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Error-free Tables for Trigonometric Function Evaluation

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1. How does a modern computer calculate sine and cosine?

- **Range reduction:** use of trigonometric identities
  - \( \sin(x - \pi) = -\sin(x) \)
  - \( \sin(x) = \pm \sqrt{1 - \cos(x)} \) with \( f_j \in \{ \sin, \cos \} \)
  - Range reduction \( \mathbb{R} \rightarrow [0, \frac{\pi}{2}] \)

- **Evaluation:** use of other trigonometric properties
  - \( \sin(x + \pi) = \sin(x) \cos(\pi) + \cos(x) \sin(\pi) \)
  - \( \cos(x + \pi) = \cos(x) \cos(\pi) - \sin(x) \sin(\pi) \)
  - Tabulated values for sine and cosine [Tan91]

- **Reconstruction:** scheme reducing the error on tabulated values [GB91]
  - Splitting: \( \cos(x) = \cos(x_1) \cos(x_2) + \sin(x_1) \sin(x_2) \)
  - Table lookup: \( \cos(x_1), \sin(x_1) \)
  - Polynomial approximations
  - Reconstruction

- \( \cos(x) = \cos(\pi) \cos(\pi) + \sin(\pi) \sin(\pi) \)
- \( \sin(x) = \sin(\pi) \cos(\pi) - \cos(x) \sin(\pi) \)

2. Pythagorean Triples

- **What is a Pythagorean triple?**
  \( (a, b, c) \in \mathbb{N}^3 \)
  - \( a^2 + b^2 = c^2 \)
  - \( \sin(\theta) = \frac{a}{c}, \cos(\theta) = \frac{b}{c} \)

- **Primitive Pythagorean Triple:** a Pythagorean triple \( (a, b, c) \) for which \( \gcd(a, b, c) = 1 \).

- **Barning-Hall ternary-tree structure:**
  \( \{1 \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow 2 \} \)

- **Several equivalent trees, easy to implement**
- **Proven to generate all primitive triples by increasing hypotenuse lengths [Bar63]**

3. Primitive Pythagorean Triple Generation

- **Barning-Hall ternary-tree structure:**
  \( \{1 \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow 2 \rightarrow 2 \} \)

- **Only one triple per entry needed**
- **Division by \( c \) removed by incorporating it in polynomial approximations**
  - **Same hypotenuse:** \( c \) needed for all entries
  - **Scale** selected PPTs to the least common multiple of their hypotenuses

4. Primitive Pythagorean Triple Selection

- A simple example for a 2-bit indexed table
- Number of PPTs/entry for a 7-bit indexed table: roughly \( 2^{20} \) combinations of 1 PPT/entry

5. Exhaustive Search for a Small Common Hypotenuse

- **Algorithm**
  1. \( n \rightarrow 4 \)
  2. repeat
  3. Generate all PPTs \( (a, b, c) \) such that \( c \leq 2^n \)
  4. Search for the LCM \( \ell \) among all generated hypotenuses \( c \).
  5. \( n \rightarrow n + 1 \)
  6. until such a \( k \) is found
  7. Build tabulated values \( (a, b, c) \) for every entry.

- **Results**

6. Heuristic Search

- **Prime factorization of found common multiples**
  - \( \ell = \prod_{P} \ell_{P} \) with
    - \( \{ P, \ell_P \} \)
    - \( \ell_P \) is the set of Pythagorean primes \( \leq \ell \)

- **Results**

7. Theoretical Gains

- **Comparison between three table-based range reductions, for \( p = 10 \).**
  - **Number of memory accesses (MA)** and the number of floating point operations (FLOP) are reported.

- **References**

8th école thématique ARChI — Lié, France — du 8 au 12 juin 2015