Error-free Tables for Trigonometric Function Evaluation
Hugues de Lassus Saint-Geniès, David Defour, Guillaume Revy

To cite this version:

HAL Id: lirmm-01273490
https://hal-lirmm.ccsd.cnrs.fr/lirmm-01273490
Submitted on 12 Feb 2016

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1. How does a modern processor calculate sine and cosine?

![Diagram of range reduction]

**Range reduction: use of trigonometric identities**
1. \( \sin(x) = \sin(x - 2\pi) \)
2. \( \sin(x) = \sin(x - \pi) \)
3. \( \sin(x) = \sin(x \cdot \frac{1}{2}) \sin(x \cdot \frac{2}{2}) \) with \( f_n \in \{\sin, \cos\} \)
4. Range reduction \( \mathbb{R}_2 \to [0, \frac{\pi}{2}] \)

**Evaluation: use of other trigonometric properties**
1. \( \sin(x) = \sin(x - \pi) = \sin(x + \pi) = \sin(x + 2\pi) \)
2. \( \cos(x) = \cos(x - \pi) = \cos(x + \pi) = \cos(x + 2\pi) \)
3. Tabulated values for sine and cosine [Tan91]

**Reconstruction: scheme reducing the error on tabulated values [GB19]**
- \( x \in [0, \frac{\pi}{2}] \)
- Splitting \( \sin(x) \to \sin(x_1) \sin(x_2) \)
- Table lookup \( \sin(x_1) \sin(x_2) \)
- Polynomial approximations

- Reconstruction

2. Pythagorean Triples

**What is a Pythagorean triple?**

\( (a, b, c) \in \mathbb{N}^3 \)

- \( a^2 + b^2 = c^2 \)
- \( \sin(\theta) = \frac{a}{c} \cos(\theta) = \frac{b}{c} \)

**Primitive Pythagorean Triple:** a Pythagorean triple \( (a, b, c) \) for which \( \gcd(a, b, c) = 1 \).

**Barning-Hall ternary-tree structure:**
- \( [1 \quad -2 \quad 2] \)
- \( [1 \quad -2 \quad 2] \)
- \( [1 \quad 2 \quad 2] \)
- \( [1 \quad 2 \quad 2] \)
- \( 5, 12, 13 \)
- \( 21, 20, 29 \)

**3. Primitive Pythagorean Triple Generation**

- Several equivalent trees, easy to implement
- Proven to generate all primitive triples by increasing hypotenuse lengths [Bar63]

**4. Primitive Pythagorean Triple Selection**

- Only one triple per entry needed
- Division by \( 3 \) removed by incorporating it in polynomial approximations
- Same hypotenuse \( c \) needed for all entries
- Scale selected PPTs to the least common multiple of their hypotenuses

3. Exhaustive Search for a Small Common Hypotenuse

**Algorithm**
1. \( n \leftarrow 4 \)
2. repeat
3. Generate all PPTs \( (a, b, c) \) such that \( c \leq 2^n \)
4. Search for the LCM \( k \) among all generated hypotenuses \( c \)
5. \( n \leftarrow n + 1 \)
6. until such a \( k \) is found
7. Build tabulated values \( (A, B, C, D) \) for every entry.

**Results**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( k )</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>843</td>
<td>1049</td>
<td>809</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>1423</td>
<td>1649</td>
<td>1229</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>2367</td>
<td>2567</td>
<td>1817</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3607</td>
<td>3807</td>
<td>2457</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>5071</td>
<td>5271</td>
<td>3351</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>6959</td>
<td>7159</td>
<td>4109</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**5. Exhaustive Search**

- Impossible to generate tables indexed by more than 7 bits.
- Up to 10-bit-indexed tables desired to optimize caching.

6. Heuristic Search

**Prime factorization of found common multiples**

- \( c = p_1^{e_1} \cdot p_2^{e_2} \)
- \( r_1 \in \{1, 2\} \) if \( p_1 \neq 5 \)
- \( r_2 \in \{1, 2\} \)
- \( r_2 \) and \( p_2 \) in \( \mathbb{P} \)

**Results**

<table>
<thead>
<tr>
<th>( \mathcal{P} )</th>
<th>( \frac{\text{Run Time (s)}}{\text{Trials}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>230</td>
</tr>
<tr>
<td>8</td>
<td>350</td>
</tr>
<tr>
<td>9</td>
<td>470</td>
</tr>
<tr>
<td>10</td>
<td>590</td>
</tr>
</tbody>
</table>

- \( \mathcal{P} = \{5, 11, 17, 19, 23, 41, 53, 61, 71\} \)

**7. Theoretical Gains**

- Comparison between three table-based range reductions, for \( p = 10 \). The number of memory accesses (MA) and the number of floating point operations (FLOP) are reported.

**References**


IEEE Computer Society Press.

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Hugues de Lassus Saint-Genies, David Defour and Guillaume Rey

Equipe DALI – Université de Perpignan Via Domitia – LIRMM – CNRS – UMR 5506

hugues.de-lassus@univ-perp.fr, david.defour@univ-perp.fr, guillaume.revy@univ-perp.fr