does not hold and for the former, it is easy to check that no element of F_7^* belongs to at least three triangles, a contradiction. So M is a 3-connected regular matroid.

By Seymour's regular matroid characterization [9], M is either graphic, cographic, isomorphic to R_{10} or is a 3-sum of smaller matroids.

Suppose that M is isomorphic to R_{10}. Note that for every element $e \in E(R_{10})$, we have that $R_{10} \setminus e$ is isomorphic to $M(K_{3,3})$. Since $M(K_{3,3})$ is triangle free then every element of R_{10} should be contained in every triangle of R_{10} implying that every triangle contains 10 elements, which is a contradiction. Thus R_{10} is triangle-free, a contradiction. Moreover by Lemmas 2 and 3, M is neither graphic nor cographic. Thus, M can be written as a 3-sum of smaller matroids. Suppose that $M = M_1 \oplus_3 M_2$. Since the only elements of M not belonging to three triangles of M are either a single element or elements that belongs to a triangle of M, then these elements are contained either in M_1 or M_2. Without loss of generality we can assume that they are contained in M_2. But then since M is 3-connected and binary then, by (6), $si(M_1)$ is a proper minor of M and, by Lemma 1, is such that every element except maybe the elements of $E(si(M_1)) \cap E(M_2)$ belong to at least 3 triangles. This contradicts the minimality of M. □

3. Proof of Theorem 2

In this section, we will prove Theorem 2 using the following theorem of Oxley [8].

(8) Any 3-connected ternary matroid with no $M(K_4)$ minor is either isomorphic to a whirl W^r, to the matroid J or to one of the 15 3-connected minors of the Steiner matroid $S(5,6,12)$.

We will first prove the following lemma about 3-connected matroids.

Lemma 4. *Let M be a 3-connected ternary matroid with no $M(K_4)$-minor with at least 2 elements such that every element belongs to at least 2 triangles, except maybe for one element e, then M contains P_7 as a minor or is isomorphic to $U_{2,4}$.*

Proof. By the (8), M is isomorphic to a whirl W^r, to J or is isomorphic to a 3-connected minor of the Steiner matroid $S(5,6,12)$. Every whirl W^r for $r \geq 3$ has at least 2 lments that does not belongs to at least two triangles and the matroid J has a P_7 minor ([8, (2,9)]). Moreover we checked by computer that all the 3-connected minors of $S(5,6,12)$ has at least 2 elements that does not belongs to at least two triangles or contain a P_7 minor. So either M contain P_7 as a minor or M is isomorphic to the whirl W^2, that is, M is isomorphic to $U_{2,4}$, and the result follows. □

We may now prove Theorem 2.

Proof of Theorem 2. Let M be a simple ternary matroid with no $M(K_4)$ minor such that every elements belongs to at least 2 triangles. If M is 3-connected then, by Lemma 4, the result follows.

Suppose now that M is not 3-connected. By (3) and (6), M can be written as $M_1 \oplus_k M_2$ where $k \leq 2$ and where M_1 and M_2 are two strict minors of M. Without loss of generality, we can suppose that M_1 is 3-connected (by taking M_1 and M_2 such that $|E(M_1)|$ is minimal). Moreover, by Lemma 1, every element of M_1 belongs to at least 2 triangles except maybe for the only element of $E(M_1) \cap E(M_2)$. So by the (8), M_1 contains P_7 as a minor or is isomorphic to $U_{2,4}$. In the first case, since M_1 is a minor of M, then M contain P_7 as a minor and we are done. In the second case, suppose by contradiction that M does not contain $U_{2,4}$ as a submatroid. If M is the direct sum of M_1 and M_2, then M_1 is a submatroid of M and thus M contain $U_{2,4}$ as a submatroid, contradicting the hypothesis. We thus deduce that M is the 2-sum of M_1 and M_2. Let p be the only element of $E(M_1) \cap E(M_2)$. We claim that every element of $E(M_1) \setminus \{p\}$ belongs to at most one triangle in M. Suppose that one element of $M_1 \setminus \{p\}$ belongs to two triangles. As $|E(M_1) \setminus \{p\}| = 3$, one of the two triangles denoted by T, can be written, by the definition of 2-sum, as $C_1 \Delta C_2$ where C_i is a circuit of M_i for $1 \leq i \leq 2$. Since $|T| = |C_1| + |C_2| - 2|C_1 \cap C_2| = 3$ and $|C_1 \cap C_2| \leq 1$, we deduce that either $|C_1| \leq 3$ and $|C_2| = 2$, either $|C_1| = 2$ and $|C_2| \leq 3$. The latter cannot happen because otherwise C_1 would be a circuit of M_1 of size 2 which is not possible since M_1 is isomorphic to $U_{2,4}$. In the former case, since $|C_2| = 2$ and $p \in C_2$ (by definition of the 2-sum), we may denote $C_2 = \{p, q\}$. Since $p \in M_2$ and q is parallel to p, $M_{|E(M_1)\setminus\{q\}}$ is isomorphic to $U_{2,4}$ and thus M contain $U_{2,4}$ as a submatroid, which is again a contradiction. Thus every element of $E(M_1) \setminus \{p\}$ belong to at most one triangle in M. Therefore all elements of $M|_{E(M_1)\setminus\{p\}}$ belong to at most one triangle, contradicting the hypothesis, and the result follows. $\square$

References

[1] B. Albar, D. Gonçalves, Triangles in K_d-minor free graphs, *arXiv:1304.5468*, (2013).
[2] T.H. Brylawski, A combinatorial model for series-parallel networks, *Trans. Amer. Math. Soc.* **154** (1971), 1-22.
[3] G.A. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, *J. London Math. Soc.* **27** (1952), 85-92.
[4] J. -C. Fournier, A characterization of binary geometries by a double elimination axiom, *J. Combin. Th. Ser. B* **31**, (1981), 249-250.
[5] W. Mader, Homomorphiesätze für Graphen, *Math. Ann.* **178** (1968), 154-168.
[6] E. Nevo, On embeddability and stresses of graphs, *Combinatorica* **27**(4) (2007), 465-472.
[7] J.G. Oxley, Matroid theory, Oxford University Press, New York, 1992.
[8] J.G. Oxley, A characterization of the ternary matroids with no $M(K_4)$-minor, *J. Combin. Th. Ser. B* **42 (2)**, (1987),212-249.
[9] P.D. Seymour, Decomposition of regular matroids, *J. Combin. Th. Ser. B* **28**, (1980), 305-359.
[10] P.D. Seymour, Matroids and multicommodity, *European J. Combin.* **2**, (1981)(3), 257-290.
[11] D.J.A. Welsh, Matroid Theory, Academic Press, London-New York, 1976.
[12] H. Whitney, Planar graphs, *Fund. Math.* **21**, (1933), 73-84.

Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France and LIRMM, Université Montpellier 2, 161 rue Ada, 34095 Cedex 05, France
E-mail address: Boris.Albar@lirmm.fr

LIRMM, Université Montpellier 2, 161 rue Ada, 34095 Cedex 05, France
E-mail address: Daniel.Goncalves@lirmm.fr

Institut de Mathématiques et de Modélisation de Montpellier, Université Montpellier 2, Case Courrier 051, Place Eugène Bataillon, 34095 Montpellier Cedex 05, France
E-mail address: jramirez@math.univ-montp2.fr