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# Error-free Tables for Trigonometric Function Evaluation

Hugues de Lassus Saint-Geniès, David Defour and Guillaume Revy

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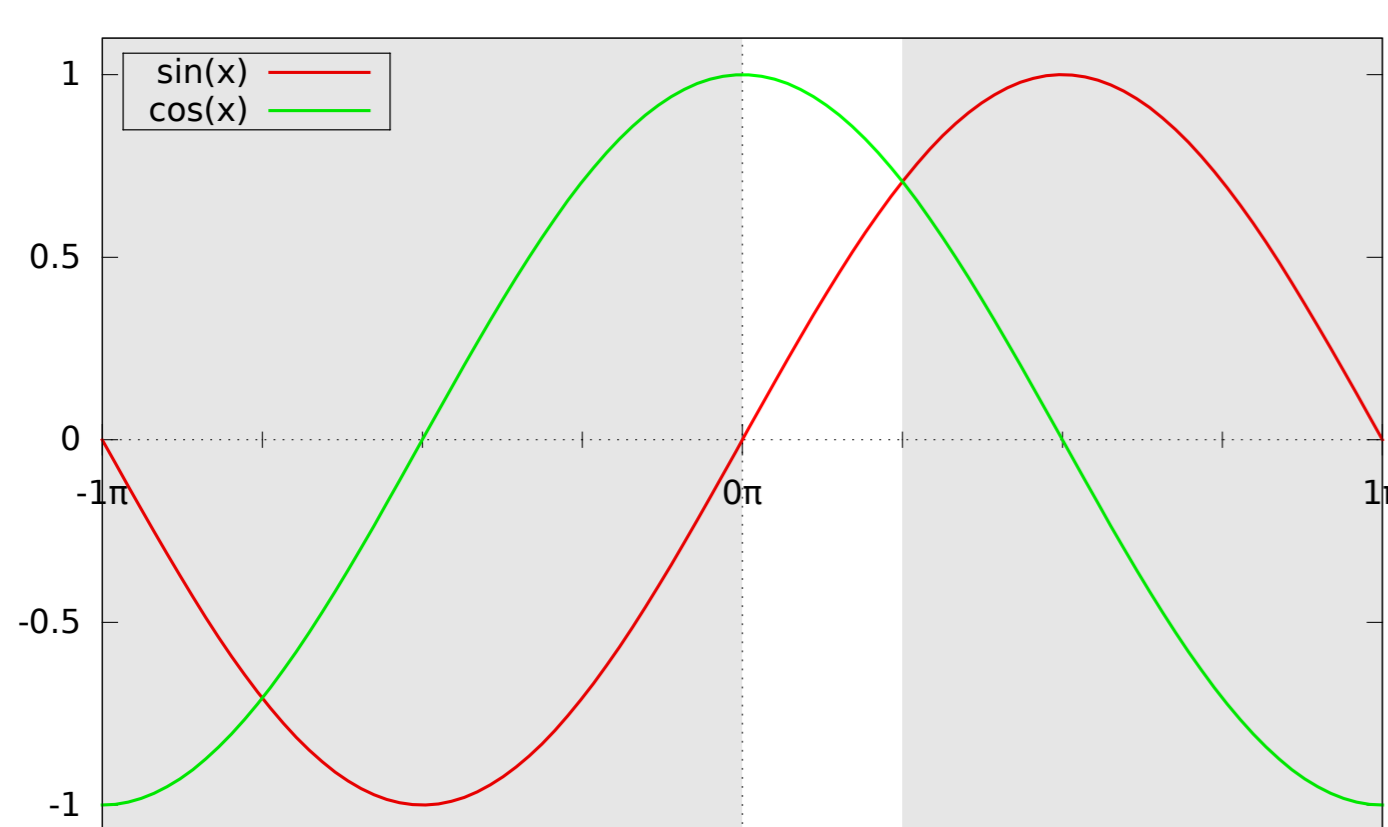


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## 1. How does a modern processor calculate sine and cosine?

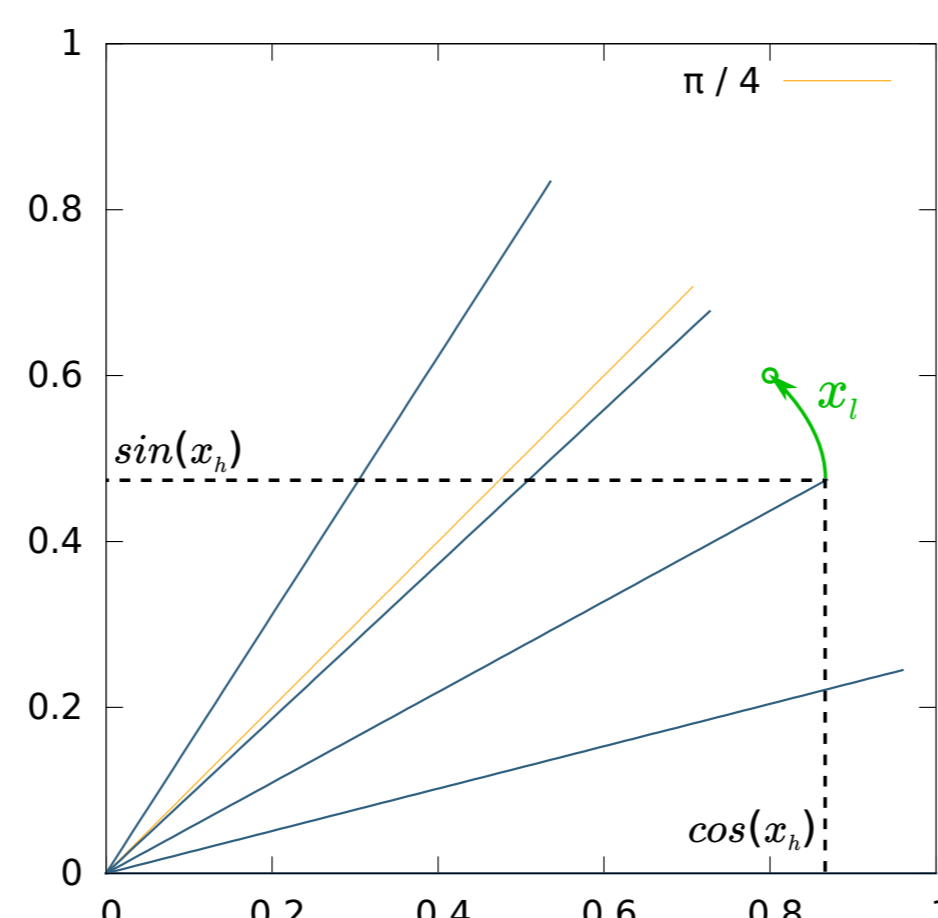
### Range reduction: use of trigonometric identities

- ▶  $\sin(-x) = -\sin(x)$
- ▶  $\sin(x) = \pm f_k(x - k \cdot \frac{\pi}{2})$  with  $f_k \in \{\sin, \cos\}$
- ⇒ Range reduction  $\mathbb{F}_{64} \mapsto [0, \frac{\pi}{4}]$

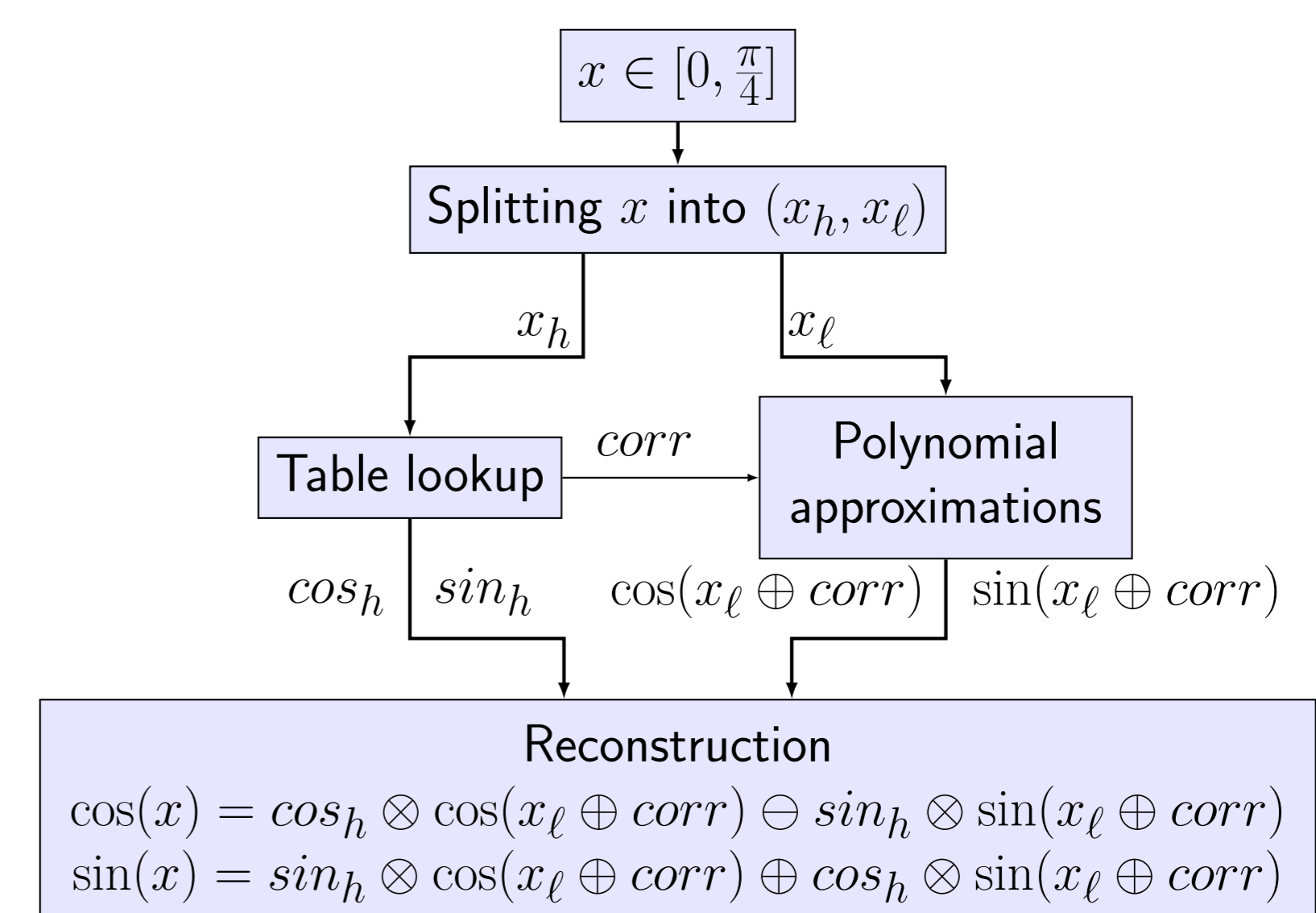


### Evaluation: use of other trigonometric properties

- ▶  $\sin(x_h + x_l) = \sin(x_h) \cdot \cos(x_l) + \cos(x_h) \cdot \sin(x_l)$
- ▶  $\cos(x_h + x_l) = \cos(x_h) \cdot \cos(x_l) - \sin(x_h) \cdot \sin(x_l)$
- ▶ **Tabulated values** for sine and cosine [Tan91]

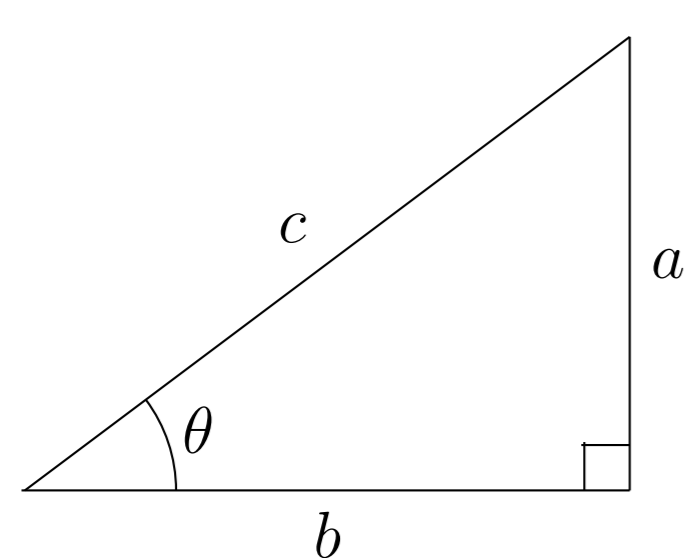


### Reconstruction: scheme reducing the error on tabulated values [GB91]



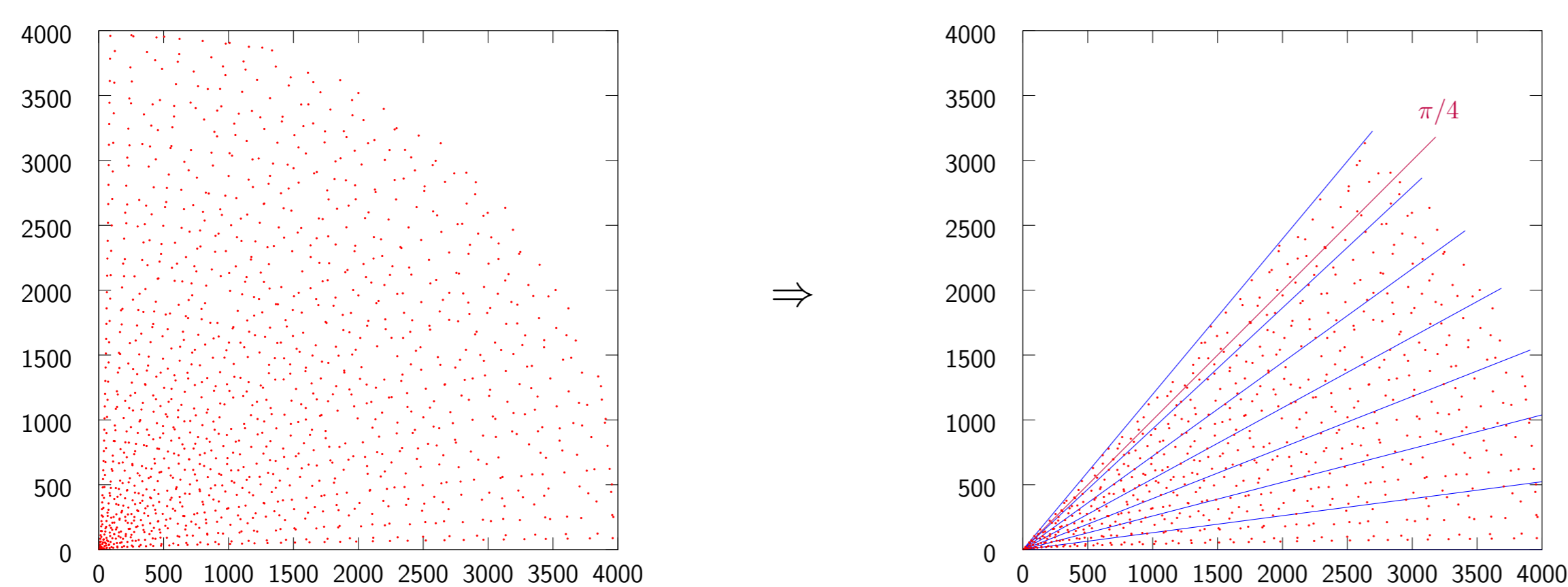
## 2. Pythagorean Triples

### What is a Pythagorean triple?



$$\begin{cases} (a, b, c) \in \mathbb{N}^3 \\ a^2 + b^2 = c^2 \\ \sin(\theta) = \frac{a}{c} \quad \cos(\theta) = \frac{b}{c} \end{cases}$$

**Primitive Pythagorean Triple**: a Pythagorean triple  $(a, b, c)$  for which  $\gcd(a, b, c) = 1$ .



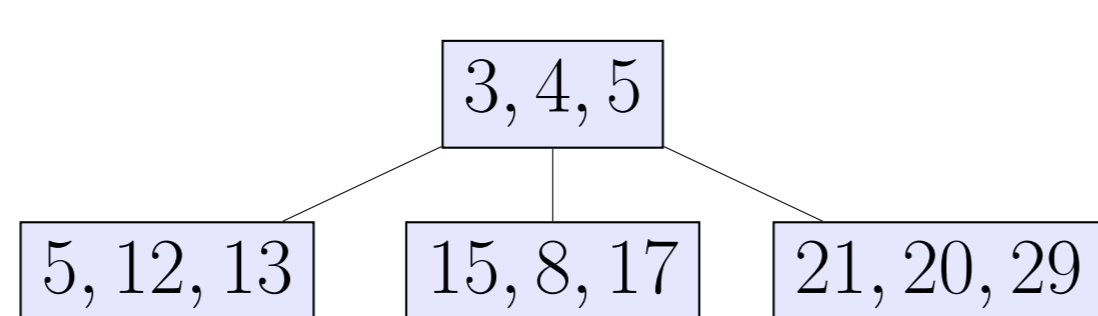
Primitive Pythagorean triples with  $c \leq 2^{12}$

Only a subset fits in a table.

## 3. Primitive Pythagorean Triple Generation

### Barning-Hall ternary-tree structure:

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$



### Several equivalent trees, easy to implement

### Proven to generate all primitive triples by increasing hypotenuse lengths [Bar63]

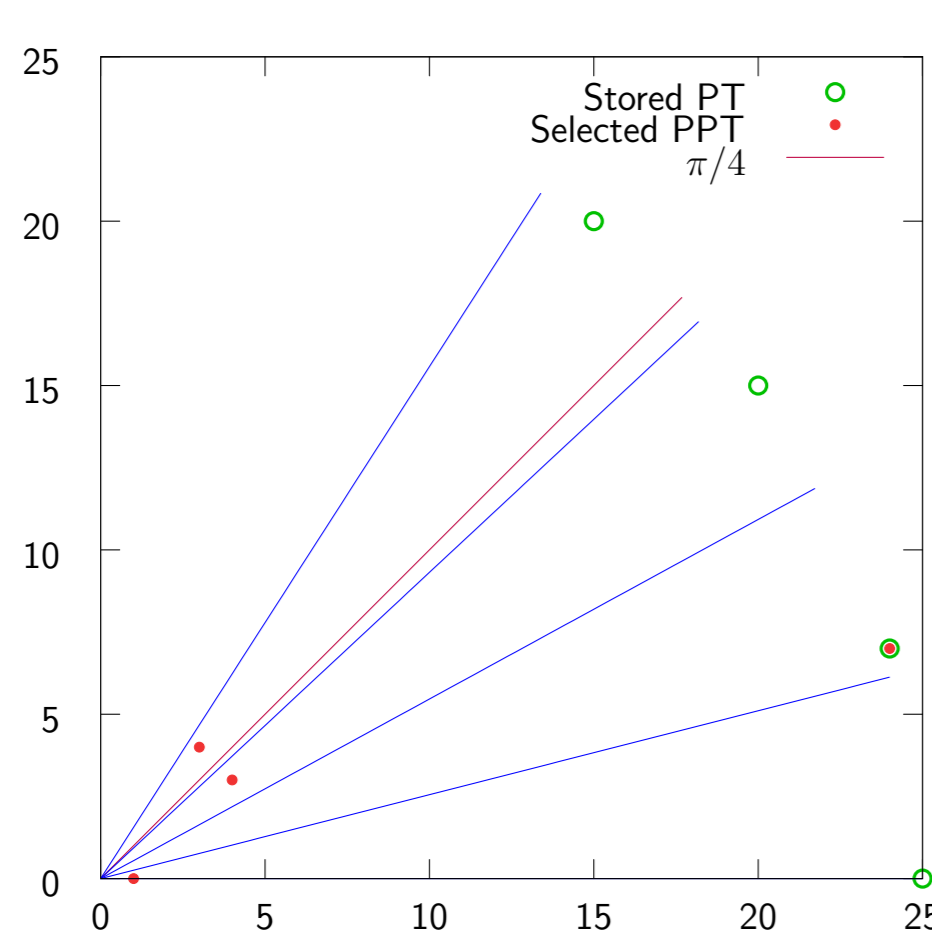
## 4. Primitive Pythagorean Triple Selection

### Only one triple per entry needed

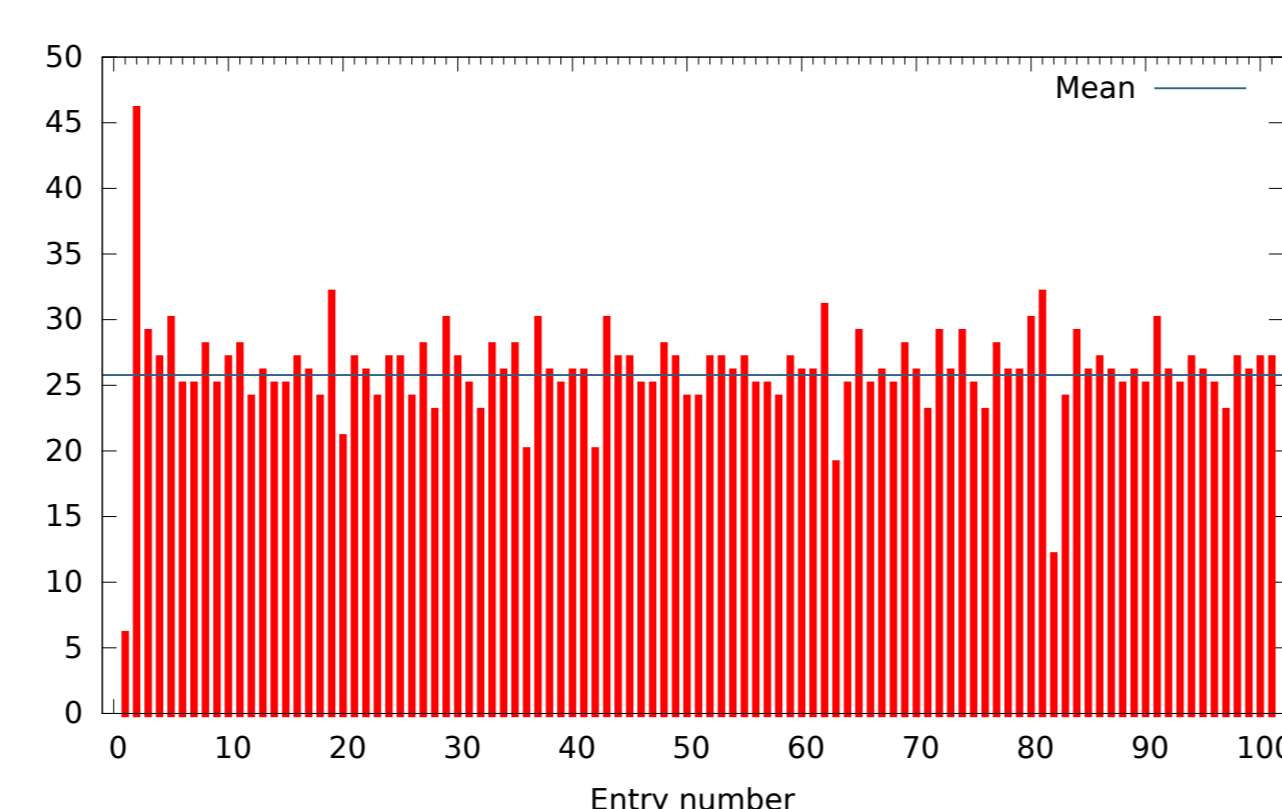
### Division by $c$ removed by incorporating it in polynomial approximations

⇒ **Same hypotenuse**  $c$  needed for all entries

⇒ **Scale** selected PPTs to the **least common multiple** of their hypotenuses



A simple example for a 2-bit indexed table



Number of PPTs/entry for a 7-bit indexed table: roughly  $26^{100}$  combinations of 1 PPT/entry.

## 5. Exhaustive Search for a Small Common Hypotenuse

### Algorithm

- 1:  $n \leftarrow 4$
- 2: **repeat**
- 3:   Generate all PPTs  $(a, b, c)$  such that  $c \leq 2^n$ .
- 4:   Search for the LCM  $k$  among all generated hypotenuses  $c$ .
- 5:    $n \leftarrow n + 1$
- 6: **until** such a  $k$  is found
- 7: Build tabulated values  $(A, B, corr)$  for every entry.

### Results

| $p$ | $k_{min}$  | $n$    | time (s)  | Triples        | Hypotenuses    |
|-----|------------|--------|-----------|----------------|----------------|
| 3   | 425        | 9      | $\ll 1$   | 86             | 66             |
| 4   | 5525       | 13     | $\ll 1$   | 1404           | 889            |
| 5   | 160,225    | 18     | 0.2       | 42,328         | 24,228         |
| 6   | 1,698,385  | 21     | 7         | 335,344        | 179,632        |
| 7   | 6,569,225  | 23     | 31        | 1,347,953      | 686,701        |
| 8   | $> 2^{27}$ | $> 27$ | $> 6700?$ | $> 21,407,992$ | $> 10,144,723$ |

▶ Impossible to generate tables indexed by **more than 7 bits**.

▶ 8 to 10 bit-indexed tables desired to **optimize caching**.

## 6. Heuristic Search

### Prime factorization of found common multiples

| $k$       | Prime factorization                       |
|-----------|---|
| 425       | $5^2 \cdot 17$                            |
| 5525      | $5^2 \cdot 13 \cdot 17$                   |
| 160,225   | $5^2 \cdot 13 \cdot 17 \cdot 29$          |
| 1,698,385 | $5 \cdot 13 \cdot 17 \cdot 29 \cdot 53$   |
| 6,569,225 | $5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 41$ |

### Heuristic: store primitive Pythagorean triples satisfying

$$c = \prod_i p_i^{r_i} \quad \text{with}$$

- ▶  $r_i \in \{0, 1\}$  if  $p_i \neq 5$
- ▶  $r_i \in \mathbb{N}^*$  else
- ▶ and  $p_i \in \mathcal{P}$

where  $\mathcal{P}$  is the set of Pythagorean primes  $\leq 73$ :

$$\mathcal{P} = \{5, 13, 17, 29, 37, 41, 53, 61, 73\}$$

### Results

| $p$ | $k_{min}$       | $n$ | time (s) | triples | hypotenuses |
|-----|-----------------|-----|----------|---------|-------------|
| 6   | 1,698,385       | 21  | 0.1      | 2171    | 66          |
| 7   | 6,569,225       | 23  | 0.4      | 3452    | 69          |
| 8   | 314,201,225     | 29  | 9.5      | 10,467  | 100         |
| 9   | 12,882,250,225  | 34  | 294      | 20,311  | 109         |
| 10  | 279,827,610,985 | 39  | 9393     | 33,056  | 110         |

▶ **> 99 % less memory usage**

▶ **> 99 % time saved at generation**

▶ **Same tables** for  $p \in \{3, 7\}$

## 7. Theoretical Gains

Comparison between three table-based range reductions, for  $p = 10$ . The number of memory accesses (MA) and the number of floating point operations (FLOP) are reported.

| Solution | Quick phase (66 bits) | Accurate phase (150 bits) | Table size (bytes) |
|----------|-----------------------|---------------------------|--------------------|
| Tang     | 4 MA + 64 FLOP        | 6 MA + 241 FLOP           | 38640              |
| Gal      | <b>3 MA + 53 FLOP</b> | 9 MA + 268 FLOP           | 57960              |
| Proposed | <b>3 MA + 53 FLOP</b> | <b>5 MA + 148 FLOP</b>    | <b>32200</b>       |

## References

- [Bar63] F. J. M. Barning. On pythagorean and quasi-pythagorean triangles and a generation process with the help of unimodular matrices. *(Dutch) Math. Centrum Amsterdam Afd. Zuivere Wisk. ZW-001*, 1963.
- [GB91] S. Gal and B. Bachelis. An accurate elementary mathematical library for the IEEE floating point standard. *ACM Trans. Math. Software*, 17:26–45, 1991.
- [Tan91] Ping Tak Peter Tang. Table-lookup algorithms for elementary functions and their error analysis. In Peter Kornerup and David W. Matula, editors, *Proceedings: 10th IEEE Symposium on Computer Arithmetic: June 26–28, 1991, Grenoble, France*, pages 232–236, pub-IEEE:adr, 1991. IEEE Computer Society Press.