Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
### Which arithmetic for computational tasks?

<table>
<thead>
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- Floating-point computations are easy and fast to implement, and easily portable. They rely only on integer instructions, making them efficient for embedded systems like µ-controllers, DSPs, and FPGAs.
- Fixed-point computations, on the other hand, are more tedious and time-consuming to implement, with over 50% of design time required according to [Wil98]. They are efficient for embedded systems but require careful consideration to ensure numerical safety for non-expert programmers.
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Embedded systems targets

- µ-controllers
- DSPs
- FPGAs

→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

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The DEFIS approach

DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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Combines conversion and IP block synthesis

- Ménard et al. (CAIRN, Univ. Rennes) [MCCS02]:
  - automatic float-to-fix conversion

- Didier et al. (PEQUAN, Univ. Paris) [LHD14]:
  - code generation for the linear filter IP block

Implementation tools
Infrastructure for the design of fixed-point systems
Algorithm level optimization
IWL Determination
Dynamic Range evaluation
FWL Determination
Back-end
Validation & Optimization
Accuracy evaluation
High level Synthesis
Compiler
Architecture model
Accuracy constraint
Fixed-point C code
Implementation tools
High level Synthesis
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System level optimization
S2S transformation
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Algorithm level optimization
Application description
Floating-point C code
Parameterized IP blocks
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  - certified fixed-point synthesis for:
    - **Fine grained IP blocks**: dot-products, polynomials, ...
    - **High level IP blocks**: matrix multiplication, triangular matrix inversion, Cholesky decomposition

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**Diagram:**

- Application description
- System level optimization
  - IWL Determination
    - Dynamic Range evaluation
  - FWL Determination
    - Validation & Optimization
- Back-end
  - High level Synthesis
  - Compiler
- Architecture
  - Fixed-point C code
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  - Floating-point C code
- Algorithm level optimization
- Implementation tools
- Infrastructure for the design of fixed-point systems

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M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS) Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks 3/25
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Long term objective: code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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How to generate certified fixed-point code for matrix inversion?

1. Specify an arithmetic model
   ▶ Contributions:
     • formalization of $\sqrt{\cdot}$ and $/\cdot$

2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   ▶ It adheres to the arithmetic model
   ▶ Contributions:
     • implementation of the arithmetic model

3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ It generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by $x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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- Example:
  - $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$  \quad $\rightarrow$  \quad $x = (100.11000)_2 = (4.75)_{10}$
**Fixed-point arithmetic numbers**

A fixed-point number $x$ is defined by two integers:

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![Diagram showing fixed-point representation with $k=8$, $i=3$, and $f=5$.]

**Notation**

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format

- **Example:**
  - $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$ $\hookrightarrow$ $x = (100.11000)_2 = (4.75)_{10}$

How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

$\text{Val}(v)$ is the range of $v$

$\text{Err}(v)$ encloses the rounding error of computing $v$
An arithmetic model for fixed-point code synthesis

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- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
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$\text{Val}(v)$ is the range of $v$
- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \overline{v}]$
  - $i = \left\lceil \log_2 \left( \max \left( |v|, |\overline{v}| \right) \right) \right\rceil + \alpha$
  - $f = k - i$

$\alpha = \begin{cases} 1, & \text{if } \mod(\log_2(\overline{v}), 1) \neq 0, \\ 2, & \text{otherwise} \end{cases}$

$\text{Err}(v)$ encloses the rounding error of computing $v$
- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [\epsilon, \overline{\epsilon}]$
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An arithmetic model for fixed-point code synthesis

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- For each coefficient or variable $v$, we keep track of 2 intervals $Val(v)$ and $Err(v)$
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**Val($v$)** is the range of $v$

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  - $i = \left\lfloor \log_2 (\max (|v|, |\bar{v}|)) \right\rfloor + \alpha$
  - $f = k - i$
  - $\alpha = \begin{cases} 1, & \text{if } \mod (\log_2(\bar{v}), 1) \neq 0, \\ 2, & \text{otherwise} \end{cases}$

**Err($v$)** encloses the rounding error of computing $v$

- a bound $\epsilon$ on rounding errors is deduced from $Err(v) = [e, \bar{e}]$
  - $\epsilon = \max (|e|, |\bar{e}|)$
An interval arithmetic based model

- For each coefficient or variable \(v\), we keep track of 2 intervals \(\text{Val}(v)\) and \(\text{Err}(v)\)
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**\(\text{Val}(v)\) is the range of \(v\)**

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**\(\text{Err}(v)\) encloses the rounding error of computing \(v\)**

- a bound \(\epsilon\) on rounding errors is deduced from \(\text{Err}(v) = [e, \bar{e}]\)
  - \(\epsilon = \max(|e|, |\bar{e}|)\)

How to propagate \(\text{Val}(v)\) and \(\text{Err}(v)\) for \(\Diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\}\)?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

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- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

$$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times$$
$$\text{Err}(v) = \text{Err} \times$$
$$+ \text{Val}(v_1) \times \text{Err}(v_2)$$
$$+ \text{Val}(v_2) \times \text{Err}(v_1)$$
$$+ \text{Err}(v_1) \times \text{Err}(v_2)$$

$$\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]$$
Fixed-point multiplication

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\[ \text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right] \]

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2) {
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

$$\text{Err} = [-2^{i_2 + f_1}, 2^{i_2 + f_1}]$$
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- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $\text{Err}/$?

\[
\text{Err} / = [-2^{f_r}, 2^{f_r}]
\]

-Sharper bound
- Risk of overflow at run-time
Our new fixed-point division

- The output integer part of \( Q_{i_1,f_1} / Q_{i_2,f_2} \) may be as large as \( i_1 + f_2 \)
- But, doubling the word-length is costly
- How to obtain sharper error bounds on \( \text{Err}/ \)?

\[
\text{Err}/ = [-2^{f_r}, 2^{f_r}]
\]

- sharper bound
- risk of overflow at run-time

How to decide on the output format of division?

- A large integer part
  - prevents overflow
  - ✓ loose error bounds and loss of precision

- A small integer part
  - X may cause overflow
  - ✓ sharp error bounds and more accurate computations
The propagation rule and implementation of division

Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/\text{Val}(v_1) \cdot \text{Err}(v_1)
\]

\[
\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/\text{Val}(v_1)} \cap \text{Val}(v_2) \text{ and } \text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]
\]
The propagation rule and implementation of division

Once the output format decided $Q_{ir.fr}$

\[ \text{Val}(v) = \text{Range}(Q_{ir.fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}] \]

\[ \text{Err}(v) = \frac{\text{Val}(v2) \cdot \text{Err}(v1) - \text{Val}(v1) \cdot \text{Err}(v2)}{\text{Val}(v2) \cdot (\text{Val}(v2) + \text{Err}(v2))} + \frac{\text{Err}}{\text{Val}(v1) \cdot \text{Err}(v2)} \]

\[ \text{Val}(v2) = \frac{\text{Val}(v1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v2) \]

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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t)V;
}
```
The propagation rule and implementation of division

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- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/\text{Val}(v_2)} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta) {
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
                || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2,30}$.
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.

- Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$. 
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\[
\begin{array}{c}
\mathbb{Q}_{2.30} \\
[-1,1] \rightarrow d \\
\mathbb{Q}_{3.29} \\
\quad / \\
\quad [-2,2] \\
\quad - \\
[-1,1] \rightarrow \times \\
\quad \times \\
\quad [-1,1] \\
\quad a \quad d \quad b \quad c
\end{array}
\]
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- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$

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![Division output format diagram]
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![Diagram of division format trade-off](image.png)
The division format trade-off: case of inverting 2 × 2 matrices

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![Diagram](image-url)
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
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1. **Computation step** $\mapsto$ **front-end**
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2. **Filtering step** $\mapsto$ **middle-end**
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency $\mapsto$ scheduling filter
     - accuracy $\mapsto$ numerical filter
     - ...

3. **Generation step** $\mapsto$ **back-end**
   - generates C codes and Gappa accuracy certificates

[Diagram showing the flow of data from front-end to back-end with intermediate filtering steps.]
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Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
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  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
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$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
                int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
                int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
                int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul (0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul (0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul (0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul (0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul (0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul (0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]

    return r17;
}
```
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A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition

Cholesky decomposition
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- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
### Cholesky decomposition and triangular matrix inversion

<table>
<thead>
<tr>
<th>Cholesky decomposition</th>
<th>Triangular matrix inversion</th>
</tr>
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<tbody>
<tr>
<td>$b_{i,j} = \begin{cases} \sqrt{c_{i,i}} &amp; \text{if } i = j \ \frac{c_{i,j}}{b_{j,j}} &amp; \text{if } i \neq j \end{cases}$ with $c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}$</td>
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Dependencies of the coefficient $b_{4,2}$ in the decomposition and inversion of a $6 \times 6$ matrix.

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[ b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases} \]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

\[ n_{i,j} = \begin{cases} \frac{1}{b_{i,i}} & \text{if } i = j \\ \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases} \]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a \( 6 \times 6 \) matrix.
FPLA (Fixed-Point Linear Algebra)

- Dot-product solver
- Matrix multiplication solver
- Triangular matrix inversion solver
- Cholesky decomposition solver

User options
Coefficients and variables
FPLA-CGPE interface
Codes
Certificates
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \left\lfloor \frac{(i_1 + i_2)}{2} \right\rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.
How fast is generating triangular matrix inversion codes?

We use $f_4(i_1, i_2) = \left\lfloor (i_1 + i_2)/2 \right\rfloor + 1$ to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \left\lfloor \frac{i_1 + i_2}{2} \right\rfloor + 1 \) to set the output format of division

![Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.](image)

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

**Ill-conditioned matrices tend to overflow more often**
- similar behaviour in floating-point arithmetic

**The decompositions of KMS and Lehmer are highly accurate**
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{}$ and $/$
Conclusions and perspectives

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- Formalization and implementation of an arithmetic model
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  - instruction selection
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Perspectives

- Integrate the matrix inversion flow
Conclusions and perspectives

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Perspectives

- Integrate the matrix inversion flow
Fixed-point code synthesis for linear algebra basic blocks

         Fridge: a fixed-point design and simulation environment.

         IEEE Standard for Floating-Point Arithmetic.

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         GUSTO: An Automatic Generation and Optimization 
         Tool for Matrix Inversion Architectures.

[LHD12]  Benoît Lopez, Thibault Hilaire, and Laurent-Stéphane 
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         Sum-of-products evaluation schemes with fixed-point 
         arithmetic, and their application to IIR filter 
         implementation.

[FRC03]  Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen.
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         Hilaire, Benoît Lopez, Eric Goubault, Sylvie Putot, 
         Franck Vedrine, Amine Najahi, Guillaume Revy, Laurent 
         Fangain, Christian Samoyeau, Fabrice Lemonnier, and 
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         French ANR Project.

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         Formatting bits to better implement signal processing 
         algorithms.

         Implementation of binary floating-point arithmetic on 
         embedded integer processors - Polynomial 
         evaluation-based algorithms and certified code 
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         Polynomial Evaluation.

[KG08]  David R. Koes and Seth C. Goldstein.
         Near-optimal instruction selection on DAGs.

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         inversion based on cholesky decomposition.

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         for Polynomial Evaluation in Fixed-Point Arithmetic.

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[LV09]  Dong-U Lee and John D. Villasenor.
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