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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
**Which arithmetic for computational tasks?**

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<th>Floating-point computations</th>
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DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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Combines conversion and IP block synthesis

- Ménard et al. (CAIRN, Univ. Rennes) [MCCS02]:
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Long term objective: code synthesis for matrix inversion
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Our road-map

How to generate certified fixed-point code for matrix inversion?

1. Specify an arithmetic model

Contributions:
• formalization of $p$ and $/2$

2. Build a synthesis tool, CGPE, for fine grained IP blocks:

Contributions:
• implementation of the arithmetic model

3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:

Contributions:
• trade-off implementations for matrix multiplication
• code synthesis for Cholesky decomposition and triangular matrix inversion

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS) Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by:

$$x = \frac{X}{2^f} = \sum_{\ell = -f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$

Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.

Example:

$x$ in $Q_{3,5}$ and $X = (1001 \ 1000)_2 = (152)_{10}$

$\rightarrow x = (100.11000)_2 = (4.75)_{10}$
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![Diagram showing $k=8$, $i=3$, $f=5$]

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How to compute with fixed-point numbers?
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
- Our model assumes a fixed word-length \( k \)

\( \text{Val}(v) \) is the range of \( v \)

\( \text{Err}(v) \) encloses the rounding error of computing \( v \)
An interval arithmetic based model

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**Val($v$) is the range of $v$**

- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \bar{v}]$
  
  - $i = \left\lceil \log_2(\max(\left|v\right|, \left|\bar{v}\right|)) \right\rceil + \alpha$
  - $f = k - i$

  $\alpha = \begin{cases} 
  1, & \text{if } \text{mod}(\log_2(\bar{v}), 1) \neq 0, \\
  2, & \text{otherwise}
  \end{cases}$

**Err($v$) encloses the rounding error of computing $v$**

- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \bar{e}]$
  
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- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [\epsilon, \bar{\epsilon}]$
  - $\epsilon = \max (|\epsilon|, |\bar{\epsilon}|)$

How to propagate $\text{Val}(v)$ and $\text{Err}(v)$ for $\diamond \in \{+,-,\times,\ll,\gg,\sqrt{},/\}$?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

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- But, doubling the word-length is costly

$\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]$
**Fixed-point multiplication**

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\[
\begin{align*}
\text{Val}(v) &= \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \\
\text{Err}(v) &= \text{Err} \\
&= + \text{Val}(v_1) \times \text{Err}(v_2) \\
&= + \text{Val}(v_2) \times \text{Err}(v_1) \\
&= + \text{Err}(v_1) \times \text{Err}(v_2)
\end{align*}
\]

\[
\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1+f_2)} \right]
\]

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

$$\text{Err}_/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]$$
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- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- How to obtain sharper error bounds on $\text{Err}/$?

$\text{Err}/ = [-2^{f_r}, 2^{f_r}]$

- Sharper bound
- Risk of overflow at run-time
An arithmetic model for fixed-point code synthesis

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How to decide of the output format of division?

- A large integer part
  - ✓ Prevents overflow
  - ✓ Loose error bounds and loss of precision
- A small integer part
  - ✗ May cause overflow
  - ✓ Sharp error bounds and more accurate computations
The propagation rule and implementation of division

- Once the output format decided $Q_{ir, fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir, fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}]$$

$$\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \text{Val}(v_2)$$

- $\overline{\text{Val}(v_2)} = \frac{\overline{\text{Val}(v_1)}}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$
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Once the output format decided $Q_{ir.fr}$

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\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/\text{Val}(v_1)\text{Err}(v_1)} \cap \text{Val}(v_2) \text{ and } \text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]
\]

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
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```
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{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFFF80000000ll) == 0xFFFFFFFFF80000000ll) || ((V & 0xFFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$.

- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$.
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![Diagram of division output format with maximum error and overflow rate](image)
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![Division format trade-off diagram]

![Graph of division output format with error and overflow rate]
The division format trade-off: case of inverting $2 \times 2$ matrices

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\[ \frac{2}{2-36} \rightarrow \frac{2}{2-31} \rightarrow \frac{2}{2-26} \rightarrow \frac{2}{2-21} \rightarrow \frac{2}{2-16} \rightarrow \frac{2}{2-11} \rightarrow 100\% \]

\[ [\frac{2}{2-42} \rightarrow \frac{2}{2-38} \rightarrow \frac{2}{2-34} \rightarrow \frac{2}{2-30} \rightarrow \frac{2}{2-26} \rightarrow \frac{2}{2-22} \rightarrow \frac{2}{2-18} \rightarrow \frac{2}{2-14} \rightarrow \frac{2}{2-10} \rightarrow \frac{2}{2-6} \rightarrow \frac{2}{2-2} \rightarrow \frac{2}{2-0} \rightarrow 0\%] \]
The division format trade-off: case of inverting $2 \times 2$ matrices

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![Division output format diagram]

- Maximum experimental error
- Maximum error
- Overflow rate
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
The CGPE tool

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1. Computation step $\rightsquigarrow$ front-end
   - computes evaluation schemes $\rightsquigarrow$ DAGs
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1. Computation step $\leadsto$ front-end
   - computes evaluation schemes $\leadsto$ DAGs

2. Filtering step $\leadsto$ middle-end
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency $\leadsto$ scheduling filter
     - accuracy $\leadsto$ numerical filter
     - ...

Front-end $\leadsto$ DAG computation $\leadsto$ Middle-end $\leadsto$ Set of DAGs $\leadsto$ Filter 1 $\leadsto$ ... $\leadsto$ Filter $n$ $\leadsto$ Decorated DAGs $\leadsto$ XML
An implementation of the arithmetic model: the CGPE tool

**The CGPE tool**

- **CGPE** (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation

1. **Computation step** $\rightarrow$ **front-end**
   - computes evaluation schemes $\rightarrow$ DAGs

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   - prunes the DAGs that do not satisfy different criteria:
     - latency $\rightarrow$ scheduling filter
     - accuracy $\rightarrow$ numerical filter
     - ...

3. **Generation step** $\rightarrow$ **back-end**
   - generates C codes and Gappa accuracy certificates
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
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</dotproduct>
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![Plot showing original signal and filtered signals](image1)

![Plot showing error bounds](image2)

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
An implementation of the arithmetic model: the CGPE tool

Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
    int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
    int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
    int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26, y1);       //Q8.24 [-2^{-24},0]
    int32_t r1 = mul (0xa6eb5908, y2);       //Q7.25 [-2^{-25},0]
    int32_t r2 = mul (0x4688a637, y3);       //Q5.27 [-2^{-27},0]
    int32_t r3 = mul (0x65718e3b, u0);       //Q2.30 [-2^{-30},0]
    int32_t r4 = mul (0x65718e3b, u3);       //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4;                    //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2;                   //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul (0x4c152aad, u1);       //Q4.28 [-2^{-28},0]
    int32_t r8 = mul (0x4c152aad, u2);       //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8;                   //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9;                  //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1;                 //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11;                 //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2;                 //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13;                 //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1;                 //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15;                 //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2;                 //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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Certified code synthesis for Cholesky decomposition

Certified code synthesis for triangular matrix inversion

Cholesky decomposition

Triangular matrix inversion
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- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
\begin{align*}
b_{i,j} &= \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
\frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j 
\end{cases} \\
\text{with } c_{i,j} &= m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}
\end{align*}
\]

**Triangular matrix inversion**

\[
\begin{align*}
n_{i,j} &= \begin{cases} 
\frac{1}{b_{i,i}} & \text{if } i = j \\
-\frac{c_{i,j}}{b_{i,i}} & \text{if } i \neq j 
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\end{align*}
\]
Cholesky decomposition and triangular matrix inversion

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\[ b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases} \]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

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where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.
FPLA (Fixed-Point Linear Algebra)

- Problem dispatcher
- Dot-product solver
- Matrix multiplication solver
- Triangular matrix inversion solver
- Cholesky decomposition solver

User options
Coefficients and variables

Codes
Certificates

FPLA-CGPE interface
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \lfloor (i_1 + i_2) / 2 \rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

(a) Cholesky 5 \( \times \) 5.

(b) Triangular 10 \( \times \) 10.

Maximum errors with various functions used to determine the output formats of division.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \left\lfloor \frac{i_1 + i_2}{2} \right\rfloor + 1 \) to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lceil (i_1 + i_2)/2 \rceil + 1$ to set the output format of division

<table>
<thead>
<tr>
<th>Matrix size</th>
<th>Certified error bound</th>
<th>Maximum experimental error</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$2^{-30}$</td>
<td>$2^{-30}$</td>
</tr>
<tr>
<td>10</td>
<td>$2^{-25}$</td>
<td>$2^{-25}$</td>
</tr>
<tr>
<td>15</td>
<td>$2^{-20}$</td>
<td>$2^{-20}$</td>
</tr>
<tr>
<td>20</td>
<td>$2^{-15}$</td>
<td>$2^{-15}$</td>
</tr>
<tr>
<td>25</td>
<td>$2^{-10}$</td>
<td>$2^{-10}$</td>
</tr>
<tr>
<td>30</td>
<td>$2^{-5}$</td>
<td>$2^{-5}$</td>
</tr>
<tr>
<td>35</td>
<td>$2^0$</td>
<td>$2^0$</td>
</tr>
<tr>
<td>40</td>
<td>$2^5$</td>
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</tr>
</tbody>
</table>

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic

The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{}$ and $/$
Conclusions and perspectives

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- Integrate the matrix inversion flow
Conclusions and perspectives

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<tr>
<td>[MCCS02]</td>
<td>Automatic floating-point to fixed-point conversion for DSP code generation.</td>
<td>Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys.</td>
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