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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

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- Easy and fast to implement
- Easily portable
  - IEEE754
- Requires dedicated hardware
- Slow if emulated in software
- Tedious and time consuming to implement
- Relies only on integer instructions

- > 50% of design time [Wil98]

Embedded systems targets: µ-controllers, DSPs, FPGAs

→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

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The DEFIS approach

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**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
  - Ménard et al. (CAIRN, Univ. Rennes) [MCCS02]:
    - automatic float-to-fix conversion
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    - code generation for the linear filter IP block

- Our approach (DALI, Univ. Perpignan):
  - certified fixed-point synthesis for:
    - Fine grained IP blocks: dot-products, polynomials, ...
    - High level IP blocks: matrix multiplication, triangular matrix inversion, Cholesky decomposition

Long term objective: code synthesis for matrix inversion

Implementation tools

Infrastructure for the design of fixed-point systems

Algorithm level optimization

IWL Determination

Dynamic Range evaluation

FWL Determination

Back-end

Application description

Floating-point C code

Parameterized IP blocks

System level optimization

S2S transformation

Specific block generation

Algorithm level optimization

Validation & Optimization

Accuracy evaluation

Architecture model

Accuracy constraint

High level Synthesis

Compiler

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Architecture

Fixed-point C code
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by

$$x = \frac{X}{2^f} = \sum_{\ell = -f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$

**Notation**

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.

**Example:**

$x$ in $Q_{3,5}$.

$$x = (100.11000)_2 = (4.75)_{10}$$
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How to compute with fixed-point numbers?
An arithmetic model for fixed-point code synthesis

**An interval arithmetic based model**

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
- Our model assumes a fixed word-length \( k \)

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\[
\text{Val}(v) = \left[ \text{Val}_1, \text{Val}_2 \right] \\
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\]
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**Val($v$) is the range of $v$**
- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \overline{v}]$
  - $i = \left\lceil \log_2(\max(|v|, |\overline{v}|)) \right\rceil + \alpha$
  - $f = k - i$
  - $\alpha = \begin{cases} 
    1, & \text{if } \mod(\log_2(\overline{v}), 1) \neq 0, \\
    2, & \text{otherwise}
  \end{cases}$

**Err($v$) encloses the rounding error of computing $v$**
- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \overline{e}]$
  - $\epsilon = \max(|e|, |\overline{e}|)$
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An arithmetic model for fixed-point code synthesis

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How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \circ \in \{+, -, \times, \ll, \gg, \sqrt{\ }, /\} \)?
**Fixed-point multiplication**

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

- The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1+i_2.f_1+f_2}$

\[
\begin{align*}
\text{Val}(v) &= \text{Val}(v_1) \times \text{Val}(v_2) \\
\text{Err}(v) &= \text{Val}(v_1) \times \text{Err}(v_2) \\
&+ \text{Val}(v_2) \times \text{Err}(v_1) \\
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**Fixed-point multiplication**

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- But, doubling the word-length is costly

\[ \text{Discarded bits} \]

\[ \text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right] \]

This multiplication is available on integer processors and DSPs

```c
int32_t mul ( int32_t v1 , int32_t v2) {
  int64_t prod = ((int64_t) v1) * ((int64_t) v2);
  return (int32_t) (prod >> 32);
}
```

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
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Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

\[
\frac{i_1}{i_1 + f_2} \rightarrow f_1 \\
\frac{i_2}{i_2 + f_2} \rightarrow f_2 \\
\frac{i_2}{i_1 + f_2} \rightarrow f_2 \\
\frac{i_2}{i_2 + f_1} \rightarrow f_2
\]

$\text{Err}_/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]$
Our new fixed-point division

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\[ \text{Err} / = [-2^{fr}, 2^{fr}] \]
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper a error bounds on $\text{Err}/$?

$$\text{Err}/ = [-2^{f_r}, 2^{f_r}]$$

-Sharper bound
- Risk of overflow at run-time
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- How to obtain sharper error bounds on $\text{Err}_/$?

\[ \text{Err}_/ = [-2^{f_r}, 2^{f_r}] \]

- sharper bound
- risk of overflow at run-time

How to decide on the output format of division?

- A large integer part
  - ✓ prevents overflow
  - ✓ loose error bounds and loss of precision

- A small integer part
  - ✓ may cause overflow
  - ✓ sharp error bounds and more accurate computations

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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/\]

- $\overline{\text{Val}(v_2)} = \frac{\overline{\text{Val}(v_1)}}{\text{Val}(v) + \text{Err}}$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$
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Once the output format decided $Q_{ir,fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}]$$

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$$\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2) \quad \text{and} \quad \text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$$

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t)V;
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- $\overline{\text{Val}(v_2)} = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err} / \text{Val}(v_2)} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{i-1}, -2^{-fr}] \cup [2^{-fr}, 2^{i-1} - 2^{fr}]$

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{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT((((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
                 || ((V & 0xFFFFFFFF80000000ll) == 0)));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.
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Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.

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[Diagram of division format trade-off]
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Diagram showing the division output format with maximum experimental error and overflow rate.

- Maximum error: $0\%$ to $100\%$
- Overflow rate: $0\%$ to $100\%$

Graph illustrating the division output format with $Q_{2.30}$ to $Q_{2.30}$.
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- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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   - prunes the DAGs that do not satisfy different criteria:
     - latency $\leadsto$ scheduling filter
     - accuracy $\leadsto$ numerical filter
     - ...

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
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3. Generation step $\mapsto$ back-end
   - generates C codes and Gappa accuracy certificates
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
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![Graph showing original signal and filtered signals using different formats](image)

- Certified error bound
- Error of the fixed-point impl. using $S_1$
- Error of the binary32 impl.
- Error of the binary64 impl.

![Graph showing log2(Err) over time](image)
An implementation of the arithmetic model: the CGPE tool

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$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
                 int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
                 int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
                 int32_t y3 /*Q6.26*/) {
    // Formats Err
    int32_t r0 = mul (0x4a5cdb26, y1); //Q8.24 [-2^{ -24},0]
    int32_t r1 = mul (0xa6eb5908, y2); //Q7.25 [-2^{ -25},0]
    int32_t r2 = mul (0x4688a637, y3); //Q5.27 [-2^{ -27},0]
    int32_t r3 = mul (0x65718e3b, u0); //Q2.30 [-2^{ -30},0]
    int32_t r4 = mul (0x65718e3b, u3); //Q5.27 [-2^{ -27},0]
    int32_t r5 = r3 + r4; //Q5.27 [-2^{ -26.9881},0]
    int32_t r6 = r5 >> 2; //Q6.26 [-2^{ -25.3561},0]
    int32_t r7 = mul (0x4c152aad, u1); //Q4.28 [-2^{ -28},0]
    int32_t r8 = mul (0x4c152aad, u2); //Q4.28 [-2^{ -28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{ -27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{ -26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{ -25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{ -25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{ -24.3853},0]
    int32_t r14 = r11 >> 1; //Q7.25 [-2^{ -24.3853},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{ -23.1978},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{ -22.5324},0]
    return r17;
}
```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

Cholesky decomposition

Triangular matrix inversion

Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
\frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

\[
n_{i,j} = \begin{cases} 
\frac{1}{b_{i,i}} & \text{if } i = j \\
-\frac{c_{i,j}}{b_{i,i}} & \text{if } i \neq j
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases}
\]

with

\[
c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}
\]

**Triangular matrix inversion**

\[
n_{i,j} = \begin{cases} 1 & \text{if } i = j \\ \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases}
\]

where

\[
c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j}
\]

Dependencies of the coefficient \(b_{4,2}\) in the decomposition and inversion of a 6 \(\times\) 6 matrix.
FPLA (Fixed-Point Linear Algebra)

- User options
- Coefficients and variables
- Problem dispatcher
  - Dot-product solver
  - Matrix multiplication solver
  - Triangular matrix inversion solver
  - Cholesky decomposition solver
- FPLA-CGPE interface
- Codes
- Certificates

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t \),
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t \),
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t \),
4. \( f_4(i_1, i_2) = \lfloor (i_1 + i_2) / 2 \rfloor + t \),

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \left\lfloor \frac{i_1 + i_2}{2} \right\rfloor + 1$ to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

We use $f_4(i_1, i_2) = \left\lfloor \frac{i_1 + i_2}{2} \right\rfloor + 1$ to set the output format of division

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles \(\sqrt{}\) and \(/\)
Conclusions and perspectives

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  - generates code for fine grained expressions
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- Development of FPLA:
  - automated and certified code synthesis for linear algebra basic block

- Cholesky decomposition and triangular matrix inversion: study of divisions' impact

- Perspectives
  - Integrate the matrix inversion flow
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Fixed-point code synthesis for linear algebra basic blocks


[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.


[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.


[LHD14] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.


