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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
### Which arithmetic for computational tasks?

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Embedded systems targets
µ-controllers DSPs FPGAs
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Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
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    - certified fixed-point synthesis for:
      - **Fine grained IP blocks:** dot-products, polynomials, ...
      - **High level IP blocks:** matrix multiplication, triangular matrix inversion, Cholesky decomposition

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*Implementation tools*

*Infrastructure for the design of fixed-point systems*
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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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1. Specify an arithmetic model
   ▶ Contributions:
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by

$$x = \frac{X}{2^f} = \sum_{\ell = -f}^{k-1-f} X_{\ell + f} \cdot 2^\ell$$

Example: $x$ in $Q_{3.5}$ and $X = (1001\ 1000)_2 = (152)_{10} \rightarrow x = (100.11000)_2 = (4.75)_{10}$
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

$\text{Val}(v)$ is the range of $v$

$\text{Err}(v)$ encloses the rounding error of computing $v$
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$\alpha = \begin{cases} 
1, & \text{if } \text{mod}(\log_2(\bar{v}), 1) \neq 0, \\
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\end{cases}$
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  $i = \left\lceil \log_2 \left( \max(\mid v \mid, \mid \bar{v} \mid) \right) \right\rceil + \alpha$
  $f = k - i$

  \[
  \alpha = \begin{cases} 
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An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
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- the format \( Q_{i,f} \) of \( v \) is deduced from
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  \[ i = \left\lfloor \log_2 \left( \max(\|v\|, \|\bar{v}\|) \right) \right\rfloor + \alpha \]

  \[ f = k - i \]

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**\( \text{Err}(v) \) encloses the rounding error of computing \( v \)**

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  \[ \text{Err}(v) = [e, \bar{e}] \]

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How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\} \)?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

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\[
\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2)
\]
\[
\text{Err}(v) = \text{Val}(v_1) \times \text{Err}(v_2) + \text{Val}(v_2) \times \text{Err}(v_1) + \text{Err}(v_1) \times \text{Err}(v_2)
\]

This multiplication is available on integer processors and DSPs:

```c
int32_t mul ( int32_t v1 , int32_t v2){
  int64_t prod = (( int64_t ) v1) * (( int64_t ) v2);
  return ( int32_t ) (prod >> 32);
}
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- But, doubling the word-length is costly.

\[ \text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times \]
\[ \text{Err}(v) = \text{Err} \times (\text{Val}(v_1) \times \text{Err}(v_2) + \text{Val}(v_2) \times \text{Err}(v_1) + \text{Err}(v_1) \times \text{Err}(v_2)) \]

\[ \text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1+f_2)} \right] \]
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\text{Err}/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]
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- How to obtain sharper error bounds on $\text{Err}/$?

$$\text{Err}/ = [-2^{f_r}, 2^{f_r}]$$

-Sharper bound
- Risk of overflow at run-time
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\[ \text{Err}/ = [-2^{f_r}, 2^{f_r}] \]

- sharper bound
- risk of overflow at run-time

How to decide on the output format of division?

- A large integer part
  - ✓ prevents overflow
  - ✓ loose error bounds and loss of precision
- A small integer part
  - ✗ may cause overflow
  - ✓ sharp error bounds and more accurate computations
The propagation rule and implementation of division

- Once the output format decided $Q_{ir.fr}$

\[
\begin{align*}
\text{Val}(v) &= \text{Range}(Q_{ir.fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}]. \\
\text{Err}(v) &= \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \\
\text{Val}(v_2) &= \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \\
\text{Val}(v) &= [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}] 
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- $$\overline{\text{Val}}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2)$$ and $$\overline{\text{Val}}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$$

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
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    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
                || ((V & 0xFFFFFFFF80000001ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$
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- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

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![Diagram showing division output format with error and overflow rates]
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- Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$.
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![Division output format diagram]

- The division output format includes:
  - Maximum experimental error
  - Maximum error
  - Overflow rate

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS) Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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1. **Computation step** $\leadsto$ **front-end**
   - computes evaluation schemes $\leadsto$ **DAGs**

![Diagram showing the process from front-end to back-end with filters and DAG computations]
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   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency ⇝ scheduling filter
     - accuracy ⇝ numerical filter
     - ...

3. **Generation step** ⇝ back-end
   - generates C codes and Gappa accuracy certificates
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Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

  $$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
An implementation of the arithmetic model: the CGPE tool

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</dotproduct>

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- Original signal
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- Filtered in binary64

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Original signal
Filtered in fixed-point using $S_1$
Filtered in binary64

Certified error bound
Error of the fixed-point impl. using $S_1$
Error of the binary32 impl.
Error of the binary64 impl.

$-16.76$
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/, int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/, int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/, int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26 , y1); //Q8.24 [-2^-24,0]
    int32_t r1 = mul (0xa6eb5908 , y2); //Q7.25 [-2^-25,0]
    int32_t r2 = mul (0x4688a637 , y3); //Q5.27 [-2^-27,0]
    int32_t r3 = mul (0x65718e3b , u0); //Q2.30 [-2^-30,0]
    int32_t r4 = mul (0x65718e3b , u3); //Q2.30 [-2^-30,0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^-29,0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^-27.6781,0]
    int32_t r7 = mul (0x4c152aad , u1); //Q4.28 [-2^-28,0]
    int32_t r8 = mul (0x4c152aad , u2); //Q4.28 [-2^-28,0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^-27,0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^-26.2996,0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^-25.9125,0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^-25.3561,0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^-24.3853,0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^-23.6601,0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^-23.1798,0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^-22.5324,0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^-22.5324,0]
    return r17;
}
```
Outline of the talk

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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Chooselesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
 b_{i,j} = \begin{cases} 
 \sqrt{c_{i,i}} & \text{if } i = j \\
 c_{i,j} & \text{if } i \neq j \\
 b_{j,j} & \text{if } i = j \\
 \end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

\[
 n_{i,j} = \begin{cases} 
 \frac{1}{b_{i,i}} & \text{if } i = j \\
 \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \\
 \end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
c_{i,j} / b_{j,j} & \text{if } i \neq j 
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

\[
n_{i,j} = \begin{cases} 
1 / b_{i,i} & \text{if } i = j \\
-c_{i,j} / b_{i,i} & \text{if } i \neq j 
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.
FPLA (Fixed-Point Linear Algebra)

- User options
- Coefficients and variables
- Problem dispatcher
  - Dot-product solver
  - Matrix multiplication solver
  - Triangular matrix inversion solver
  - Cholesky decomposition solver
- FPLA-CGPE interface
- Codes
- Certificates
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles √ and /
## Conclusions and perspectives

### Contributions

- **Formalization and implementation of an arithmetic model**
  - allows certification
  - handles $\sqrt{}$ and $/$

- **Adaptation of the CGPE tool to the model:**
  - generates code for fine grained expressions
  - instruction selection

---

**Development of FPLA:**
- Automated and certified code synthesis for linear algebra basic blocks
  - Cholesky decomposition and triangular matrix inversion: study of divisions' impact

**Perspectives**
- Integrate the matrix inversion flow
Conclusions and perspectives

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Fixed-point code synthesis for linear algebra basic blocks


[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.


[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.


[LHD14] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.


