Error-free Tables for Trigonometric Function Evaluation

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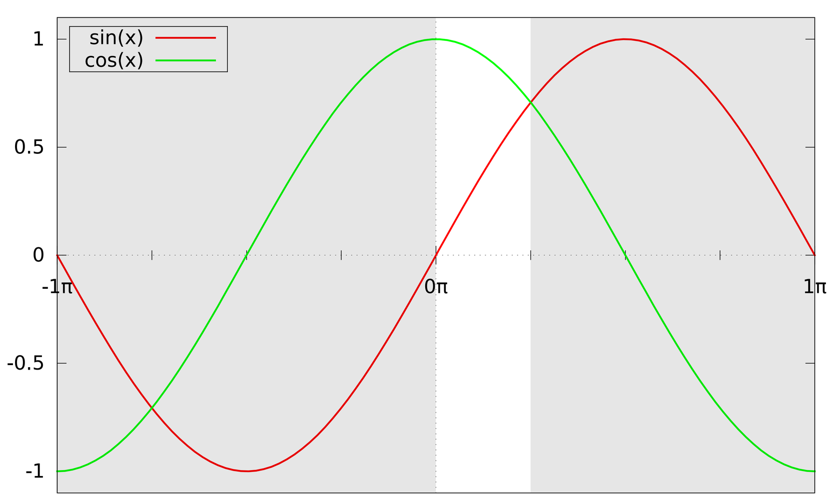
1. How does a modern processor calculate sine and cosine?

Range reduction: use of **trigonometric identities**

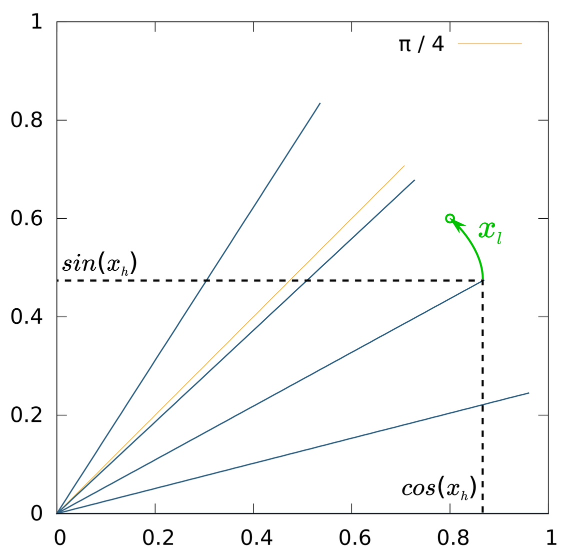
- $\sin(-x) = -\sin(x)$
- $\sin(x) = \pm f_k\left(x - k \cdot \frac{\pi}{2}\right)$ with $f_k \in \{\sin, \cos\}$

⇒ Range reduction $\mathbb{F}_{64} \mapsto [0, \frac{\pi}{4}]$

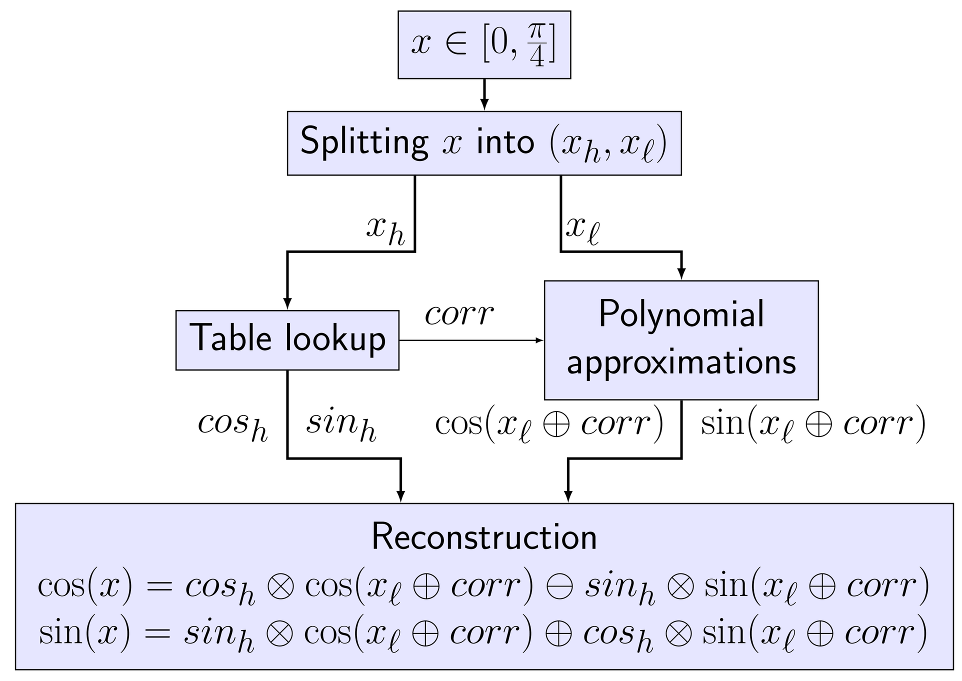


Evaluation: use of other trigonometric properties

- $\sin(x_h + x_l) = \sin(x_h) \cdot \cos(x_l) + \cos(x_h) \cdot \sin(x_l)$
- $\cos(x_h + x_l) = \cos(x_h) \cdot \cos(x_l) - \sin(x_h) \cdot \sin(x_l)$
- **Tabulated values** for sine and cosine [Tan91]

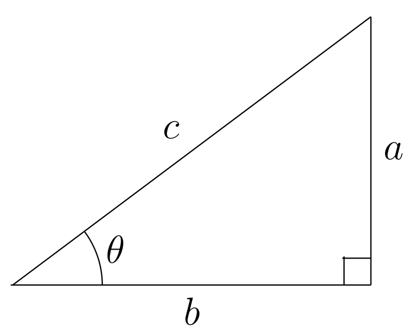


Reconstruction: scheme **reducing the error** on tabulated values [GB91]



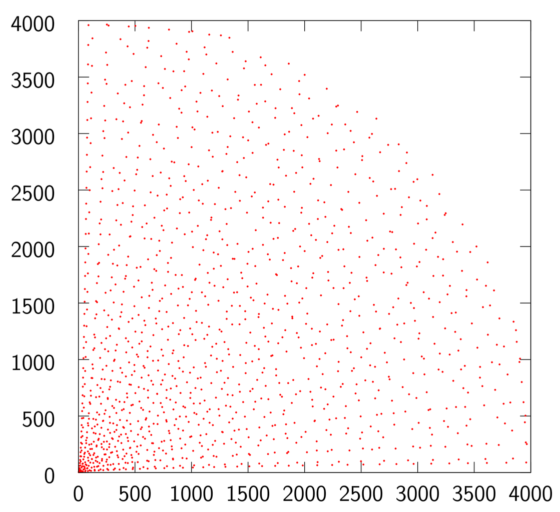
2. Pythagorean Triples

What is a **Pythagorean triple?**



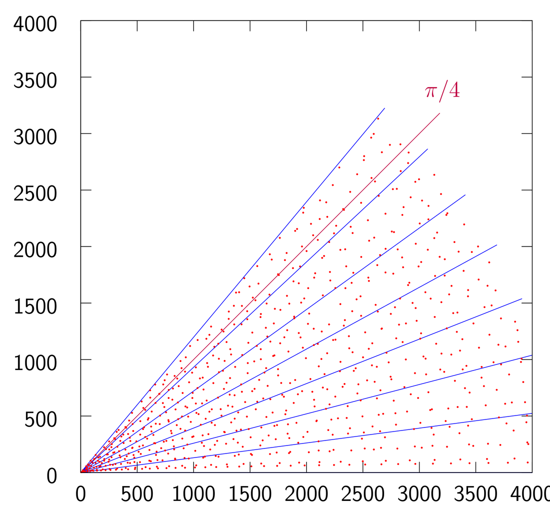
$$\begin{cases} (a, b, c) \in \mathbb{N}^3 \\ a^2 + b^2 = c^2 \\ \sin(\theta) = \dfrac{a}{c} \quad \cos(\theta) = \dfrac{b}{c} \end{cases}$$

Primitive Pythagorean Triple : a Pythagorean triple (a, b, c) for which $\gcd(a, b, c) = 1$.



Primitive Pythagorean triples with $c \leq 2^{12}$

⇒

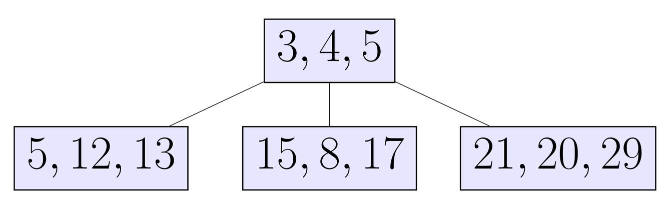


Only a subset fits in a table.

3. Primitive Pythagorean Triple Generation

Barning-Hall ternary-tree structure:

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$



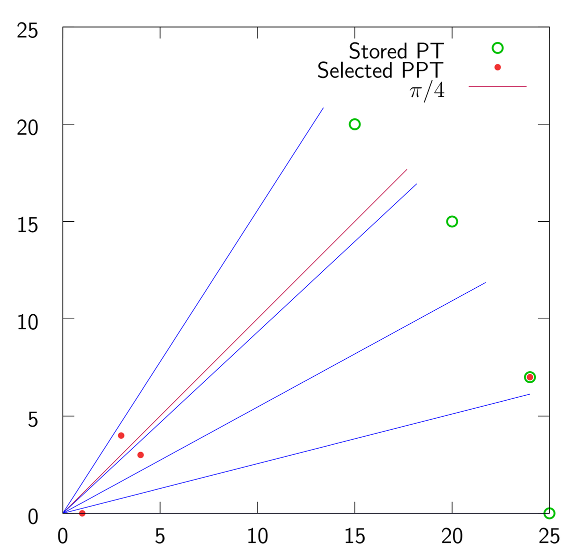
- Several equivalent trees, easy to implement
- Proven to generate all primitive triples by increasing hypotenuse lengths [Bar63]

4. Primitive Pythagorean Triple Selection

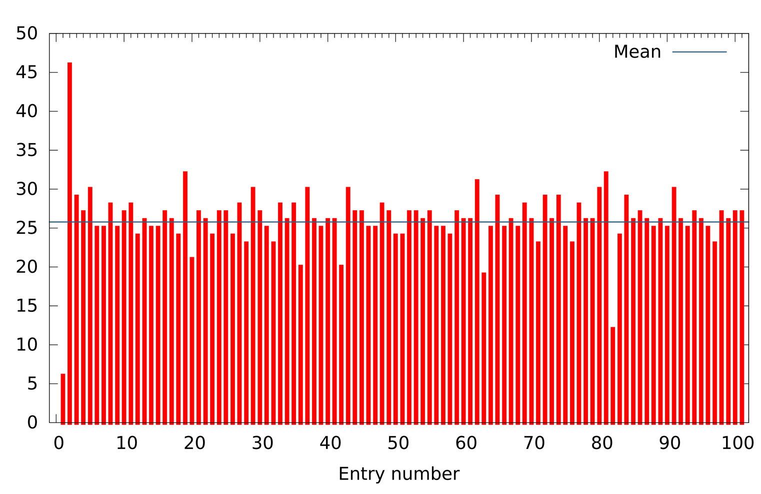
- Only **one triple per entry** needed
- Division by c removed by incorporating it in polynomial approximations

⇒ **Same hypotenuse** c needed for all entries

⇒ **Scale** selected PPTs to the **least common multiple** of their hypotenuses



A simple example for a 2-bit indexed table



Number of PPTs/entry for a 7-bit indexed table: roughly 26^{100} combinations of 1 PPT/entry.

5. Exhaustive Search for a Small Common Hypotenuse

Algorithm

```
n ← 4
repeat
    Generate all PPTs (a, b, c) such that c ≤ 2^n.
    Search for the LCM k among all generated hypotenuses c.
    n ← n + 1
until such a k is found
Build tabulated values (A, B, corr) for every entry.
```

Results

p	k_{min}	n	time (s)	Triples	Hypotenuses
3	425	9	$\ll 1$	86	66
4	5525	13	$\ll 1$	1404	889
5	160,225	18	0.2	42,328	24,228
6	1,698,385	21	7	335,344	179,632
7	6,569,225	23	31	1,347,953	686,701
8	$> 2^{27}$	> 27	$> 6700?$	$> 21,407,992$	$> 10,144,723$

- Impossible to generate tables indexed by **more than 7 bits**.
- 8 to 10 bit-indexed tables desired to **optimize caching**.

6. Heuristic Search

Prime factorization of found common multiples

k	Prime factorization
425	$5^2 \cdot 17$
5525	$5^2 \cdot 13 \cdot 17$
160,225	$5^2 \cdot 13 \cdot 17 \cdot 29$
1,698,385	$5 \cdot 13 \cdot 17 \cdot 29 \cdot 53$
6,569,225	$5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 41$

Heuristic: store primitive Pythagorean triples satisfying

$$c = \prod_i p_i^{r_i} \quad \text{with}$$

- $r_i \in \{0, 1\}$ if $p_i \neq 5$
- $r_i \in \mathbb{N}^*$ else
- and $p_i \in \mathcal{P}$

where $\mathcal{P}$ is the set of Pythagorean primes ≤ 73:

$$\mathcal{P} = \{5, 13, 17, 29, 37, 41, 53, 61, 73\}$$

Results

p	k_{min}	n	time (s)	triples	hypotenuses
6	1,698,385	21	0.1	2171	66
7	6,569,225	23	0.4	3452	69
8	314,201,225	29	9.5	10,467	100
9	12,882,250,225	34	294	20,311	109
10	279,827,610,985	39	9393	33,056	110

- > 99 % **less memory usage**
- > 99 % **time saved at generation**
- **Same tables** for $p \in [\![3, 7]\!]$

7. Theoretical Gains

Comparison between three table-based range reductions, for $p = 10$. The number of memory accesses (MA) and the number of floating point operations (FLOP) are reported.

Solution	Quick phase (66 bits)	Accurate phase (150 bits)	Table size (bytes)
Tang	4 MA + 64 FLOP	6 MA + 241 FLOP	38640
Gal	**3 MA + 53 FLOP**	9 MA + 268 FLOP	57960
Proposed	**3 MA + 53 FLOP**	**5 MA + 148 FLOP**	**32200**

References

[Bar63] F. J. M. Barning. On pythagorean and quasi-pythagorean triangles and a generation process with the help of unimodular matrices. *(Dutch) Math. Centrum Amsterdam Afd. Zuivere Wisk. ZW-001*, 1963.

[GB91] S. Gal and B. Bachelis. An accurate elementary mathematical library for the IEEE floating point standard. *ACM Trans. Math. Software*, 17:26–45, 1991.

[Tan91] Ping Tak Peter Tang. Table-lookup algorithms for elementary functions and their error analysis. In Peter Kornerup and David W. Matula, editors, *Proceedings: 10th IEEE Symposium on Computer Arithmetic: June 26–28, 1991, Grenoble, France*, pages 232–236, pub-IEEE:adr, 1991. IEEE Computer Society Press.

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