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# Error-free Tables for Trigonometric Function Evaluation

Hugues de Lassus Saint-Geniès, David Defour and Guillaume Revy

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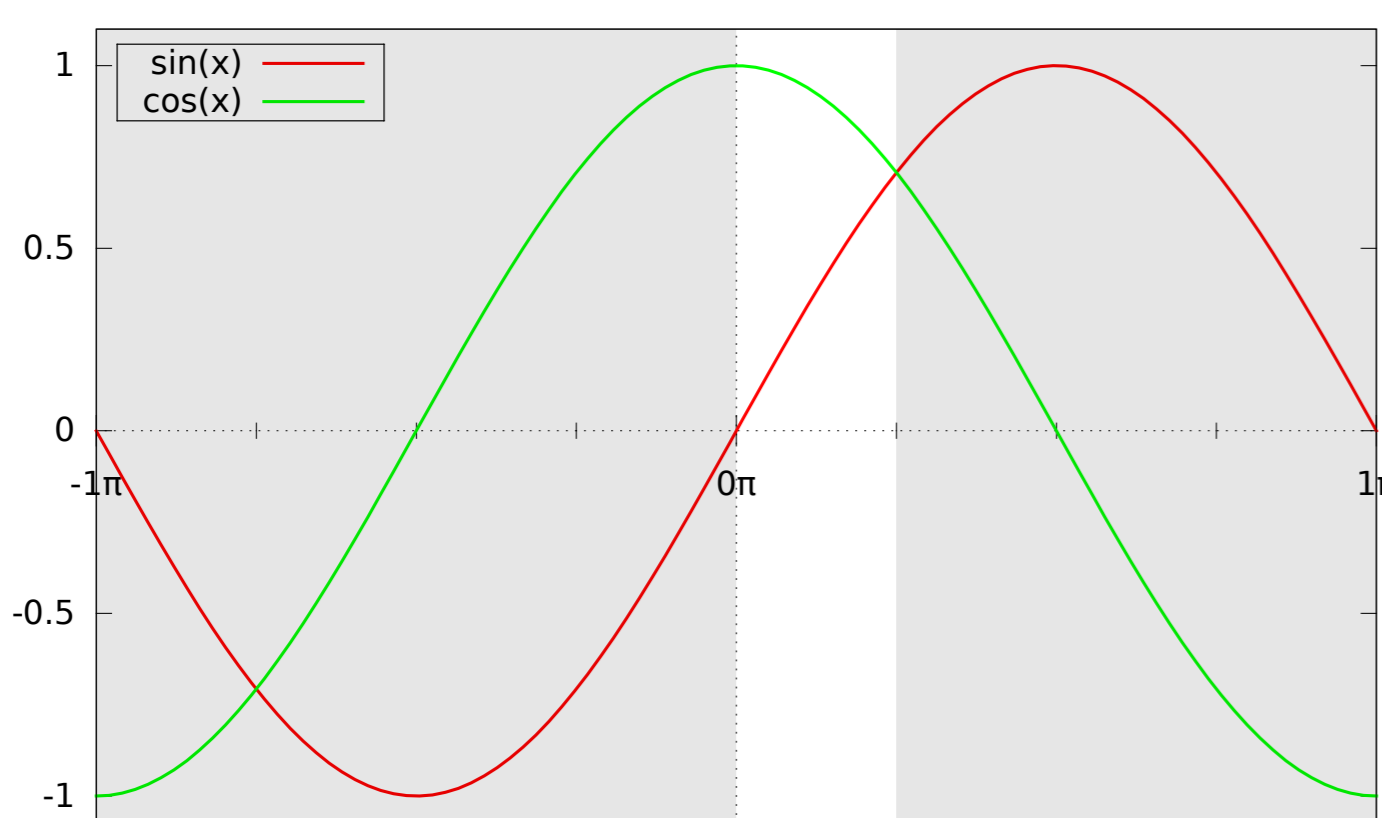


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## 1. How does a modern processor calculate sine and cosine?

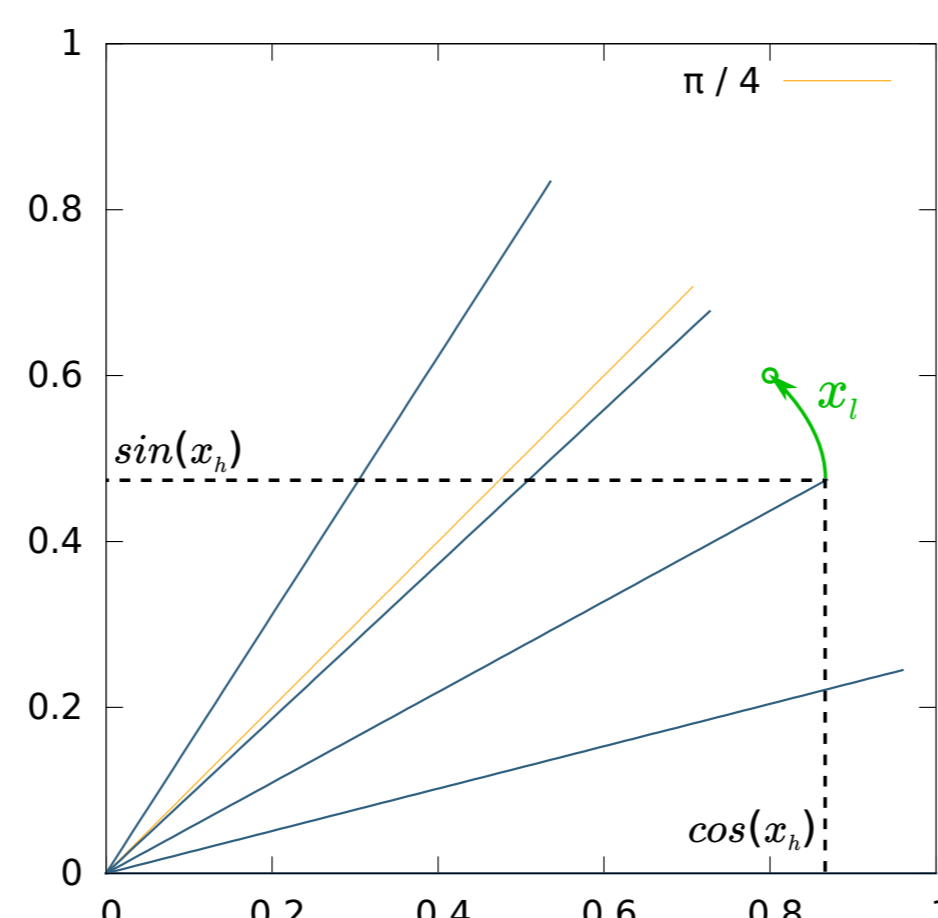
### Range reduction: use of trigonometric identities

- ▶  $\sin(-x) = -\sin(x)$
- ▶  $\sin(x) = \pm f_k(x - k \cdot \frac{\pi}{2})$  with  $f_k \in \{\sin, \cos\}$
- ⇒ Range reduction  $\mathbb{F}_{64} \mapsto [0, \frac{\pi}{4}]$

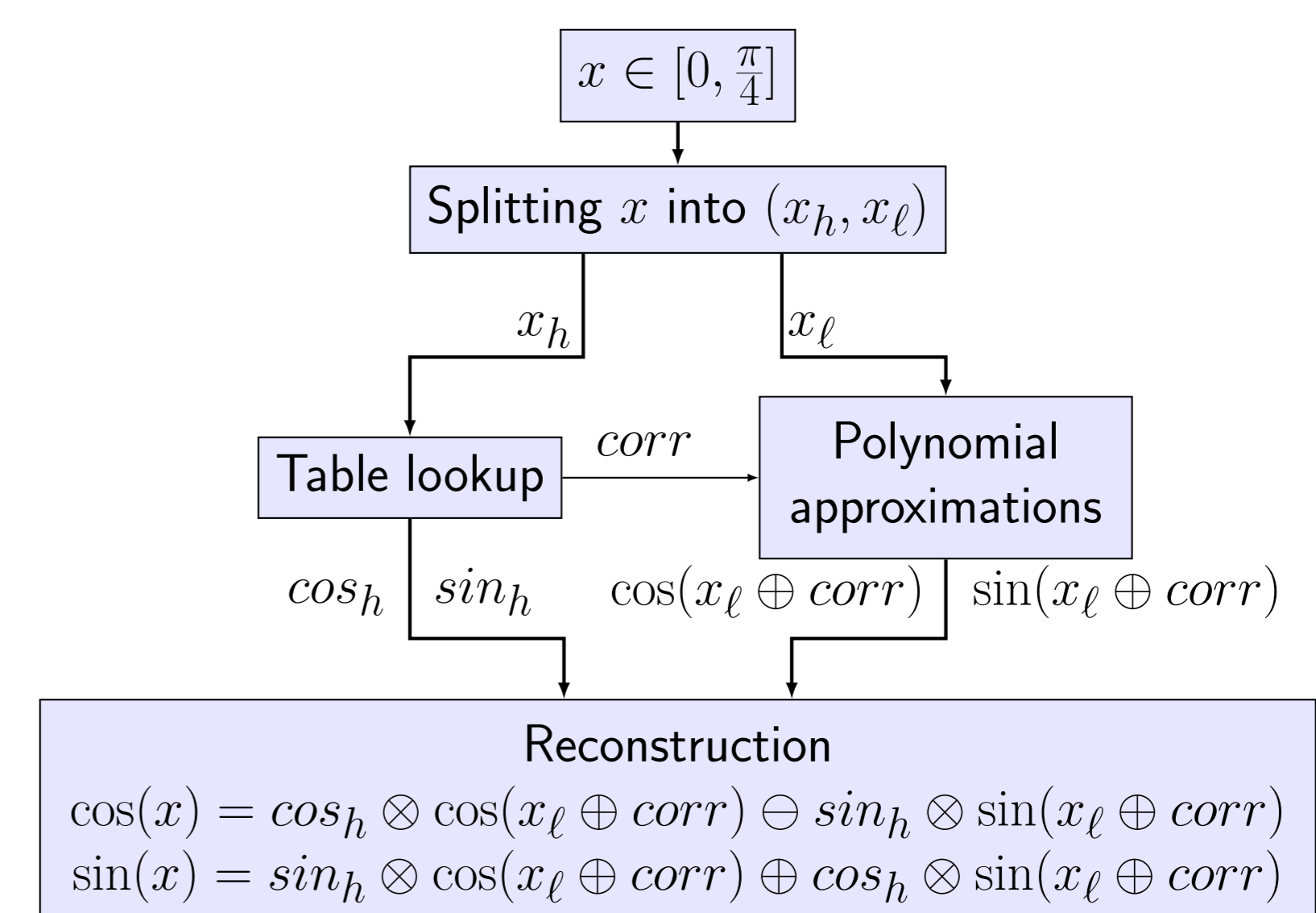


### Evaluation: use of other trigonometric properties

- ▶  $\sin(x_h + x_l) = \sin(x_h) \cdot \cos(x_l) + \cos(x_h) \cdot \sin(x_l)$
- ▶  $\cos(x_h + x_l) = \cos(x_h) \cdot \cos(x_l) - \sin(x_h) \cdot \sin(x_l)$
- ▶ **Tabulated values** for sine and cosine [Tan91]

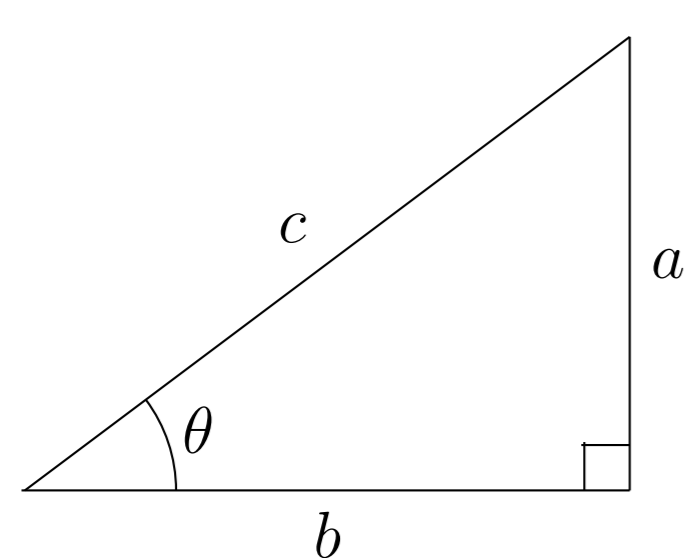


### Reconstruction: scheme reducing the error on tabulated values [GB91]



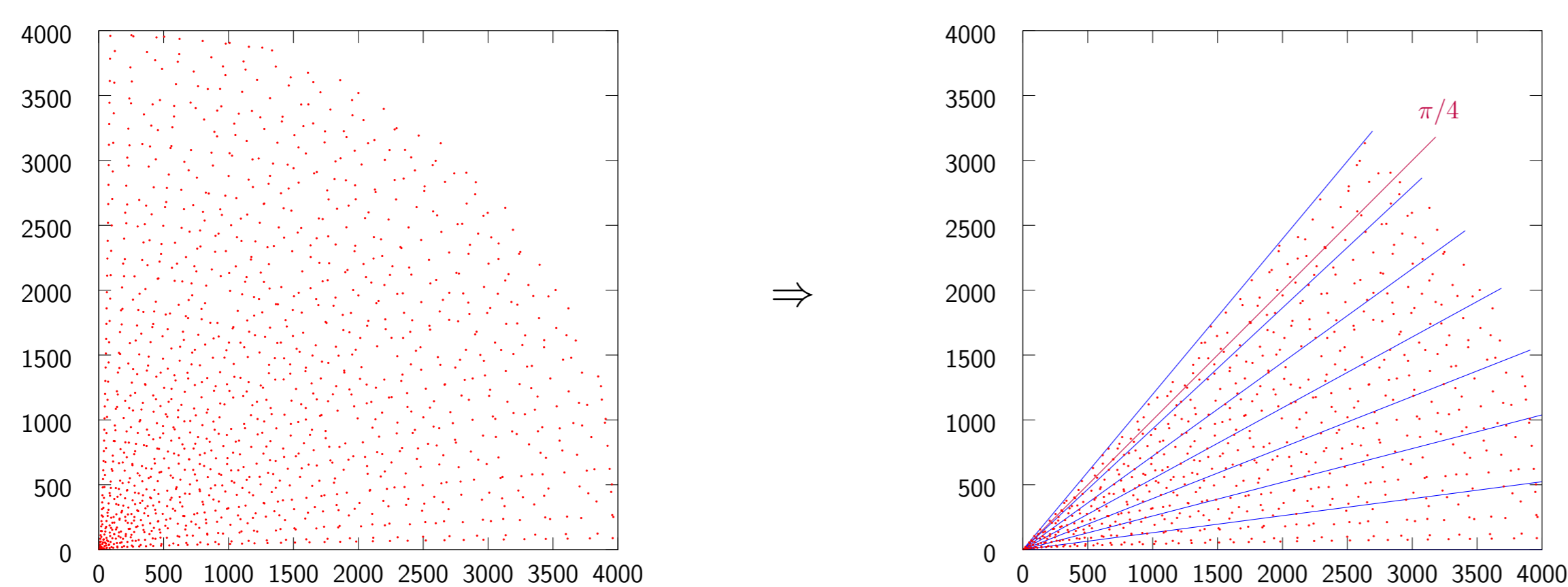
## 2. Pythagorean Triples

### What is a Pythagorean triple?



$$\begin{cases} (a, b, c) \in \mathbb{N}^3 \\ a^2 + b^2 = c^2 \\ \sin(\theta) = \frac{a}{c} \quad \cos(\theta) = \frac{b}{c} \end{cases}$$

**Primitive Pythagorean Triple**: a Pythagorean triple  $(a, b, c)$  for which  $\gcd(a, b, c) = 1$ .



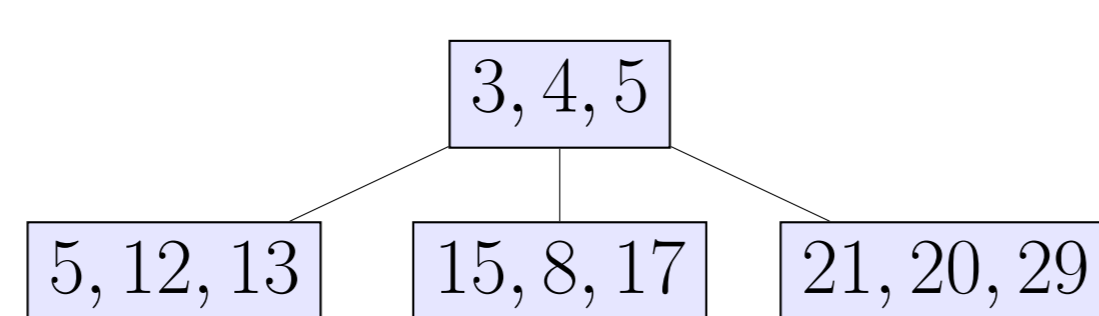
Primitive Pythagorean triples with  $c \leq 2^{12}$

Only a subset fits in a table.

## 3. Primitive Pythagorean Triple Generation

### Barning-Hall ternary-tree structure:

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$



### Several equivalent trees, easy to implement

### Proven to generate all primitive triples by increasing hypotenuse lengths [Bar63]

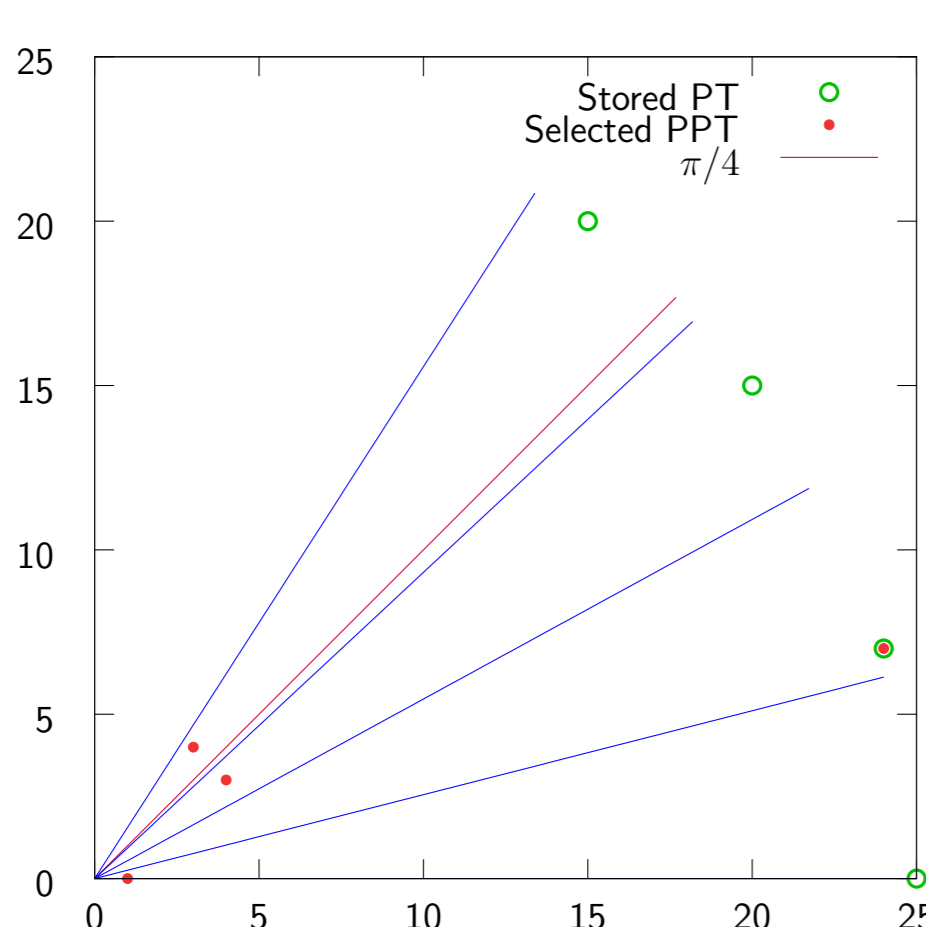
## 4. Primitive Pythagorean Triple Selection

### Only one triple per entry needed

### Division by $c$ removed by incorporating it in polynomial approximations

⇒ **Same hypotenuse**  $c$  needed for all entries

⇒ **Scale** selected PPTs to the **least common multiple** of their hypotenuses



A simple example for a 2-bit indexed table



Number of PPTs/entry for a 7-bit indexed table: roughly  $26^{100}$  combinations of 1 PPT/entry.

## 5. Exhaustive Search for a Small Common Hypotenuse

### Algorithm

- 1:  $n \leftarrow 4$
- 2: **repeat**
- 3:   Generate all PPTs  $(a, b, c)$  such that  $c \leq 2^n$ .
- 4:   Search for the LCM  $k$  among all generated hypotenuses  $c$ .
- 5:    $n \leftarrow n + 1$
- 6: **until** such a  $k$  is found
- 7: Build tabulated values  $(A, B, \text{corr})$  for every entry.

### Results

$p$	$k_{min}$	$n$	time (s)	Triples	Hypotenuses
3	425	9	$\ll 1$	86	66
4	5525	13	$\ll 1$	1404	889
5	160,225	18	0.2	42,328	24,228
6	1,698,385	21	7	335,344	179,632
7	6,569,225	23	31	1,347,953	686,701
8	$> 2^{27}$	$> 27$	$> 6700?$	$> 21,407,992$	$> 10,144,723$

▶ Impossible to generate tables indexed by **more than 7 bits**.

▶ 8 to 10 bit-indexed tables desired to **optimize caching**.

## 6. Heuristic Search

### Prime factorization of found common multiples

$k$	Prime factorization
425	$5^2 \cdot 17$
5525	$5^2 \cdot 13 \cdot 17$
160,225	$5^2 \cdot 13 \cdot 17 \cdot 29$
1,698,385	$5 \cdot 13 \cdot 17 \cdot 29 \cdot 53$
6,569,225	$5^2 \cdot 13 \cdot 17 \cdot 29 \cdot 41$

### Heuristic: store primitive Pythagorean triples satisfying

$$c = \prod_i p_i^{r_i} \quad \text{with}$$

- ▶  $r_i \in \{0, 1\}$  if  $p_i \neq 5$
- ▶  $r_i \in \mathbb{N}^*$  else
- ▶ and  $p_i \in \mathcal{P}$

where  $\mathcal{P}$  is the set of Pythagorean primes  $\leq 73$ :

$$\mathcal{P} = \{5, 13, 17, 29, 37, 41, 53, 61, 73\}$$

### Results

$p$	$k_{min}$	$n$	time (s)	triples	hypotenuses
6	1,698,385	21	0.1	2171	66
7	6,569,225	23	0.4	3452	69
8	314,201,225	29	9.5	10,467	100
9	12,882,250,225	34	294	20,311	109
10	279,827,610,985	39	9393	33,056	110

▶ **> 99 % less memory usage**

▶ **> 99 % time saved at generation**

▶ **Same tables** for  $p \in \{3, 7\}$

## 7. Theoretical Gains

Comparison between three table-based range reductions, for  $p = 10$ . The number of memory accesses (MA) and the number of floating point operations (FLOP) are reported.

Solution	Quick phase (66 bits)	Accurate phase (150 bits)	Table size (bytes)
Tang	4 MA + 64 FLOP	6 MA + 241 FLOP	38640
Gal	<b>3 MA + 53 FLOP</b>	9 MA + 268 FLOP	57960
Proposed	<b>3 MA + 53 FLOP</b>	<b>5 MA + 148 FLOP</b>	<b>32200</b>

## References

- [Bar63] F. J. M. Barning. On pythagorean and quasi-pythagorean triangles and a generation process with the help of unimodular matrices. *(Dutch) Math. Centrum Amsterdam Afd. Zuivere Wisk. ZW-001*, 1963.
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