A survey on
The Active Bijection
in Graphs, Hyperplane Arrangements, and Oriented Matroids

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Joint work with
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A longer online version of the presentation given at:
Workshop on New Directions for the Tutte Polynomial, London, July 2015

Abstract.

The active bijection maps any directed graph, resp. signed hyperplane arrangement or oriented matroid, on a linearly ordered edge set, resp. ground set, onto one of its spanning trees, resp. bases. It relates all spanning trees to all orientations of a graph, all bases to all reorientations of an hyperplane arrangement or, more generally, an oriented matroid. It preserves activities: for bases in the sense of Tutte, for orientations in the sense of Las Vergnas, yielding a bijective interpretation of the equality of two expressions of the Tutte polynomial. It can be mathematically defined in a short way, and can be built, characterized, particularized, or refined in several ways.

For instance, we get a bijection between bounded regions (bipolar orientations in the case of a graph) and bases with internal activity one and external activity zero, which can be seen as an elaboration on linear programming. We get a decomposition of into bounded regions of minors of the primal and the dual, which also has a counterpart for decomposing matroid bases, and yields an expression of the Tutte polynomial using only beta invariants of minors. We also get activity preserving bijections between all reorientations and all subsets (related to a four-variable expansion of the Tutte polynomial), between regions and no-broken-circuit subsets (acyclic case), between increasing trees and permutations (complete graph case)...

It is the subject of a series of paper.

0

The active bijection: short mathematical definition

The active bijection (in one slide) **Aim of the talk: explain this slide!**

For every oriented matroid M on a linearly ordered set E,
$\alpha(M)$ is the basis of M defined by the three following properties:

- If M is acyclic and $min(E)$ is contained in every positive cocircuit of M, then $\alpha(M)$ is the unique *(fully optimal)* basis B of M such that:
  - for all $b \in M \setminus min(E)$, the signs of b and $min(C^*(B;b))$ are opposite in $C^*(B;b)$;
  - for all $e \notin B$, the signs of e and $min(C(B;e))$ are opposite in $C(B;e)$.
- $\alpha(M^*) = E \setminus \alpha(M)$
- $\alpha(M) = \alpha(M/F) \uplus \alpha(M(F))$
  where F is the [complementary of the] union of all positive [co]circuits of M whose smallest element is the greatest possible smallest element of a positive [co]circuit of M; [...]=equivalent dual formulation

The mapping α yields an activity preserving bijection:
- between all activity classes of reorientations and all bases of M,
- and between all reorientations and all subsets of M.

The active bijection in graphs

For every directed graph $\overrightarrow{G}$ on a linearly ordered set of edges E,
$\alpha(\overrightarrow{G})$ is the spanning tree of G defined by:

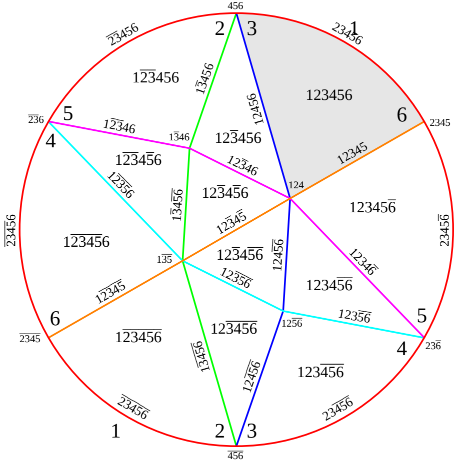
- If $\overrightarrow{G}$ is acyclic and $min(E)$ is contained in every directed cocycle of $\overrightarrow{G}$, then $\alpha(\overrightarrow{G})$ is the unique *(fully optimal)* spanning tree B of G such that:
  - for all $b \in E \setminus min(E)$, the signs of b and $min(C^*(B;b))$ are opposite in $C^*(B;b)$;
  - for all $e \notin B$, the signs of e and $min(C(B;e))$ are opposite in $C(B;e)$.
- $\cancel{\alpha(M^*) = E \setminus \alpha(M)}$ If $\overrightarrow{G}$ is strongly connected and $min(E)$ is contained in every directed cycle, then... [dual formulation]
- $\alpha(\overrightarrow{G}) = \alpha(\overrightarrow{G}/F) \uplus \alpha(\overrightarrow{G}(F))$ where F can be the [complementary of the] union of all directed [co]cycles of $\overrightarrow{G}$ whose smallest element is the greatest possible smallest element of a directed [co]cycle of $\overrightarrow{G}$; [...]= ~~equivalent~~ required dual formulation

The mapping α yields an activity preserving bijection:
- between all activity classes of orientations and all spanning trees of G,
- and between all orientations and all subsets of G.

1

Graphs, hyperplane arrangements, and oriented matroids

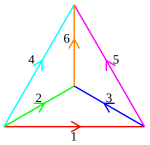
Hyperplane arrangement $\longrightarrow$ oriented matroid

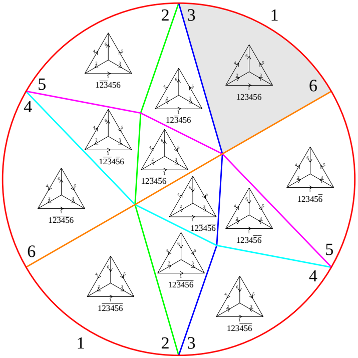


Matroid: incidence properties and flat intersection lattice

Oriented matroid: convexity properties and face relative positions

Directed graph $\longrightarrow$ associated arrangement





edge $ij \longrightarrow$ hyperplane with equation $v_j - v_i = 0$

spanning tree $\longrightarrow$ basis

orientation of edge $ij \longrightarrow$ half-space $v_j - v_i > 0$

directed cut $\longrightarrow$ vertex of the region (positive cocircuit)

cut $\longrightarrow$ vertex (cocircuit)

acyclic orientation $\longrightarrow$ region

strongly connected orientation $\longrightarrow$ region of the dual arrangement

Duality

Every oriented matroid M has a dual M^*.

circuits of M	$=$	cocircuits of M^*
cocircuits of M	$=$	circuits of M^*
acyclic orientations of M (or regions)	$=$	totally cyclic orientations of M^* (or dual regions) (or strongly connected orientations)
bases of M	$=$	complementary of bases of M^*

In the realizable case: duality $\sim$ orthogonality

In the graphical case: duality $=$ cycles/cocycles duality
(extends planar graph duality)

Arrangements of pseudolines $\sim$ Rank 3 oriented matroids

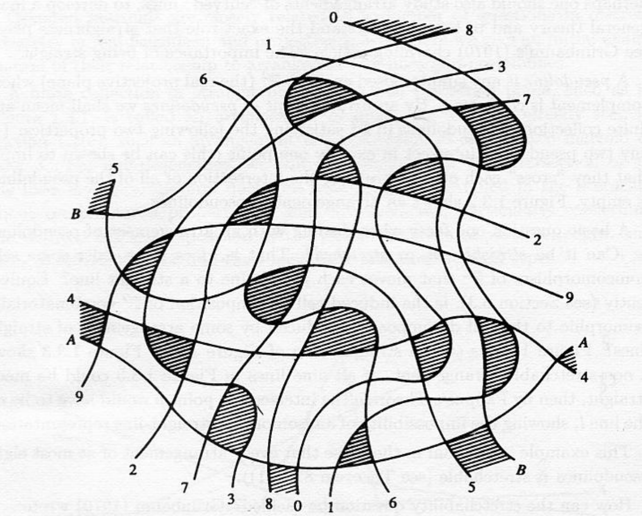


Figure 1.3.4: A pseudoline arrangement without adjacent triangles.

Oriented matroids

Combinatorial axiomatics

Circuits (or cocircuits) are signed subsets satisfying simple combinatorial axioms...

(There are about 20 known equivalent combinatorial axiomatics!)

Topological representation theorem [Folkman & Lawrence 1978]

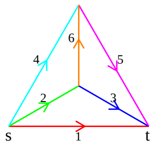
$$\text{Oriented matroids on } E$$
$$\sim$$
$$\text{Signed pseudosphere arrangements } (S_e)_{e\in E} \text{ up to homeomorphism}$$

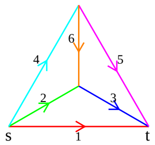
2

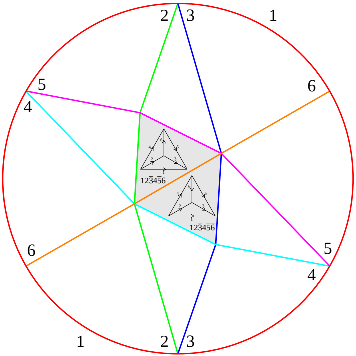
The Tutte polynomial in oriented matroids and directed graphs

Bipolar orientations and bounded regions

graph$\longrightarrow$ hyperplane arrangement







bipolar orientations w.r.t. $p = 1$:
acyclic orientations
with unique source and unique sink
extremities of $p = 1$

$\longrightarrow$

bounded regions w.r.t. $p = 1$:
regions that do not touch $p = 1$

The β invariant

M underlying matroid

- $\beta(M) = \#$ bounded regions w.r.t. e (on one side of e)
- $\beta(M) = \#$ bipolar orientations w.r.t. e
  (for a given orientation of e)
- $\beta(M)$ does not depend on e: it is an invariant
- $\beta(M) \ = \ t_{1,0}(M)$ coefficient of x (or y) of the
  Tutte polynomial $t_M(x, y)$ of M
- $\beta(M) = \#$ acyclic (re)orientations such that e belongs to every positive cocircuit (directed cocycle)
  Other coefficients of t_M can also be interpreted a similar way...

Activities of orientations

Let M be an oriented matroid on a linearly ordered set E

(or a directed graph $\overrightarrow{G} = (V, E)$).

- An element of E is *active* if it is the smallest of a positive circuit of M (or: ... the smallest edge of a directed cycle)

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Theorem [Las Vergnas 1984]

$$t(M; x, y) = \sum_{i,j} \frac{o_{i,j}}{2^{i+j}} \; x^i \; y^j$$

where $o_{i,j}$ is the number of reorientations of M with i dual-active elements and j active elements.

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Related classical results

$o_{1,0} = \#$ bounded regions $= \#$bipolar orientations w.r.t. $e \in E$ (with given orientation for e)

[Crapo 1969, Zaslavsky 1975, Las Vergnas 1977]

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where $o_{i,j}$ is the number of reorientations of M with i dual-active elements and j active elements.

Related classical results

$t(M; 2, 0) = \#$ regions $= \#$ acyclic orientations

[Winder 1966, Stanley 1973, Las Vergnas 1977]

Activities of orientations

Let M be an oriented matroid on a linearly ordered set E

(or a directed graph $\overrightarrow{G} = (V, E)$).

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Theorem [Las Vergnas 1984]

$$t(M; x, y) = \sum_{i,j} \frac{o_{i,j}}{2^{i+j}} x^i y^j$$

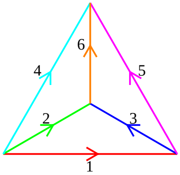
where $o_{i,j}$ is the number of reorientations of M with i dual-active elements and j active elements.

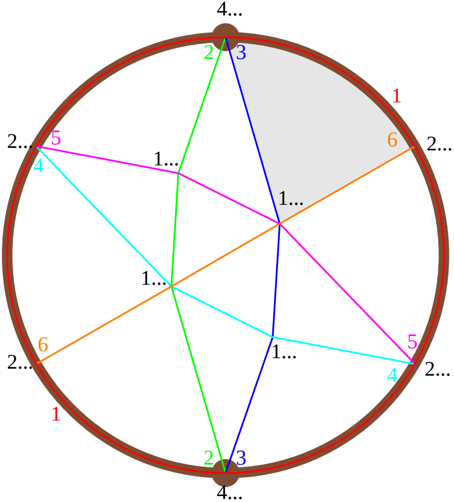
Related classical results

$t(M; 0, 2) = \#$ dual regions $= \#$ totally cyclic orientations

(strongly connected orientations)

Dual activity of a region (or an acyclic orientation)



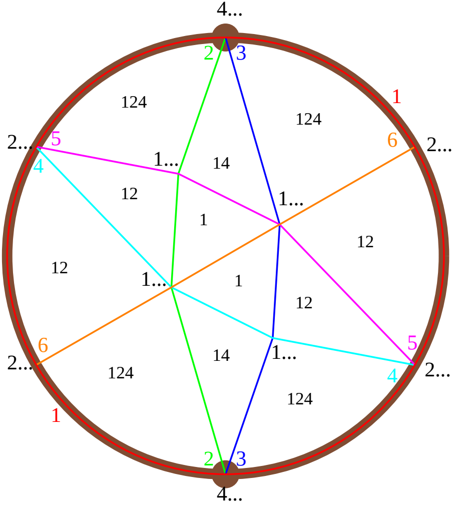


Consider the smallest element of each cocircuit

They correspond to the sequence of nested faces induced by the minimal basis: $1 \cap 2 \quad \subset 1$

The grey region (digraph) has dual-active elements: 124

Dual activity of a region (or an acyclic orientation)



$t(K_4; x, y) = 8.(\frac{x}{2})^3 + +12.(\frac{x}{2})^2 + 4.(\frac{x}{2}) + \dots$
$t(K_4; 2, 0) = 24$

Activities of bases (spanning trees)

Let M be a matroid on a linearly ordered set E

(or a graph $G = (V, E)$).

B a basis (spanning tree) of M

- $e \in E \setminus B$ is externally active w.r.t. B if it is the smallest element of $C_e = C(B; e)$, the unique circuit (cycle) contained in $B \cup \{e\}$
- $b \in B$ is internally active w.r.t. B if it is the smallest element of $C^*_b = C^*(B; b)$, the unique cocircuit (cocycle) contained in $(E \setminus B) \cup \{b\}$

Theorem [Tutte 1954 & Crapo 1969]

$$t(M; x, y) = \sum_{i,j} \; b_{i,j} \; x^i \; y^j$$

où $b_{i,j}$ is the number of bases of M with i internally active elements and j externally active elements.

Tutte polynomial $t(M; x, y)$ of an ordered oriented matroid M

Theorem. [Tutte 1954 & Crapo 1969]

$$t(M; x, y) = \sum_{i,j} b_{i,j} x^i y^j$$

où $b_{i,j}$ = number of bases with activities (i, j)

Theorem. [Las Vergnas 1984]

$$t(M; x, y) = \sum_{i,j} o_{i,j} \left(\frac{x}{2}\right)^i \left(\frac{y}{2}\right)^j$$

where $o_{i,j}$ = number of reorientations with activities (i, j)

$$\boxed{o_{i,j} = 2^{i+j} b_{i,j}}$$

The active bijection is a canonical underlying bijection...

3

The fully optimal basis
(The fully optimal spanning tree)
# of a bounded region
(of a bipolar orientation)

M an oriented matroid on a linearly ordered set

$$E = e_1 < \ldots < e_n$$

We look for a bijection between

bounded regions $-_A M$ w.r.t. e_1

$$o^*(-_A M) = 1 \text{ and } o(-_A M) = 0$$

and

(1,0)-active bases B of M

$$\iota(B) = 1 \text{ and } \varepsilon(B) = 0$$

Fully Optimal Basis (in an oriented matroid)

M an oriented matroid on a linearly ordered set

$$E = e_1 < \ldots < e_n$$

$B \subseteq E$ a basis of M

C_e = fundamental circuit of $e \notin B$ w.r.t. B

= unique circuit in $B \cup e$

C_b^* = fundamental cocircuit of $b \in B$ w.r.t. B

= unique cocircuit in $(E \setminus B) \cup b$

The basis B is *fully optimal* if

- b and *min* C_b^* have opposite signs in C_b^* for all $b \in B \setminus e_1$
- e and *min* C_e have opposite signs in C_e for all $e \in E \setminus B$

Remark

if M has a fully optimal basis B then M is bounded w.r.t. e_1
and B is $(1, 0)$-active

Active bijection: main theorem in the bounded case

- An ordered bounded oriented matroid w.r.t. $min(E)$ M has a unique fully optimal basis denoted

$$\alpha(M)$$

Active bijection: main theorem in the bounded case

- An ordered bounded oriented matroid w.r.t. $min(E)$ M has a unique fully optimal basis denoted

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- The mapping α is a bijection between (pairs of opposite) bounded regions of M and $(1,0)$-active bases of M.

Active bijection: main theorem in the bounded case

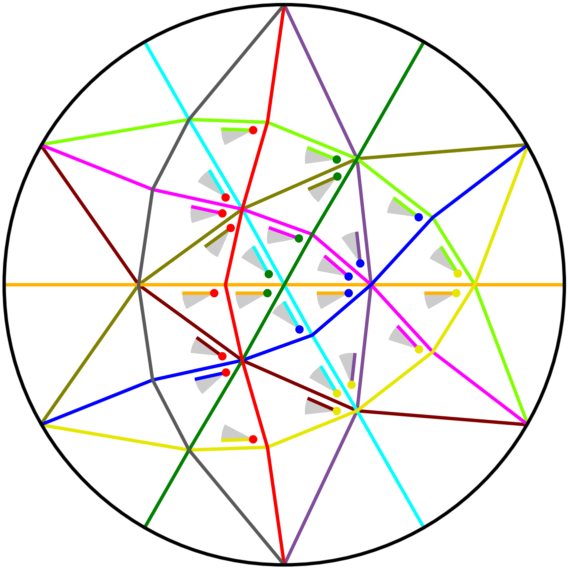
- An ordered bounded oriented matroid w.r.t. $min(E)$ M has a unique fully optimal basis denoted

$$\alpha(M)$$

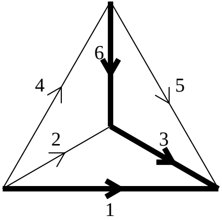
- The mapping α is a bijection between (pairs of opposite) bounded regions of M and $(1,0)$-active bases of M.

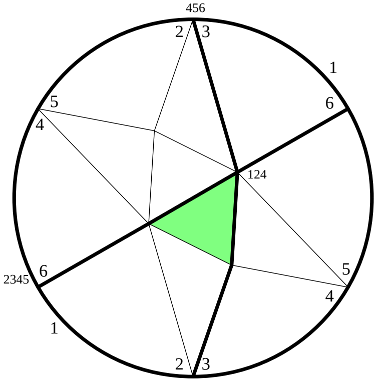
Computational remarks

- The fully optimal basis $\alpha(M)$ is difficult to compute (the problem contains the real/combinatorial linear programming problem).
- But the unique reorientation $\alpha^{-1}(B)$ associated with the basis B is easy to compute (just sign the elements from the first to the last so that the criterion is satisfied).



Ex: 136 is the fully optimal basis of the green region.

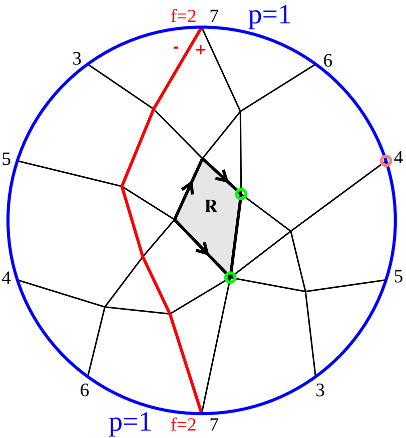




	C_1^*	2	C_3^*	4	5	C_6^*
1	$+$					
C_2	$+$	$-$	$-$			
3			$+$			
C_4	$+$		$-$	$-$		$-$
C_5			$+$		$-$	$+$
6						$+$

(fundamental tableau of the basis)

Usual Linear Programming Optimal Bases (simplex criterion)



137

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3			+				
4	+		−	−			+
5	+		−		−		+
6	+		+			−	−
7							+

145

	1	2	3	4	5	6	7
1	+						
2	+	−		−	+		
3	+		−	+	−		
4				+			
5					+		
6	+			−	+	−	
7				+	−		−

147

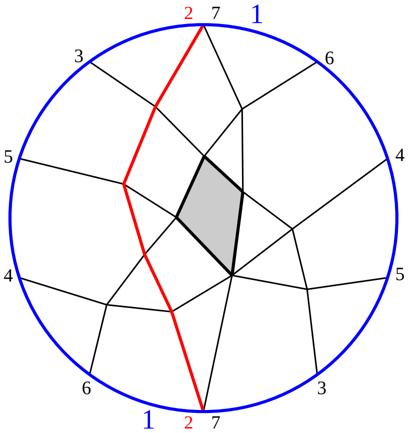
	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			−		−	−
7							+

157

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−		−		+
4				−	+		+
5					+		
6	+				−	−	−
7							+

Only takes the first column and first row into account.

The Fully Optimal Basis



137

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3			+				
4	+		−	−			+
5	+		−		−		+
6	+		+			−	−
7							+

147

	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−	−			+
4				+			
5				+	−		−
6	+			−		−	−
7							+

157

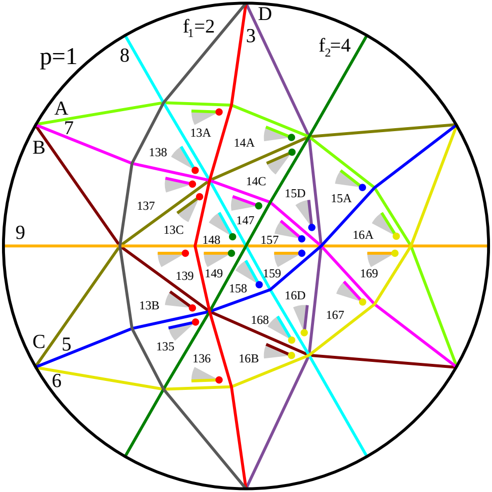
	1	2	3	4	5	6	7
1	+						
2	+	−					−
3	+		−		−		+
4				−	+		+
5					+		
6	+				−	−	−
7							+

Takes all columns and rows into account.

Two elaborations on linear programming:

- multiobjective programming: unique optimal vertex
- flag programming: unique sequence of nested faces (induction)

A “Bijective Characterization” of LP Optimality



In the real case: the active bijection is the unique bijection associating $(1,0)$-active flags and bounded regions with the adjacency property.
In the graphical case: multiobjective LP $\sim$ a cocycle weight function
In the general case: we need the dual adjacency property,
and it implies the bijection.

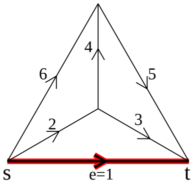
Duality for bounded regions (bipolar orientations)

Bounded region M w.r.t. e $\xleftrightarrow{\text{canonical bijection}}$ Dual-bounded region M^*

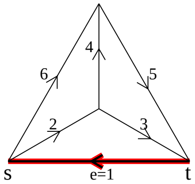
$o^*(M) = 1$, $o(M) = 0$ — $o^*(M^*) = 0$, $o(M^*) = 1$

idem $\xleftrightarrow{\text{canonical bijection}}$ Dual-bounded region $-_e M$

$o^*(-_e M) = 0$, $o(-_e M) = 1$



$\xleftrightarrow{\text{canonical bijection}}$



Bipolar orientation $\overrightarrow{G}$ — Cyclic-bipolar orientation $-_e \overrightarrow{G}$

Strong duality property (refines LP duality):

$$\alpha(-_e M) = \alpha(M) \setminus \{e\} \cup \{e'\}$$

for M bounded w.r.t. e, where $E = e < e' < \ldots$.

(means that the active bijection is compatible with the above bijections)

4

Decompositions of activities and Tutte polynomial in terms of beta invariants of minors

Active bijection: inductive decomposition construction

For every oriented matroid M on a linearly ordered set E,
$\alpha(M)$ defined by the three following properties:

- $\alpha(M)$ is the fully optimal basis of M if it is a bounded region
- $\alpha(M^*) = E \setminus \alpha(M)$
- $\alpha(M) = \alpha(M/F) \uplus \alpha(M(F))$
  where F is the union of all positive circuits of M whose smallest element is the greatest possible active element of M

Lemma. *If $F \neq \emptyset$, then $M(F)$ is dual-bounded, and M/F has one less active element than M.*

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- $\alpha(M^*) = E \setminus \alpha(M)$
- $\alpha(M) = \alpha(M/F) \uplus \alpha(M(F))$
  where F is the complementary of the union of all positive cocircuits of M whose smallest element is the greatest possible dual-active element of M;

(equivalent dual formulation)

Lemma. *If $F \neq \emptyset$, then M/F is bounded, and $M(F)$ has one less dual-active element than M.*

Active bijection: direct decomposition construction

Let M be an ordered oriented matroid on E
with $\iota = o^*(M)$ dual-active elements $a_1 < ... < a_\iota$
and $\varepsilon = o(M)$ active elements $a'_1 < ... < a'_\varepsilon$.

The *active decomposing sequence of M* is

$$\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c = F_0 \subset ... \subset F_\iota = E$$

- It corresponds to the active partition of M:

$$E = \; F'_{\varepsilon-1} \setminus F'_\varepsilon \;\uplus \ldots \uplus F'_0 \setminus F'_1 \;\uplus\; F_1 \setminus F_0 \uplus \ldots \uplus \; F_\iota \setminus F_{\iota-1}$$

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- It corresponds to the active partition of M:

$$E = F'_{\varepsilon-1} \setminus F'_\varepsilon \uplus \ldots \uplus F'_0 \setminus F'_1 \uplus F_1 \setminus F_0 \uplus \ldots \uplus F_\iota \setminus F_{\iota-1}$$

- For $0 \leq k \leq \varepsilon - 1$, we have $F'_k = \bigcup_{\substack{D \text{ positive circuit} \\ \text{Min } D > a'_k}} D$.
- Dually, for $0 \leq k \leq \iota - 1$, we have $F_k = E \setminus \bigcup_{\substack{D \text{ positive cocircuit} \\ \text{Min } D > a_k}} D$.
- F_c is the union of all positive circuits of M (directed cycles), and $E \setminus F_c$ is the union of all positive cocircuits (directed cocycles).
  F_c is a cyclic flat.

Active bijection: direct decomposition construction

Let M be an ordered oriented matroid on E
with $\iota = o^*(M)$ dual-active elements $a_1 < ... < a_\iota$
and $\varepsilon = o(M)$ active elements $a'_1 < ... < a'_\varepsilon$.

The *active decomposing sequence of M* is

$$\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c = F_0 \subset ... \subset F_\iota = E$$

- It corresponds to the active partition of M:

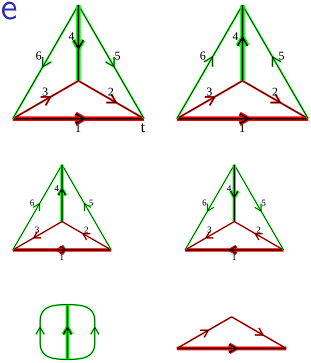
$$E = F'_{\varepsilon-1} \setminus F'_\varepsilon \uplus \ldots \uplus F'_0 \setminus F'_1 \uplus F_1 \setminus F_0 \uplus \ldots \uplus F_\iota \setminus F_{\iota-1}$$

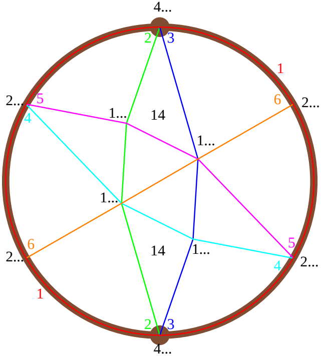
- for $1 \leq k \leq \iota$, the minor $M(F_k)/F_{k-1}$ is dual-bounded (cyclic-bipolar)
- for $1 \leq k \leq \varepsilon$, the minor $M(F'_{k-1})/F'_k$ is bounded (bipolar)

Direct definition of α:

$$\alpha(M) = \biguplus_{1\leq k\leq\iota} \alpha(M(F_k)/F_{k-1}) \uplus \biguplus_{1\leq k\leq\varepsilon} \alpha(M(F'_{k-1})/F'_k)$$

Example





Dual-active elements:		1		4		
Active partition:		123	$\uplus$	456		
Active decomposing sequence:	$\emptyset$	$\subset$	123	$\subset$	123456	
Bounded (bipolar) minors :		$M(123)$	and	$M/123$		

$$\alpha(\overrightarrow{G}) = 134$$

Activity class: $\overrightarrow{G}$, $-_{123}\overrightarrow{G}$, $-_{456}\overrightarrow{G}$, $-_{123456}\overrightarrow{G}$.

Reorientation activity classes of ordered oriented matroids

Activity class of M: the set of $2^{o(M)+o^*(M)}$ reorientations of M obtained by reorienting independently the $o(M) + o^*(M)$ parts of the active partition of M (all its elements have same active partition)

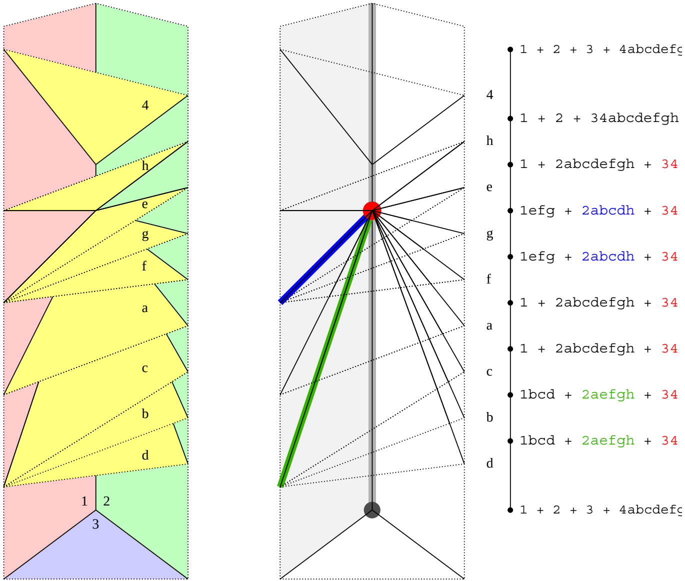
They are interesting on their own:

- Activity classes form a partition of the set 2^E of reorientations of M
- Refine the acyclic / totally cyclic usual decomposition
- Describe the intersections of regions w.r.t. a given (minimal) flag
- Wide generalization of components of acyclic orientations of graphs defined on particular linear orderings of the vertices, as related to
  - Non-commutative monoids [Cartier & Foata 1969]
  - Heaps of pieces [Viennot 1986]
  - Acyclic orientations and the chromatic polynomial [Lass 2001]
- There is one and only one unique sink acyclic orientation in each activity class of acyclic orientation of a graph, as related to

$$t(G; 1, 0) = \#\text{acyclic orientations of } G \text{ with unique sink}$$

[Zaslavsky 1975, Greene & Zaslavsky 1983, Gebhard & Sagan 2000]

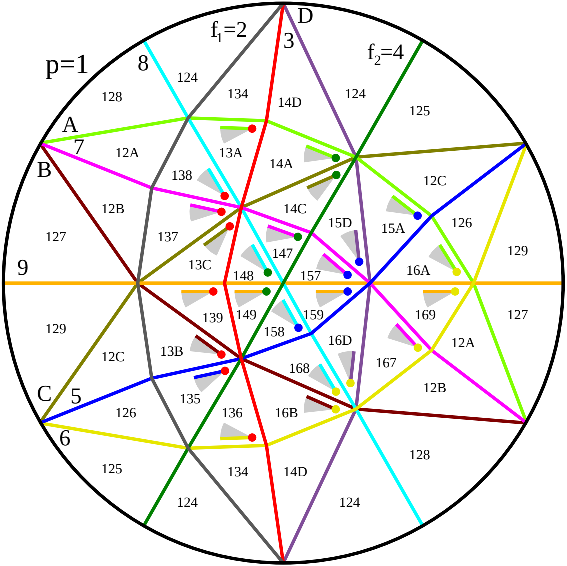
Example of active partitions of rank 4 regions



Active bijection main theorem

- $\alpha(M)$ is a basis of M
- active (resp. dual-active) elements of M are externally (resp. internally) active elements of $\alpha(M)$
- more precisely, α also preserves active partitions (they exist for bases too and ensure that $\alpha(M)$ is a basis)
- the basis $\alpha(M)$ is also associated with the $2^{o(M)+o^*(M)}$ reorientations in the activity class of M
- α is a bijection between all activity classes of reorientations of M and all bases of M.

Active bijection between region activity classes and internal bases



Decomposing sequences

Let E be a finite linearly ordered set.

We call *abstract decomposing sequence of E* a sequence of subsets of E such that:

- $\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c = F_0 \subset ... \subset F_\iota = E$
- the sequence $min(F_k \setminus F_{k-1})$, $1 \leq k \leq \iota$ is increasing with k
- the sequence $min(F'_{k-1} \setminus F'_k)$, $1 \leq k \leq \varepsilon$, is increasing with k

Let M be a matroid on E.

A *decomposing sequence of M* is an abstract decomposing sequence of E such that:

- for every $1 \leq k \leq \iota$, the minor $M(F_k)/F_{k-1}$ is either a single isthmus, or is connected (2-connected for a graph)
- for every $1 \leq k \leq \varepsilon$, the minor $M(F'_{k-1})/F'_k$ is either a single loop, or is connected (2-connected for a graph)

Nota bene. For $|E| > 1$, M connected (2-connected) if and only if $\beta(M) \neq 0$

Theorem: decomposing oriented matroids into bounded regions

Let M be an oriented matroid on a linearly ordered set E.

$$2^E = \{\text{reorientations } A \subseteq E \text{ of } M\}$$

$$= \biguplus \Big\{ A \subseteq E \;\mid\; -_A M(F_k)/F_{k-1},\ 1 \leq k \leq \iota,\ \text{bounded w.r.t. } min(F_k \setminus F_{k-1}),$$

$$\text{and } -_A M(F'_{k-1})/F'_k,\ 1 \leq k \leq \varepsilon,\ \text{dual-bounded w.r.t. } min(F'_{k-1} \setminus F'_k) \Big\}$$

where the disjoint union is over all decomposing sequences of M
$\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c = F_0 \subset ... \subset F_\iota = E$

(the decomposing sequence of M associated in the second term to a reorientation A is the active decomposing sequence of $-_A M$.)

Theorem: decomposing matroid bases into $(1,0)$-activity bases

Let M be a matroid on a linearly ordered set E.

$$\{\text{bases of } M\} = \biguplus_{\substack{\emptyset=F'_\varepsilon\subset\ldots\subset F'_0=F_c \\ F_c=F_0\subset\ldots\subset F_\iota=E \\ \text{decomposing sequence of } M}} \{B'_1\uplus\ldots\uplus B'_\varepsilon\uplus B_1\uplus\ldots\uplus B_\iota \mid$$

for all $1\leq k\leq\varepsilon,\ B'_k$ base of $M(F'_{k-1})/F'_k$ with $\iota(B'_k)=0$ and $\varepsilon(B'_k)=1$,

for all $1\leq k\leq\iota,\ B_k$ base of $M(F_k)/F_{k-1}$ with $\iota(B_k)=1$ and $\varepsilon(B_k)=0\}$

Then $B=B'_1\uplus\ldots\uplus B'_\varepsilon\uplus B_1\uplus\ldots\uplus B_\iota$ it the active partition of B and

$$Int(B)=\cup_{1\leq k\leq\iota} min(F_k\setminus F_{k-1})=\cup_{1\leq k\leq\iota} Int(B_k),$$

$$Ext(B)=\cup_{1\leq k\leq\varepsilon} min(F'_{k-1}\setminus F'_k)=\cup_{1\leq k\leq\varepsilon} Ext(B'_k).$$

Theorem:
Tutte polynomial in terms of β invariants of minors

Let M be a matroid on a linearly ordered set E.

$$t(M;x,y) = \sum \Big(\prod_{1\leq k\leq \iota} \beta\big(M(F_k)/F_{k-1}\big)\Big) \Big(\prod_{1\leq k\leq \varepsilon} \beta'\big(M(F'_{k-1})/F'_k\big)\Big)\ x^\iota\ y^\varepsilon$$

where $\beta'(M) = \beta(M)$ if $|E| > 1$, $\beta'(\text{isthmus}) = 0$, and $\beta'(\text{loop}) = 1$

and where the sum can be equally:

- either over all decomposing sequences of M
- or over all abstract decomposing sequences of E
  $\emptyset = F'_\varepsilon \subset ... \subset F'_0 = F_c = F_0 \subset ... \subset F_\iota = E$

Classical results refined by this formula

Theorem. [Tutte 1954]

$$t(M;x,y) = \sum_{i,j} b_{i,j} x^i y^j$$

where $b_{i,j} = \#\ (i,j)$-active bases

Theorem. [Las Vergnas 1984]

$$t(M;x,y) = \sum_{i,j} o_{i,j} \left(\frac{x}{2}\right)^i \left(\frac{y}{2}\right)^j$$

where $o_{i,j} = \#\ (i,j)$-active reorientations

Theorem. [Etienne & Las Vergnas 1998, Kook Reiner & Stanton 1999]

$$t(M;x,y) = \sum t(M/F;x,0)\ t(M(F);0,y)$$

where the sum can be equally either over all subsets F of E, or over all cyclic flats F of M.

5

Further results

Refined bijection between (re)orientations and subsets

Let M be an (oriented) matroid on a linearly ordered set E.

Partition of 2^E in terms of reorientation activity classes

$$2^E = \biguplus_{\substack{\text{activity classes of } M \\ \text{(one } -_A M \text{ chosen in each class)}}} \left\{ 2^{o(-_A M)+o^*(-_A M)} \ \begin{array}{l}\text{reorientations obtained by}\\ \text{active partition reorienting}\end{array} \right\}$$

Partition of 2^E in terms of matroid basis activity intervals

[Crapo 1969, Dawson 1981, ...]

$$2^E = \biguplus_{B \text{ basis of } M} \left[\ B \setminus Int(B), \ B \cup Ext(B) \ \right]$$

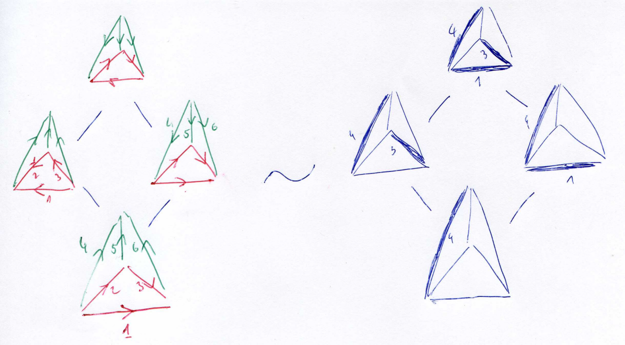
Active bijection

One activity class (2^{i+j} elements)	$\longleftrightarrow$	one basis	$\longleftrightarrow$	one interval (2^{i+j} elements)

Refined bijection between (re)orientations and subsets

Active bijection

One activity class	$\longleftrightarrow$	one basis	$\longleftrightarrow$	one interval
(2^{i+j} elements)				(2^{i+j} elements)



The activity class of $-_A M$ and the matroid basis interval of $B = \alpha(-_A M)$ are isomorphic boolean lattices.

Refined bijection between (re)orientations and subsets

Refined active bijection

For $A \subseteq E$ and $B = \alpha(-_A M)$, set

$$\alpha^{\emptyset}_M(A) = B \ \setminus \ (A \cap Int(B)) \ \cup \ (A \cap Ext(B))$$

Theorem

- $\alpha^{\emptyset}_M$ is a bijection between 2^E (reorientations) and 2^E (subsets)
- restricted to acyclic reorientations, $\alpha^{\emptyset}_M$ is a bijection between regions (acyclic orientations) and no-broken-circuit subsets (i.e. subsets of bases with external activity zero)
- it preserves activities, active partitions, and also some four-variable refined activities (that take into acount the positions in the boolean lattices)

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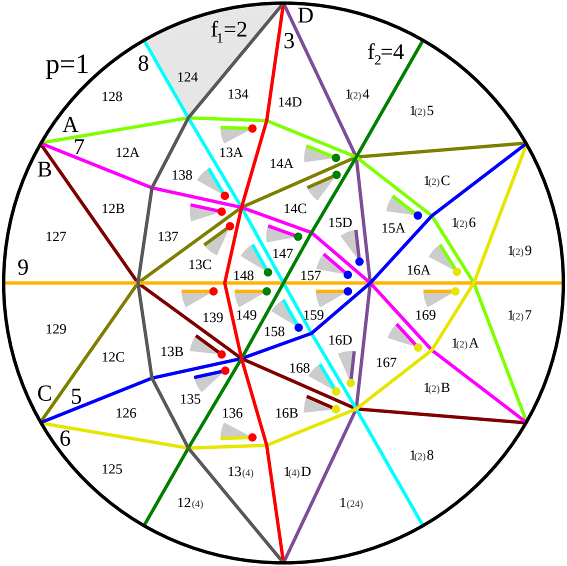
  (i.e. subsets of bases with external activity zero)
- it preserves activities, active partitions, and also some four-variable refined activities
  (that take into acount the positions in the boolean lattices)

Remark. $\alpha^{\emptyset}_{M}$ depends on M and applies to A (on the contrary with α).
The $^{\emptyset}$ symbol is a parameter that can be changed to get other similar refined active bijections, with other boolean lattice bijections.

Active bijection between regions and no-broken-circuit subsets

(acyclic restriction of the refined active bijection w.r.t. the grey region)



Refined basis activities

Let M be a matroid on a linearly ordered set E. Let $B \subseteq E$ be a basis of M. Let A be in the boolean interval $[B \setminus Int_M(B), B \cup Ext_M(B)]$. Set:

$$\begin{aligned} Int_M(A) &= Int_M(B) \cap A; \\ P_M(A) &= Int_M(B) \setminus A; \\ Ext_M(A) &= Ext_M(B) \setminus A; \\ Q_M(A) &= Ext_M(B) \cap A. \end{aligned}$$

Theorem. [Gordon & Traldi 1990, Las Vergnas 2013]
Let M be a matroid on a linearly ordered set E.

$$T(M; x+u, y+v) = \sum_{A \subseteq E} x^{|Int_M(A)|} u^{|P_M(A)|} y^{|Ext_M(A)|} v^{|Q_M(A)|}$$

Refined orientation activities

Let M be an oriented matroid on a linearly ordered set E. Let $A \subseteq E$. Set:

$$\Theta_M(A) = O(-_A M) \setminus A, \qquad \theta_M(A) = |\Theta_M(A)|,$$
$$\bar{\Theta}_M(A) = \Theta_M(E \setminus A) = O(-_A M) \cap A, \qquad \bar{\theta}_M(A) = |\bar{\Theta}_M(A)|.$$

$$\Theta^*_M(A) = \Theta_{M^*}(A) = O^*(-_A M) \setminus A, \qquad \theta^*_M(A) = |\Theta^*_M(A)|,$$
$$\bar{\Theta}_M(A) = \bar{\Theta}_{M^*}(A) = O^*(-_A M) \cap A, \qquad \bar{\theta}^*_M(A) = |\bar{\Theta}^*_M(A)|.$$

Theorem.

$$T(M; x+u, y+v) = \sum_{A \subseteq E} x^{\theta^*_M(A)} u^{\bar{\theta}^*_M(A)} y^{\theta_M(A)} v^{\bar{\theta}_M(A)}.$$

Proof: by the partition of 2^E into activity classes.

Refined bijection between (re)orientations and subsets

For $B = \alpha(-_A M)$,

$$\alpha_M^{\emptyset}(A) = B \ \setminus \ (A \cap Int(B)) \ \cup \ (A \cap Ext(B))$$

Theorem (continued)

- $\alpha_M^{\emptyset}$ is a bijection between 2^E (reorientations) and 2^E (subsets)
- It preserves the four refined activities, i.e. for all $A \subseteq E$:

$$\begin{array}{rcl} Int_M(\alpha_M^{\emptyset}(A)) & = & \Theta_M^*(A) \\ P_M(\alpha_M^{\emptyset}(A)) & = & \bar{\Theta}_M^*(A) \\ Ext_M(\alpha_M^{\emptyset}(A)) & = & \Theta_M(A) \\ Q_M(\alpha_M^{\emptyset}(A)) & = & \bar{\Theta}_M(A) \end{array}$$

(bijection for the equality of the two expressions of $t(M; x+u, y+v)$)

And now for something completely different

Deletion/contraction construction of various bijections

M oriented matroid on a linearly ordered set E with $max(E) = \omega$.

Choice at each step:

$$\left\{ \alpha(M),\ \alpha(-_\omega M) \right\} = \left\{ \alpha(M \setminus \omega),\ \alpha(M/\omega) \cup \omega \right\}$$

Theorem.
With suitable choices, we get whole classes of bijections between
bases: all / subsets / internal / no broken circuit subsets / ...
and
reorientations: classes / all / specific / acyclic / ...

Various properties can be demanded: activities / adjacency / ...

Specifying choices yield: THE active bijection.

Deletion/contraction construction of the active bijection

M oriented matroid on a linearly ordered set E with $max(E) = \omega$.

Choice at each step:

$$\Big\{\ \alpha(M),\ \alpha(-_\omega M)\ \Big\}\ =\ \Big\{\ \alpha(M\setminus\omega),\ \alpha(M/\omega)\cup\omega\ \Big\}$$

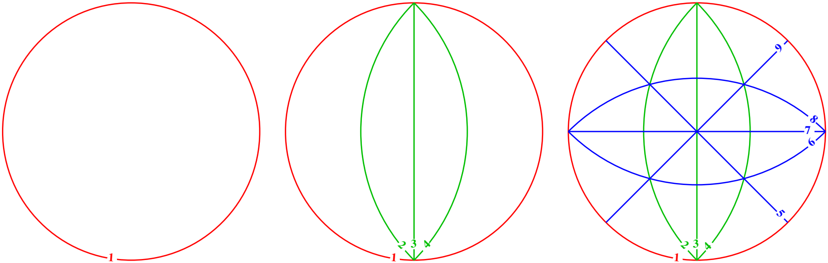
One way of defining the active bijection:

- If M is a region (acyclic orientation):
  - If M is not cut by ω then $\alpha(M) = \alpha(M\setminus\omega)$
    (and $\alpha(-_\omega M) = \alpha(M/\omega)\cup\omega$)
  - If M is cut by ω then $-_\omega M$ is also a region and
    - If M and $-_\omega M$ do not have same active partition then the choice is determined
    - Otherwise then compare the flags associated to each region in the bounded minor containing ω in order to preserve the fully optimal basis adjacency property
      (refines the usual LP induction by variable/constraint deletion)
- If M is a dual-region, then use dual rules.
- If M is not a region nor a dual-region, then use the same rules in region/dual-region minors (can be done also directly)

Supersolvable arrangement (generalizes chordal graph)

Inductive construction: at each new dimension, the new hyperplanes cut each other in lower dimension hyperplanes

(roughly said)



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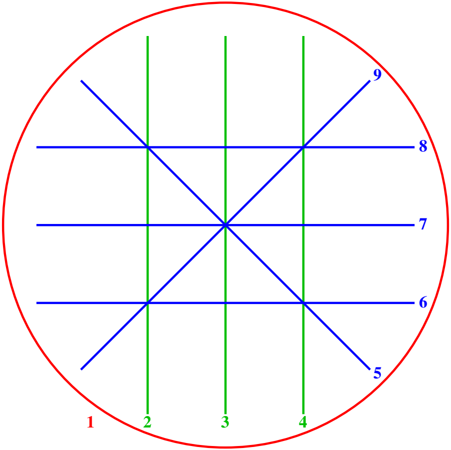


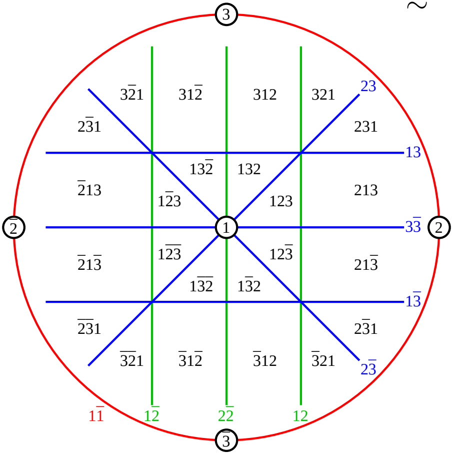
Allows a construction in nested "fibers" dimension by dimension, that fits well with the deletion/contraction construction...

Signed permutations

Regions of the hyperoctahedral arrangement $\sim$ Signed permutations

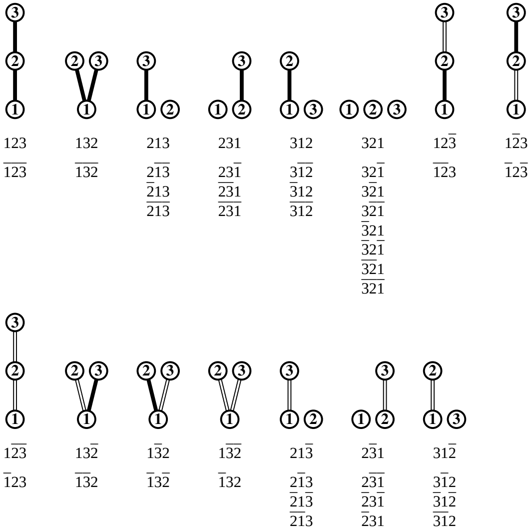
(it is supersolvable) $\sim$ Cube symmetries

$\sim$ Coxeter group B_n



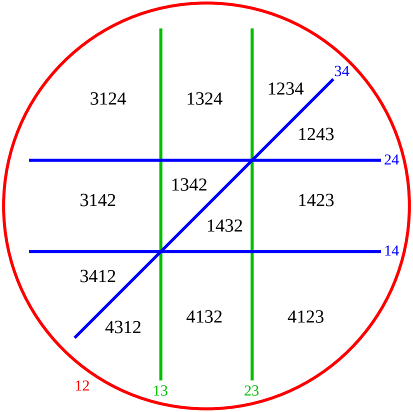
No-broken-circuit base subsets $\sim$ edge-signed increasing forests

Active bijection between signed permutations and edge-signed increasing forests



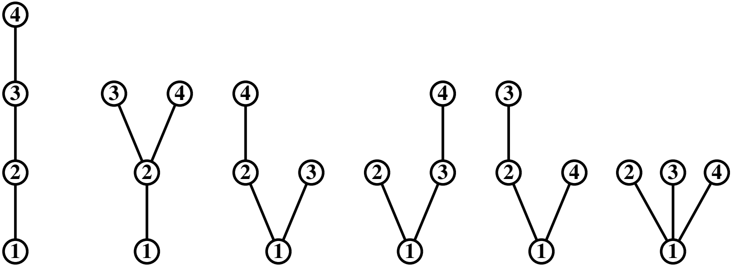
Permutations (acyclic complete graph case)

Regions of the braid arrangement	$\sim$	Permutations
(it is supersolvable)	$\sim$	Simplex symmetries
	$\sim$	Coxeter group A_n
	$\sim$	Acyclic orientations of K_{n+1}



No-broken-circuit base subsets $\sim$ increasing trees

Active bijection between permutations and increasing trees



123	132	213	231	312	321
1432	1342	1423	1243	1324	1234
2341	2431	2413	2143	2314	2134
		3142	3412	4132	3124
		3241	3421	4231	3214
					4123
					4213
					4312
					4321

It is equal to a classical bijection!

oriented matroids	
activity classes of reorientations act. cl. of acyclic reorientations act. cl. of totally cyclic reor. bounded acyclic reorientations reorientations acyclic reorientations totally cyclic reorientations	bases bases B with $\varepsilon(B) = 0$ bases B with $\iota(B) = 0$ bases B with $\iota(B) = 1$ and $\varepsilon(B) = 0$ subsets no-broken-circuit subsets supsets of bases B with $\iota(B) = 0$
hyperplane arrangements	
reorientations $\sim$ signatures *acyclic reorientations $\sim$ regions*	*bases $\sim$ simplices*
graphs	
reorientations $\sim$ orientations unique sink acyclic orientations bipolar orientations source-sink reversed bipolar orientations	*bases $\sim$ spanning trees* spanning trees B with $\varepsilon(B) = 0$ sp. trees B with $\iota(B) = 1$ and $\varepsilon(B) = 0$ sp. trees B with $\iota(B) = 0$ and $\varepsilon(B) = 1$
uniform oriented matroids	
bounded regions	linear programming optimal vertices
braid arrangement or complete graph or Coxeter arrangement A_n	
permutations	increasing trees
hyperoctahedral arrangement or Coxeter arrangement B_n	
signed permutations	signed increasing trees

6

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  (almost) ready for submission and available on demand or at Arxiv (2015).

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