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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

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Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
### Which arithmetic for computational tasks?

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- Embedded systems targets: µ-controllers, DSPs, FPGAs

- Fixed-point arithmetic is well suited for embedded systems

- But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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*Embedded systems targets µ-controllers DSPs FPGAs → have efficient integer instructions*

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Which arithmetic for computational tasks?

**Floating-point computations**

- Easy and fast to implement
- Easily portable [IEEE754]
- Requires dedicated hardware
- Slow if emulated in software

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The DEFIS approach

DEFIS (ANR, 2011-2015)

Goal: develop techniques and tools to automate fixed-point programming
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  **Goal:** develop techniques and tools to automate fixed-point programming

- Combines conversion and IP block synthesis
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
Our road-map

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1. Specify an arithmetic model
   ▶ Contributions:
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   ▶ it adheres to the arithmetic model
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by

$$x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $\mathbb{Q}_{i,f}$ format.
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Example:

\( x \) in \( \mathbb{Q}_{3,5} \) and \( X = (1001\ 1000)_{\text{2}} = (152)_{\text{10}} \) \( \rightarrow \) \( x = (100.11000)_{\text{2}} = (4.75)_{\text{10}} \)
An arithmetic model for fixed-point code synthesis

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\[ x = (100.11000)_2 = (4.75)_{10} \]

How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

$\text{Val}(v)$ is the range of $v$

$\text{Err}(v)$ encloses the rounding error of computing $v$
An interval arithmetic based model

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**Val\( (v) \) is the range of \( v \)**

- the format \( Q_{i,f} \) of \( v \) is deduced from \( \text{Val}(v) = [v, \overline{v}] \)

\[
\begin{align*}
  i &= \left\lfloor \log_2(\max(|v|, |\overline{v}|)) \right\rfloor + \alpha \\
  f &= k - i
\end{align*}
\]

\[
\alpha = \begin{cases} 
  1, & \text{if } \mod(\log_2(\overline{v}), 1) \neq 0, \\
  2, & \text{otherwise}
\end{cases}
\]

**Err\( (v) \) encloses the rounding error of computing \( v \)**

- a bound \( \varepsilon \) on rounding errors is deduced from \( \text{Err}(v) = [e, \overline{e}] \)

\[
\varepsilon = \max(|e|, |\overline{e}|)
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  - \( i = \left\lfloor \log_2(\max(|v|, |\bar{v}|)) \right\rfloor + \alpha \)
  - \( f = k - i \)

\[ \alpha = \begin{cases} 1, & \text{if mod} \left( \log_2(\bar{v}), 1 \right) \neq 0, \\ 2, & \text{otherwise} \end{cases} \]

**Err(\( v \)) encloses the rounding error of computing \( v \)**
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- a bound \( \epsilon \) on rounding errors is deduced from

\[ \text{Err}(v) = [e, \bar{e}] \]

\[ \epsilon = \max(|e|, |\bar{e}|) \]

How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+,-,\times,\ll,\gg,\sqrt{\cdot},/\} \)?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

- The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1 + i_2.f_1 + f_2}$

\[
\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) \\
\text{Err}(v) = \text{Val}(v_1) \times \text{Err}(v_2) + \text{Val}(v_2) \times \text{Err}(v_1) + \text{Err}(v_1) \times \text{Err}(v_2)
\]
Fixed-point multiplication

The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1+i_2.f_1+f_2}$.

But, doubling the word-length is costly.

Discarded bits

$\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1+f_2)} \right]$
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1+i_2.f_1+f_2}$
- But, doubling the word-length is costly

- $\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1+f_2)} \right]$
- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$.
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

$$\text{Err}_/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]$$
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Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $\text{Err}\, /$?

\[ \text{Err}\, / = [-2^{f_r}, 2^{f_r}] \]

-Sharper bound
- Risk of overflow at run-time
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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$$
\text{Err}/ = [-2^{f_r}, 2^{f_r}]
$$
-Sharper bound
- Risk of overflow at run-time

How to decide of the output format of division?

- A large integer part
  - ✔️ prevents overflow
  - ✔️ loose error bounds and loss of precision
- A small integer part
  - ✗ may cause overflow
  - ✔️ sharp error bounds and more accurate computations
The propagation rule and implementation of division

Once the output format decided $Q_{ir,fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].$$

$$\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/\text{Val}(v_1) \cdot \text{Err}(v_1)$$

$$\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/\text{Val}(v_2)} \cap \text{Val}(v_2)$$

$$\text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$$
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\]

- $\overline{\text{Val}(v_2)} = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t)V;
}
```
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int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll) || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2,30}$.

Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & -\frac{b}{\Delta} \\ -\frac{c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$.
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[Diagram of the division format trade-off: case of inverting $2 \times 2$ matrices]
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\[\text{DIVISION OUTPUT FORMAT}\]

\[\text{Maximum experimental error}\]

\[\text{Maximum error}\]

\[\text{Overflow rate}\]
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![Diagram of division output format with error and overflow rates]
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![Diagram showing division output format with maximum experimental error and overflow rate]
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The division format trade-off: case of inverting $2 \times 2$ matrices

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![Diagram of division output format](image)
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- **CGPE (Code Generation for Polynomial Evaluation):** initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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     - latency ⇔ scheduling filter
     - accuracy ⇔ numerical filter
     - ...

Diagram:
- Front-end
- DAG computation
- Set of DAGs
- Filter 1
- ... Filter n
- Decorated DAGs
- XML
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3. Generation step \(\rightarrow\) back-end
   - generates C codes and Gappa accuracy certificates
An implementation of the arithmetic model: the CGPE tool

Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
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\text{
\ldots}
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\text{<\dotproduct>}
\]

\[
\text{-60 -50 -40 -30 -20 -10 0 10 20 30 40 50 60 70 80}
\text{log}_2(\text{Err})
\text{Time}
\text{Certified error bound}
\text{Error of the fixed-point impl. using S_1}
\text{Error of the binary32 impl.}
\text{Error of the binary64 impl.}
\]

\[
\text{M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)}
\text{Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks}
\]
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/, int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/, int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/, int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul(0x4a5cdb26, y1); //Q8.24 [-2^-24,0]
    int32_t r1 = mul(0xa6eb5908, y2); //Q7.25 [-2^-25,0]
    int32_t r2 = mul(0x4688a637, y3); //Q5.27 [-2^-27,0]
    int32_t r3 = mul(0x65718e3b, u0); //Q2.30 [-2^-30,0]
    int32_t r4 = mul(0x65718e3b, u3); //Q2.30 [-2^-30,0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^-29,0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^-27.6781,0]
    int32_t r7 = mul(0x4c152aad, u1); //Q4.28 [-2^-28,0]
    int32_t r8 = mul(0x4c152aad, u2); //Q4.28 [-2^-28,0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^-27,0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^-26.2996,0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^-25.9125,0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^-25.3561,0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^-24.3853,0]
    int32_t r14 = r13 >> 2; //Q7.25 [-2^-23.6601,0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^-23.1798,0]
    int32_t r16 = r15 + r15; //Q8.24 [-2^-22.5324,0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^-22.5324,0]
    return r17;
}
```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
A strategy to synthesize code for matrix inversion

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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition

- Certified code synthesis for triangular matrix inversion

- Certified code synthesis for matrix multiplication
A strategy to synthesize code for matrix inversion

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Certified code synthesis for Cholesky decomposition

Triangular matrix inversion
A strategy to synthesize code for matrix inversion

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Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
# Cholesky decomposition and triangular matrix inversion

### Cholesky decomposition

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b_{i,j} = \begin{cases} \sqrt{c_{i,i}} &amp; \text{if } i = j \ c_{i,j} / b_{j,j} &amp; \text{if } i \neq j \end{cases} )</td>
<td>( b_{i,j} ) is the coefficient in the Cholesky decomposition.</td>
</tr>
</tbody>
</table>
| \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \) | \( c_{i,j} \) is computed based on the coefficients of the matrix.

### Triangular matrix inversion

<table>
<thead>
<tr>
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<tr>
<td>( n_{i,j} = \begin{cases} 1 / b_{i,i} &amp; \text{if } i = j \ -c_{i,j} / b_{i,i} &amp; \text{if } i \neq j \end{cases} )</td>
<td>( n_{i,j} ) is the coefficient in the triangular matrix inversion.</td>
</tr>
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| \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \) | \( c_{i,j} \) is computed based on the coefficients of the matrix.

**Dependencies of the coefficient**

The coefficient \( b_{4,2} \) in the decomposition and inversion of a 6x6 matrix.
Cholesky decomposition and triangular matrix inversion

### Cholesky decomposition

$$b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases}$$

with $c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}$

### Triangular matrix inversion

$$n_{i,j} = \begin{cases} \frac{1}{b_{i,i}} & \text{if } i = j \\ -\frac{c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases}$$

where $c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j}$

---

Dependencies of the coefficient $b_{4,2}$ in the decomposition and inversion of a $6 \times 6$ matrix.
FPLA (Fixed-Point Linear Algebra)

User options

Problem dispatcher

Dot-product solver

Matrix multiplication solver

Triangular matrix inversion solver

Cholesky decomposition solver

FPLA-CGPE interface

Codes

Certificates

Coefficients and variables
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \left\lfloor (i_1 + i_2) / 2 \right\rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lceil (i_1 + i_2)/2 \rceil + 1$ to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \left\lfloor \frac{i_1 + i_2}{2} \right\rfloor + 1 \) to set the output format of division

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

![Graph showing condition numbers for different matrices of varying sizes](image-url)
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{}$ and $/$
Conclusions and perspectives

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    → Cholesky decomposition and triangular matrix inversion: study of divisions’ impact
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Perspectives

- Integrate the matrix inversion flow
Conclusions and perspectives

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Perspectives

- Integrate the matrix inversion flow
Fixed-point code synthesis for linear algebra basic blocks


[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.


[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.


[LHD14] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.


