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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

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Embedded systems targets: µ-controllers, DSPs, FPGAs → have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
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  - Our approach (DALI, Univ. Perpignan):
    - certified fixed-point synthesis for:
      - **Fine grained IP blocks**: dot-products, polynomials, ...
      - **High level IP blocks**: matrix multiplication, triangular matrix inversion, Cholesky decomposition

---

**Implementation tools**

1. Infrastructure for the design of fixed-point systems
2. Algorithm level optimization
   - IWL Determination
   - FWL Determination
3. Back-end
   - S2S transformation
   - Specific block generation
4. Application description
5. Floating-point C code
6. Accuracy constraint
7. Architecture model
8. Validation & Optimization
9. Accuracy evaluation

---

**System level optimization**

- High level Synthesis
- Compiler
- Implementation tools
- Architecture

---

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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   ▶ Contributions:
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by $x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$
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A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10} \quad \rightarrow \quad x = (100.11000)_2 = (4.75)_{10}$
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How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
- Our model assumes a fixed word-length \( k \)

\[
\text{Val}(v) \quad \text{is the range of } v
\]

\[
\text{Err}(v) \quad \text{encloses the rounding error of computing } v
\]
An arithmetic model for fixed-point code synthesis

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\( \text{Val}(v) \) is the range of \( v \)

- the format \( Q_{i,f} \) of \( v \) is deduced from \( \text{Val}(v) = [v, \bar{v}] \)

\[ i = \left\lceil \log_2 \left( \max(|v|, |\bar{v}|) \right) \right\rceil + \alpha \]

\[ f = k - i \]

\[ \alpha = \begin{cases} 1, & \text{if } \mod \left( \log_2(\bar{v}), 1 \right) \neq 0, \\ 2, & \text{otherwise} \end{cases} \]

\( \text{Err}(v) \) encloses the rounding error of computing \( v \)

- a bound \( \epsilon \) on rounding errors is deduced from \( \text{Err}(v) = [e, \bar{e}] \)

\[ \epsilon = \max(|e|, |\bar{e}|) \]
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- $$\epsilon = \max(|e|, |\bar{e}|)$$

How to propagate $\text{Val}(v)$ and $\text{Err}(v)$ for $\diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\}$?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$

![Diagram of fixed-point multiplication]

This multiplication is available on integer processors and DSPs:

```c
int32_t mul ( int32_t v1 , int32_t v2) {
    int64_t prod = (( int64_t ) v1) * (( int64_t ) v2);
    return ( int32_t ) (prod >> 32);
}
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\[ \text{Err}_\times = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right] \]
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$$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times$$
$$\text{Err}(v) = \text{Err} \times$$
$$+ \text{Val}(v_1) \times \text{Err}(v_2)$$
$$+ \text{Val}(v_2) \times \text{Err}(v_1)$$
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Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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$$\text{Err}/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]$$
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- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- How to obtain sharper error bounds on $\text{Err}/$?

$\text{Err}/ = [-2^{f_r}, 2^{f_r}]$
-Sharper bound
- Risk of overflow at run-time
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How to decide of the output format of division?

- A large integer part
  - ✔ prevents overflow
  - X loose error bounds and loss of precision
- A small integer part
  - X may cause overflow
  - ✔ sharp error bounds and more accurate computations
The propagation rule and implementation of division

Once the output format decided $Q_{ir.fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir.fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].$$

$$\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \text{Val}(v_1) \cdot \text{Err}(v_1) \cdot \text{Val}(v_2) \cdot \text{Err}(v_2)$$

$$\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2) \text{ and } \text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$$
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\end{align*}$$

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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t)V;
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The propagation rule and implementation of division

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- $\overline{Val(v_2)} = \frac{\overline{Val(v_1)}}{\overline{Val(v)} + Err/V}$

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- $Err(v) = \frac{Val(v_2) \cdot Err(v_1) - Val(v_1) \cdot Err(v_2)}{Val(v_2) \cdot (Val(v_2) + Err(v_2))} + \frac{Err}{Val(v_1)} \cdot \overline{Val(v_2)} \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$

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int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll) || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$. 
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$

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![Diagram illustrating the division format trade-off for inverting $2 \times 2$ matrices]
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```
Q_{2.30} / Q_{3.29}
[-1,1] d
[-1,1] - [-2,2]

[-1,1] × [-1,1] a d b c
```

```
Maximum experimental error

Maximum error

Overflow rate

Division output format
```
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} / \Delta$

![Diagram](image)
The division format trade-off: case of inverting $2 \times 2$ matrices

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\[ \text{Division output format} \]

Maximum experimental error: 0% to 100%

Overflow rate: 0% to 100%
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- **CGPE** (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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1. Computation step $\leadsto$ front-end
   - computes evaluation schemes $\leadsto$ DAGs
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1. **Computation step** ⇝ **front-end**
   - computes evaluation schemes ⇝ **DAGs**

2. **Filtering step** ⇝ **middle-end**
   - applies the arithmetic model
   - prunes the **DAGs** that do not satisfy different criteria:
     - latency ⇝ **scheduling filter**
     - accuracy ⇝ **numerical filter**
     - ...

---

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)  
Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
The CGPE tool

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  - synthesizes fixed-point code for polynomial evaluation

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3. Generation step $\rightsquigarrow$ back-end
   - generates C codes and Gappa accuracy certificates
An implementation of the arithmetic model: the CGPE tool

Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency \(0.3 \cdot \pi\):

\[
y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]
\]

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
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![Graph showing original signal and filtered signals using different precisions]
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$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/, int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/, int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/, int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul (0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul (0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul (0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul (0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul (0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul (0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
Outline of the talk

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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

### Cholesky decomposition

\[
b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ c_{i,j} & \text{if } i \neq j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

### Triangular matrix inversion

\[
n_{i,j} = \begin{cases} \frac{1}{b_{i,i}} & \text{if } i = j \\ \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases}
\]

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Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

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where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.
FPLA (Fixed-Point Linear Algebra)

- User options
- Coefficients and variables
- Problem dispatcher
  - Dot-product solver
  - Matrix multiplication solver
  - Triangular matrix inversion solver
  - Cholesky decomposition solver
- Codes
- Certificates
- FPLA-CGPE interface
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \left\lfloor \frac{(i_1 + i_2)}{2} \right\rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.

Maximum errors with various functions used to determine the output formats of division.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1 \) to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \left\lfloor (i_1 + i_2)/2 \right\rfloor + 1$ to set the output format of division

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{}$ and $/$

Adaptation of the CGPE tool to the model:
- generates code for fine grained expressions
- instruction selection

Development of FPLA:
- automated and certified code synthesis for linear algebra basic block

→ Cholesky decomposition and triangular matrix inversion: study of divisions’ impact

Perspectives

Integrate the matrix inversion flow
Conclusions and perspectives

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- **Integrate the matrix inversion flow**
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<td>[MCCS02]</td>
<td>Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.</td>
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<tr>
<td>[FRC03]</td>
<td>Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.</td>
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<tr>
<td>[LHD14]</td>
<td>Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.</td>
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