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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
### Which arithmetic for computational tasks?

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- Tedious and time consuming to implement
  - > 50% of design time \[\text{Wil98}\]
- Relies only on integer instructions
- Efficient

Embedded systems targets
- µ-controllers
- DSPs
- FPGAs

→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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- Combines conversion and IP block synthesis
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
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M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS) Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks 3/25
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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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1. Specify an arithmetic model
   - Contributions:
     - formalization of $\sqrt{\cdot}$ and $\div$

2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   - it adheres to the arithmetic model
   - Contributions:
     - implementation of the arithmetic model

3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   - it generates code using CGPE
   - Contributions:
     - trade-off implementations for matrix multiplication
     - code synthesis for Cholesky decomposition and triangular matrix inversion
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by

$$x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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Example:

- $x$ in $Q_{3,5}$ and $X = (1001\ 1000)_{2} = (152)_{10}$
  $$x = (100.11000)_{2} = (4.75)_{10}$$
Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

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- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$ $\implies$ $x = (100.11000)_2 = (4.75)_{10}$

How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

$\text{Val}(v)$ is the range of $v$

$\text{Err}(v)$ encloses the rounding error of computing $v$
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
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**$\text{Val}(v)$ is the range of $v$**

- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \bar{v}]$
  
  $i = \lceil \log_2(\max(|v|, |\bar{v}|)) \rceil + \alpha$
  $f = k - i$

  $\alpha = \begin{cases} 
  1, & \text{if } \mod(\log_2(\bar{v}), 1) \neq 0, \\
  2, & \text{otherwise}
  \end{cases}$

**$\text{Err}(v)$ encloses the rounding error of computing $v$**

- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \bar{e}]$
  
  $\epsilon = \max(|e|, |\bar{e}|)$
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\( \text{Val}(v) \) is the range of \( v \)

- the format \( Q_{i,f} \) of \( v \) is deduced from
  \[ \text{Val}(v) = [v, \overline{v}] \]
  - \( i = \left\lfloor \log_2 \left( \max(|v|, |\overline{v}|) \right) \right\rfloor + \alpha \)
  - \( f = k - i \)
  
  \[ \alpha = \begin{cases} 
  1, & \text{if } \text{mod} \left( \log_2(|v|), 1 \right) \neq 0, \\
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An arithmetic model for fixed-point code synthesis

**An interval arithmetic based model**

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\]

\[
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\]

\[
\text{Err}(v) \text{ encloses the rounding error of computing } v
\]

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\[
\epsilon = \max (|e|, |\bar{e}|)
\]

How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+,-,\times,\ll,\gg,\sqrt{},/\} \)?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1+i_2.f_1+f_2}$
Fixed-point multiplication

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- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times$
$\text{Err}(v) = \text{Err} \times$
$\text{Val}(v_1) \times \text{Err}(v_2) + \text{Val}(v_2) \times \text{Err}(v_1) + \text{Err}(v_1) \times \text{Err}(v_2)$

$\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]$
Fixed-point multiplication

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$\text{Err}(v_1) \times \text{Err}(v_2)$

$\text{Err}_v = \left[ 0, 2^{-f_r} - 2^{-(f_1+f_2)} \right]$ 

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

\[
\frac{\left\lceil i_1 \right\rceil}{f_1} \quad \left\lceil i_2 \right\rceil \quad f_2
\]

\[
\div \quad i_1 + f_2 \quad i_2 + f_1
\]

\[
\text{Err} = \left[ -2^{i_2 + f_1}, 2^{i_2 + f_1} \right]
\]
Our new fixed-point division

- The output integer part of $Q_{i_1,f_1} / Q_{i_2,f_2}$ may be as large as $i_1 + f_2$
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- How to obtain sharper error bounds on $\text{Err}/$?

$\text{Err}/ = [-2^{f_r}, 2^{f_r}]$

- sharper bound
- risk of overflow at run-time
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How to decide of the output format of division?

- A large integer part
  - prevents overflow
  - loose error bounds and loss of precision

- A small integer part
  - may cause overflow
  - sharp error bounds and more accurate computations
The propagation rule and implementation of division

- Once the output format decided $Q_{ir, fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir, fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/\text{Val}(v_1) \cdot \text{Err}(v_1).
\]

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/\text{Val}(v_2)} \cap \text{Val}(v_2)$ and $\text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$
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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t)V;
}
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int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFFF80000000ll) == 0xFFFFFFFFF80000000ll)
               || ((V & 0xFFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$.
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

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An arithmetic model for fixed-point code synthesis

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\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Division output format}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Maximum experimental error}
\end{array}
\end{array}
\end{array}
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![Diagram showing division output format]
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The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$

Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$
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![Diagram showing division output format with maximum experimental error and overflow rate.](image)
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- **CGPE (Code Generation for Polynomial Evaluation):** initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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1. **Computation step** $\rightarrow$ front-end
   - computes evaluation schemes $\rightarrow$ DAGs

![DAG computation diagram]
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1. **Computation step** $\rightarrow$ front-end
   - computes evaluation schemes $\rightarrow$ DAGs

2. **Filtering step** $\rightarrow$ middle-end
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency $\rightarrow$ scheduling filter
     - accuracy $\rightarrow$ numerical filter
     - ...

3. **Generation step** $\rightarrow$ back-end
   - generates C codes and Gappa accuracy certificates
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An implementation of the arithmetic model: the CGPE tool

Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```
<dotproduct inf="0xb1e91685" sup="0xe16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0xe16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
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![Graph showing original signal and filtered signals](image)

- Error analysis:
  - Certified error bound
  - Error of the fixed-point impl. using $S_1$
  - Error of the binary32 impl.
  - Error of the binary64 impl.
An implementation of the arithmetic model: the CGPE tool

**Code synthesis for an IIR filter using CGPE**

- **Low-pass Butterworth filter with cutoff frequency** $0.3 \cdot \pi$:
  
  $$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

$$\begin{align*}
  \text{Original signal} & \quad \text{Filtered in fixed-point using } S_1 \\
  \text{Filtered in binary64} & \\
\end{align*}$$

$$\begin{align*}
  \text{Certified error bound} & \quad \text{Error of the fixed-point impl. using } S_1 \\
  \text{Error of the binary32 impl.} & \quad \text{Error of the binary64 impl.} \\
\end{align*}$$

- **Amplitude vs. Time**
- **log}_{2}(Err) vs. Time**
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter(int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
                int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
                int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
                int32_t y3 /*Q6.26*/) {

    int32_t r0 = mul(0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul(0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul(0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul(0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul(0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul(0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul(0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r13 >> 1; //Q8.24 [-2^{-24.1798},0]
    int32_t r15 = r15 >> 1; //Q8.24 [-2^{-22.5324},0]
    return r17;
}
```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables,
to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
A strategy to synthesize code for matrix inversion

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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
# Cholesky decomposition and triangular matrix inversion

## Cholesky decomposition

\[
b_{i,j} = \begin{cases} 
  \sqrt{c_{i,i}} & \text{if } i = j \\
  \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j 
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

## Triangular matrix inversion

\[
n_{i,j} = \begin{cases} 
  \frac{1}{b_{i,i}} & \text{if } i = j \\
  -\frac{c_{i,j}}{b_{i,i}} & \text{if } i \neq j 
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\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a \( 6 \times 6 \) matrix.
FPLA (Fixed-Point Linear Algebra)

- User options
- Coefficients and variables
- Problem dispatcher
  - Dot-product solver
  - Matrix multiplication solver
  - Triangular matrix inversion solver
  - Cholesky decomposition solver
- FPLA-CGPE interface
- Codes
- Certificates
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t \),
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t \),
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t \),
4. \( f_4(i_1, i_2) = \left\lfloor \frac{(i_1 + i_2)}{2} \right\rfloor + t \),

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

(a) Cholesky 5 \times 5.

(b) Triangular 10 \times 10.

Maximum errors with various functions used to determine the output formats of division.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \left\lfloor \frac{i_1 + i_2}{2} \right\rfloor + 1$ to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic

The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{\ }$ and $/$
Conclusions and perspectives

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  - generates code for fine grained expressions
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Perspectives

- Integrate the matrix inversion flow
Conclusions and perspectives

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Perspectives

- Integrate the matrix inversion flow
Fixed-point code synthesis for linear algebra basic blocks

M. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Fridge: a fixed-point design and simulation environment.

IEEE Standard for Floating-Point Arithmetic.

[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieux.
Automatic floating-point to fixed-point conversion for DSP code generation.

GUSTO: An Automatic Generation and Optimization Tool for Matrix Inversion Architectures.

Sum-of-products evaluation schemes with fixed-point arithmetic, and their application to IIR filter implementation.

[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tszan Chen.
Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.

Design of Fixed-Point Embedded Systems (defis) French ANR Project.

Formatting bits to better implement signal processing algorithms.

Implementation of binary floating-point arithmetic on embedded integer processors - Polynomial evaluation-based algorithms and certified code generation.

Approach based on instruction selection for fast and certified code generation.


[KG08] David R. Koes and Seth C. Goldstein.
Near-optimal instruction selection on DAGs.

Toward the synthesis of fixed-point code for matrix inversion based on cholesky decomposition.

[MNR14c] Christophe Mouilleron, Amine Najahi, and Guillaume Revy.
Automated Synthesis of Target-Dependent Programs for Polynomial Evaluation in Fixed-Point Arithmetic.

Code Size and Accuracy-Aware Synthesis of Fixed-Point Programs for Matrix Multiplication.

Evaluation of static analysis techniques for fixed-point precision optimization.

[LV09] Dong-U Lee and John D. Villasenor.
Optimized custom precision function evaluation for embedded processors.