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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

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<thead>
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<th>Floating-point computations</th>
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- > 50% of design time [Wil98]

- Relies only on integer instructions

- Efficient

- Embedded systems targets: µ-controllers, DSPs, FPGAs

- → have efficient integer instructions

- Fixed-point arithmetic is well suited for embedded systems

- But, how to make it easy, fast, and numerically safe to use by non-expert programmers?

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

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- Combines conversion and IP block synthesis
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
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**Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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   ▶ Contributions:
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   - it adheres to the arithmetic model
   - Contributions:
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   - it generates code using CGPE
   - Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by

$$x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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How to compute with fixed-point numbers?
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
- Our model assumes a fixed word-length \( k \)

- \( \text{Val}(v) \) is the range of \( v \)
- \( \text{Err}(v) \) encloses the rounding error of computing \( v \)

\[
\text{Val}(v) = \left[ v, v \right] \\
\text{Err}(v) = \left[ e, e \right] \\
\text{Val}(v) = g(\text{Val}(v_1), \text{Val}(v_2), \text{Err}(v_1), \text{Err}(v_2)) \\
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- the format \( Q_{i,f} \) of \( v \) is deduced from
  \[ \text{Val}(v) = [v, \bar{v}] \]
  - \( i = \lceil \log_2(\max(|v|, |\bar{v}|)) \rceil + \alpha \)
  - \( f = k - i \)

\[ \begin{align*}
\alpha &= \begin{cases} 
1, & \text{if } \text{mod}(\log_2(\bar{v}), 1) \neq 0, \\
2, & \text{otherwise}
\end{cases}
\end{align*} \]

\[ \text{Err}(v) \text{ encloses the rounding error of computing } v \]

- a bound \( \epsilon \) on rounding errors is deduced from
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$\text{Val}(v)$ is the range of $v$

- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \bar{v}]$
  
  - $i = \left\lfloor \log_2 \left( \max (|v|, |\bar{v}|) \right) \right\rfloor + \alpha$
  - $f = k - i$

  $$\alpha = \begin{cases} 
    1, & \text{if } \mod (\log_2(\bar{v}), 1) \neq 0, \\
    2, & \text{otherwise}
  \end{cases}$$

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How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\} \)?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$

\[
\begin{align*}
\text{Val}(v) &= \text{Val}(v_1) \times \text{Val}(v_2) \\
\text{Err}(v) &= \text{Val}(v_1) \times \text{Err}(v_2) \\
&\quad + \text{Val}(v_2) \times \text{Err}(v_1) \\
&\quad + \text{Err}(v_1) \times \text{Err}(v_2)
\end{align*}
\]

This multiplication is available on integer processors and DSPs.

```c
int32_t mul(int32_t v1, int32_t v2) {
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Fixed-point multiplication

- The output format of a $\mathbf{Q}_{i_1.f_1} \times \mathbf{Q}_{i_2.f_2}$ is $\mathbf{Q}_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

- $\text{Err}_x = \left[0, 2^{-f_r} - 2^{-(f_1 + f_2)}\right]$
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Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
Our new fixed-point division

- The output integer part of $\frac{Q_{i_1.f_1}}{Q_{i_2.f_2}}$ may be as large as $i_1 + f_2$.

Err/ = $[-2^{i_2+f_1}, 2^{i_2+f_1}]$
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly

\[
\text{Err} = [-2^{f_r}, 2^{f_r}]
\]
Our new fixed-point division

- The output integer part of $Q_{i_1 \cdot f_1} / Q_{i_2 \cdot f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $\text{Err}/$?

\[ \text{Err}/ = \left[ -2^{f_r}, 2^{f_r} \right] \]

- sharper bound
- risk of overflow at run-time
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- Sharper bound
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How to decide of the output format of division?

- A large integer part
  - Prevents overflow
  - May cause overflow
  - Loose error bounds and loss of precision
  - Sharp error bounds and more accurate computations

- A small integer part
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  - Sharp error bounds and more accurate computations
The propagation rule and implementation of division

Once the output format decided $Q_{ir,fr}$

\[ \text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}]. \]

\[ \text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/. \]

\[ \overline{\text{Val}}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/} \cap \text{Val}(v_2) \text{ and } \overline{\text{Val}}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}] \]
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\]

\[
\text{Val}(v_1) = \frac{\text{Val}(v_2) \cdot \text{Val}(v)}{\text{Val}(v)} + \text{Err}/\text{Val}(v_2)
\]

\[
\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v)} \cap \text{Val}(v_2) \quad \text{and} \quad \text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]
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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t)V;
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```
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$\Val(v) = \Range(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}]$.

$\Err(v) = \frac{\Val(v_2) \cdot \Err(v_1) - \Val(v_1) \cdot \Err(v_2)}{\Val(v_2) \cdot (\Val(v_2) + \Err(v_2))} + \Err/\Val(v_1)$

$\Val(v_2) = \frac{\Val(v_1)}{\Val(v) + \Err}$

and

$\Val(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
                 || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$

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\[
\begin{align*}
\text{division output format} \\
\end{align*}
\]
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The division format trade-off: case of inverting $2 \times 2$ matrices

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- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
An implementation of the arithmetic model: the CGPE tool

The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
The CGPE tool

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1. **Computation step** $\mapsto$ **front-end**
   - computes evaluation schemes $\mapsto$ **DAGs**

Diagram:
- Front-end
- DAG computation
- Set of DAGs
- XML
The CGPE tool

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1. **Computation step** $\rightarrow$ front-end
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2. **Filtering step** $\rightarrow$ middle-end
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency $\rightarrow$ scheduling filter
     - accuracy $\rightarrow$ numerical filter
     - ...

3. **Generation step** $\rightarrow$ back-end
   - generates C codes and Gappa accuracy certificates
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---

[Diagram: Front-end $\rightarrow$ DAG computation $\rightarrow$ Set of DAGs $\rightarrow$ Filter 1 $\rightarrow$ Filter n $\rightarrow$ Back-end $\rightarrow$ C, Gappa, VHDL]
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
An implementation of the arithmetic model: the CGPE tool

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- **Low-pass Butterworth filter with cutoff frequency** $0.3 \cdot \pi$:

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\[
\begin{align*}
\text{Original signal} & \quad \text{Filtered in fixed-point using } S_1 \\
& \quad \text{Filtered in binary32} \\
& \quad \text{Filtered in binary64}
\end{align*}
\]

\[
\begin{align*}
\text{Amplitude} & \quad \text{Time} \\
-15 & \quad 0 \quad 10 \quad 20 \quad 30 \quad 40 \quad 50 \quad 60 \quad 70 \quad 80 \quad 90
\end{align*}
\]

\[
\begin{align*}
\text{Amplitude} & \quad \text{Time} \\
-16.76 & \quad 0 \quad 10 \quad 20 \quad 30 \quad 40 \quad 50 \quad 60 \quad 70 \quad 80
\end{align*}
\]

\[
\begin{align*}
\text{M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)}
\end{align*}
\]
Code synthesis for an IIR filter using CGPE

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![Graph showing original signal, filtered signal, and error bounds for different implementations.](image)
**Code synthesis for an IIR filter using CGPE**

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

  \[ y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i] \]

  ```c
  int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/, int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/, int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/, int32_t y3 /*Q6.26*/ )
  {
    int32_t r0 = mul (0x4a5cdb26, y1); //Q8.24 [-2^{ -24},0]
    int32_t r1 = mul (0xa6eb5908, y2); //Q7.25 [-2^{ -25},0]
    int32_t r2 = mul (0x4688a637, y3); //Q5.27 [-2^{ -27},0]
    int32_t r3 = mul (0x65718e3b, u0); //Q2.30 [-2^{ -30},0]
    int32_t r4 = mul (0x65718e3b, u3); //Q2.30 [-2^{ -30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{ -29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{ -27.6781},0]
    int32_t r7 = mul (0x4c152aad, u1); //Q4.28 [-2^{ -28},0]
    int32_t r8 = mul (0x4c152aad, u2); //Q4.28 [-2^{ -28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{ -27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{ -26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{ -25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{ -25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{ -24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{ -23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{ -23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{ -22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{ -22.5324},0]
    return r17;
  }
  ```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
A strategy to synthesize code for matrix inversion

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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
A strategy to synthesize code for matrix inversion

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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
### Cholesky decomposition and triangular matrix inversion

#### Cholesky decomposition

\[
 b_{i,j} = \begin{cases} 
 \sqrt{c_{i,i}} & \text{if } i = j \\
 \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j 
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

#### Triangular matrix inversion

\[
 n_{i,j} = \begin{cases} 
 1 & \text{if } i = j \\
 \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j 
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)
Fixed-point code synthesis for linear algebra basic blocks

**Cholesky decomposition and triangular matrix inversion**

### Cholesky decomposition

\[
b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ c_{i,j} / b_{j,j} & \text{if } i \neq j \end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

### Triangular matrix inversion

\[
n_{i,j} = \begin{cases} 1 / b_{i,i} & \text{if } i = j \\ -c_{i,j} / b_{i,i} & \text{if } i \neq j \end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

**Dependencies of the coefficient** \( b_{4,2} \) **in the decomposition and inversion of a 6 × 6 matrix.**
FPLA (Fixed-Point Linear Algebra)
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t \),
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t \),
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t \),
4. \( f_4(i_1, i_2) = \left\lfloor \frac{(i_1 + i_2)}{2} \right\rfloor + t \),

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \lfloor \frac{(i_1 + i_2)}{2} \rfloor + 1 \) to set the output format of division

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  > similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{}$ and $/$
## Conclusions and perspectives

### Contributions

- **Formalization and implementation of an arithmetic model**
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- **Adaptation of the CGPE tool to the model:**
  - generates code for fine grained expressions
  - instruction selection
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Perspectives

- Integrate the matrix inversion flow
Conclusions and perspectives

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Perspectives

- Integrate the matrix inversion flow
M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)  Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks 26/25


[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.


[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.


[LHD14] Benoît Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.


