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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

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<th>Fixed-point computations</th>
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Embedded systems targets
- µ-controllers
- DSPs
- FPGAs

→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
## Which arithmetic for computational tasks?

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- Tedious and time consuming to implement
  - > 50% of design time [Wil98]
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The DEFIS approach

- DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
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      - **Fine grained IP blocks**: dot-products, polynomials, ...
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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
Our road-map

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1. Specify an arithmetic model
   ▶ Contributions:
     • formalization of $\sqrt{}$ and $/$
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   ▶ it adheres to the arithmetic model
   ▶ Contributions:
     • implementation of the arithmetic model

3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by

$$x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$
An arithmetic model for fixed-point code synthesis

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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.

Example: $x$ in $Q_{3,5}$.

$X = (1001 1000)_2 = (152)_{10}$

$X = (100.11000)_2 = (4.75)_{10}$
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

$\implies$ The value of $x$ is given by $x = \frac{X}{2^f} = \sum_{\ell = -f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$

Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format

Example:

- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$ $\implies$ $x = (100.11000)_2 = (4.75)_{10}$

How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
- Our model assumes a fixed word-length \( k \)

\[ \text{Val}(v) \text{ is the range of } v \]

\[ \text{Err}(v) \text{ encloses the rounding error of computing } v \]
An arithmetic model for fixed-point code synthesis

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\[ \text{Val}(v) \text{ is the range of } v \]

- the format \( Q_{i,f} \) of \( v \) is deduced from \( \text{Val}(v) = [v, \bar{v}] \)
  
\[ i = \left\lceil \log_2 (\max(\|v\|, |\bar{v}|)) \right\rceil + \alpha \]
  
\[ f = k - i \]

\[ \alpha = \begin{cases} 
1, & \text{if } \mod (\log_2(\bar{v}), 1) \neq 0, \\
2, & \text{otherwise} 
\end{cases} \]

\[ \text{Err}(v) \text{ encloses the rounding error of computing } v \]

- a bound \( \epsilon \) on rounding errors is deduced from \( \text{Err}(v) = [e, \bar{e}] \)
  
\[ \epsilon = \max (|e|, |\bar{e}|) \]
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

### $\text{Val}(v)$ is the range of $v$
- The format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \overline{v}]
- i = \left\lfloor \log_2 \left( \max (|v|, |\overline{v}|) \right) \right\rfloor + \alpha$
- $f = k - i$

$$
\alpha = \begin{cases} 
1, & \text{if } \text{mod} \left( \log_2(\overline{v}), 1 \right) \neq 0, \\
2, & \text{otherwise}
\end{cases}
$$

### $\text{Err}(v)$ encloses the rounding error of computing $v$
- A bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \overline{e}]$
- $\epsilon = \max (|e|, |\overline{e}|)$
An arithmetic model for fixed-point code synthesis

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- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
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**Val(\( v \))** is the range of \( v \)
- the format \( Q_{i,f} \) of \( v \) is deduced from
  \[ \text{Val}(v) = [v, \bar{v}] \]
  \[ i = \left\lfloor \log_2 \left( \max(|v|, |\bar{v}|) \right) \right\rfloor + \alpha \]
- \( f = k - i \)
- \( \alpha = \begin{cases} 1, & \text{if } \text{mod} \left( \log_2(\bar{v}), 1 \right) \neq 0, \\ 2, & \text{otherwise} \end{cases} \)

**Err(\( v \))** encloses the rounding error of computing \( v \)
- a bound \( \epsilon \) on rounding errors is deduced from
  \[ \text{Err}(v) = [e, \bar{e}] \]
- \( \epsilon = \max(|e|, |\bar{e}|) \)

How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+,-,\times,\ll,\gg,\sqrt{},/\} \)?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1+i_2.f_1+f_2}$
Fixed-point multiplication

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Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1+i_2,f_1+f_2}$
- But, doubling the word-length is costly

$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times$
$\text{Err}(v) = \text{Err} \times$
$+ \text{Val}(v_1) \times \text{Err}(v_2)$
$+ \text{Val}(v_2) \times \text{Err}(v_1)$
$+ \text{Err}(v_1) \times \text{Err}(v_2)$

$\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1+f_2)} \right]$
Fixed-point multiplication

- The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

$$\text{Discarded bits}$$

- $\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]$  
- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2)
{
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$. 

$\begin{array}{c}
\text{i_1} \quad \text{i_2} \\
\text{f_1} \quad \text{f_2} \\
\hline
\end{array}$
Our new fixed-point division

- The output integer part of $\mathbb{Q}_{i_1.f_1}/\mathbb{Q}_{i_2.f_2}$ may be as large as $i_1 + f_2$

\[
\text{Err}/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]
\]
Our new fixed-point division

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$\text{Err}_/ = [-2^{f_r}, 2^{f_r}]$
Our new fixed-point division

- The output integer part of $Q_{i_1,f_1} / Q_{i_2,f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $\text{Err}_/^?$

\[
\begin{align*}
\text{Err}_/^ & = \left[-2^{f_r}, 2^{f_r}\right] \\
& \text{Sharper bound} \\
& \text{Risk of overflow at run-time}
\end{align*}
\]
Our new fixed-point division

- The output integer part of $\text{Q}_{i_1.f_1}/\text{Q}_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- How to obtain sharper error bounds on $\text{Err}/$?

$$\text{Err}/ = [-2^{f_r}, 2^{f_r}]$$

-Sharper bound
- Risk of overflow at run-time

How to decide of the output format of division?

- **A large integer part**
  - ✓ prevents overflow
  - ✗ loose error bounds and loss of precision

- **A small integer part**
  - ✗ may cause overflow
  - ✓ sharp error bounds and more accurate computations
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [2^{-i_r-1}, 2^{i_r-1} - 2^{f_r}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/\text{Val}(v_1) \cdot \text{Err}(v_2)
\]

- $\overline{\text{Val}(v_2)} = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/\text{Val}(v)} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{i_r-1}, -2^{-f_r}] \cup [2^{-f_r}, 2^{i_r-1} - 2^{f_r}]$
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}]$$

$$\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \text{Val}(v_1) \text{Err}(v_1) \text{Val}(v_2) \text{Err}(v_2)$$

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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t)V;
}
```
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].$$

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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
                || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$. 

Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. 

Division output format:

- Maximum experimental error
- Maximum error
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- Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$

![Diagram with division output format and maximum experimental error graph]
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d & -b \\ c & a \end{pmatrix} / \Delta$

[Diagram of division output format with error and overflow rates]
The division format trade-off: case of inverting $2 \times 2$ matrices

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Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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1. **Computation step** $\leadsto$ **front-end**
   - computes evaluation schemes $\leadsto$ **DAGs**

![Diagram of DAG computation process]

- **Front-end**
- **DAG computation**
- **Set of DAGs**
- **XML**
The CGPE tool

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1. **Computation step** ⇝ front-end
   - computes evaluation schemes ⇝ DAGs

2. **Filtering step** ⇝ middle-end
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency ⇝ scheduling filter
     - accuracy ⇝ numerical filter
     - ...

3. **Generation step** ⇝ back-end
   - generates C codes and Gappa accuracy certificates
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Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
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An implementation of the arithmetic model: the CGPE tool

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- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter(int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/, int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/, int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/, int32_t y3 /*Q6.26*/) {
    // Formats Err
    int32_t r0 = mul(0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul(0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul(0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul(0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul(0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul(0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul(0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
## Cholesky decomposition and triangular matrix inversion

### Cholesky decomposition

\[
\begin{align*}
    b_{i,j} &= \begin{cases} 
        \sqrt{c_{i,i}} & \text{if } i = j \\
        \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j 
    \end{cases} \\
    \text{with } c_{i,j} &= m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}
\end{align*}
\]

### Triangular matrix inversion

\[
\begin{align*}
    n_{i,j} &= \begin{cases} 
        \frac{1}{b_{i,i}} & \text{if } i = j \\
        -\frac{c_{i,j}}{b_{i,i}} & \text{if } i \neq j 
    \end{cases} \\
    \text{where } c_{i,j} &= \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j}
\end{align*}
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Cholesky decomposition and triangular matrix inversion

Cholesky decomposition

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
c_{i,j} / b_{j,j} & \text{if } i \neq j 
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

Triangular matrix inversion

\[
n_{i,j} = \begin{cases} 
1 / b_{i,i} & \text{if } i = j \\
-c_{i,j} / b_{i,i} & \text{if } i \neq j 
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6×6 matrix.
FPLA (Fixed-Point Linear Algebra)

User options → Problem dispatcher →
- Dot-product solver
- Matrix multiplication solver
- Triangular matrix inversion solver
- Cholesky decomposition solver

FPLA-CGPE interface →
- Codes
- Certificates

Coefficients and variables
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t \),
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t \),
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t \),
4. \( f_4(i_1, i_2) = \left\lfloor (i_1 + i_2)/2 \right\rfloor + t \),

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 \times 5.

(b) Triangular 10 \times 10.
How fast is generating triangular matrix inversion codes?

We use \( f_4(i_1, i_2) = \lceil (i_1 + i_2) / 2 \rceil + 1 \) to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \left\lfloor \frac{i_1 + i_2}{2} \right\rfloor + 1 \) to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles √ and /
Conclusions and perspectives

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  - generates code for fine grained expressions
  - instruction selection
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**Contributions**

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  - automated and certified code synthesis for linear algebra basic block
    - Cholesky decomposition and triangular matrix inversion: study of divisions’ impact
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Perspectives

- Integrate the matrix inversion flow
Conclusions and perspectives

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- Integrate the matrix inversion flow
Merging information from the text:

- Fixed-point code synthesis for linear algebra basic blocks

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