Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
Mohamed Amine Najahi

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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
## Which arithmetic for computational tasks?

<table>
<thead>
<tr>
<th>Floating-point computations</th>
<th>Fixed-point computations</th>
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<tr>
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</tr>
<tr>
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Embedded systems targets

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Fixed-point arithmetic is well suited for embedded systems

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Which arithmetic for computational tasks?

Embedded systems targets

- µ-controllers
- DSPs
- FPGAs

→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
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Floating-point computations are easy and fast to implement and easily portable, but they require dedicated hardware and can be slow if emulated in software. Fixed-point computations are tedious and time-consuming to implement, but they rely only on integer instructions and are efficient. They are well suited for embedded systems such as μ-controllers, DSPs, and FPGAs, which have efficient integer instructions.
Which arithmetic for computational tasks?

**Floating-point computations**

- Easy and fast to implement
- Easily portable [IEEE754]
- Requires dedicated hardware
- Slow if emulated in software

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  - > 50% of design time [Wil98]
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The DEFIS approach

- DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
  
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
    - automatic float-to-fix conversion
  
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  - Our approach (DALI, Univ. Perpignan):
    - certified fixed-point synthesis for:
      - **Fine grained IP blocks**: dot-products, polynomials, ...
      - **High level IP blocks**: matrix multiplication, triangular matrix inversion, Cholesky decomposition

---

**Diagram Description**

- Application description
- System level optimization
- S2S transformation
- Specific block generation
- Algorithm level optimization
- FWL Determination
- Dynamic Range evaluation
- IWL Determination
- Back-end
- High level Synthesis
- Compiler
- Implementation tools
- Architecture model
- Validation & Optimization
- Accuracy evaluation
- Accuracy constraint
- Architecture
- Fixed-point C code
- Parameterized IP blocks
- Floating-point C code
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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
Our road-map

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1. Specify an arithmetic model
   - Contributions:
     - formalization of $\sqrt{}$ and $/$
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   ▶ it adheres to the arithmetic model
   ▶ Contributions:
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
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1. An arithmetic model for fixed-point code synthesis

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Fixed-point arithmetic numbers

A fixed-point number \( x \) is defined by two integers:

- \( X \) the \( k \)-bit integer representation of \( x \)
- \( f \) the implicit scaling factor of \( x \)

The value of \( x \) is given by

\[
x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell
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**Fixed-point arithmetic numbers**

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$\implies$ The value of $x$ is given by $x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$

**Notation**

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $\mathbb{Q}_{i,f}$ format
Fixed-point arithmetic numbers

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$$x = \frac{X}{2^f} = \sum_{\ell = -f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$

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Example:
- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$

  $$x = (100.11000)_2 = (4.75)_{10}$$
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Example:

- $x$ in $Q_{3.5}$ and $X = (10011000)_2 = (152)_{10}$ $\rightarrow$ $x = (100.11000)_2 = (4.75)_{10}$

How to compute with fixed-point numbers?
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $Val(v)$ and $Err(v)$
- Our model assumes a fixed word-length $k$

$Val(v)$ is the range of $v$

$Err(v)$ encloses the rounding error of computing $v$
An interval arithmetic based model

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$\text{Val}(v)$ is the range of $v$

- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \bar{v}]$

  $i = \left\lceil \log_2 \left( \max \left( |v|, |\bar{v}| \right) \right) \right\rceil + \alpha$

  $f = k - i$

  $\alpha = \begin{cases} 
  1, & \text{if } \mod \left( \log_2(\bar{v}), 1 \right) \neq 0, \\
  2, & \text{otherwise}
  \end{cases}$

$\text{Err}(v)$ encloses the rounding error of computing $v$

- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \bar{e}]$

  $\epsilon = \max \left( |e|, |\bar{e}| \right)$
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
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**Val($v$) is the range of $v$**
- the format $Q_{i,f}$ of $v$ is deduced from
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**Err($v$) encloses the rounding error of computing $v$**
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An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
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\[ \text{Val}(v) \] is the range of \( v \)

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  \[ \text{Val}(v) = [v, \overline{v}] \]
  
  - \( i = \lceil \log_2 (\max(|v|, |\overline{v}|)) \rceil + \alpha \)
  
  \[ \alpha = \begin{cases} 
  1, & \text{if } \mod (\log_2(\overline{v}), 1) \neq 0, \\
  2, & \text{otherwise} 
  \end{cases} \]

\[ f = k - i \]

\[ \text{Err}(v) \] encloses the rounding error of computing \( v \)

- a bound \( \epsilon \) on rounding errors is deduced from
  \[ \text{Err}(v) = [\epsilon, \overline{\epsilon}] \]
  
  - \( \epsilon = \max (|\epsilon|, |\overline{\epsilon}|) \)

How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\} \)?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

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- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

\[ \text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right] \]
**Fixed-point multiplication**

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1+i_2.f_1+f_2}$
- But, doubling the word-length is costly

\[ \text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times \]
\[ \text{Err}(v) = \text{Err} \times \]
\[ + \text{Val}(v_1) \times \text{Err}(v_2) \]
\[ + \text{Val}(v_2) \times \text{Err}(v_1) \]
\[ + \text{Err}(v_1) \times \text{Err}(v_2) \]

\[ \text{Err}_x = [0, 2^{-f_r} - 2^{-(f_1+f_2)}] \]

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of \( Q_{i_1.f_1} / Q_{i_2.f_2} \) may be as large as \( i_1 + f_2 \).
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

\[
\text{Err}/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]
\]
Our new fixed-point division

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- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper a error bounds on $\text{Err}_/$?

$$\text{Err}_/ = [-2^{f_r}, 2^{f_r}]$$
- Sharper bound
- Risk of overflow at run-time
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- How to obtain sharper error bounds on $\text{Err}/$?

\[ \text{Err}/ = [-2^{f_r}, 2^{f_r}] \]

-Sharper bound
- Risk of overflow at run-time

How to decide of the output format of division?

- A large integer part
  - Prevents overflow
  - \( \checkmark \) loose error bounds and loss of precision

- A small integer part
  - \( \times \) may cause overflow
  - Sharp error bounds and more accurate computations
The propagation rule and implementation of division

Once the output format decided $Q_{ir,fr}$

\[ \text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}] \]

\[ \text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \]

\[ \text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \] \cap \text{Val}(v_2) \text{ and } \overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}] \]
The propagation rule and implementation of division

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$$\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].$$

$$\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) \cdot \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/\text{Val}(v_1) \cdot \text{Err}(v_1).$$

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/\text{Val}(v)} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
}
```
The propagation rule and implementation of division

- Once the output format decided $Q_{ir.fr}$

\[
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\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}.
\]

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err} / V}$ and $\text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll) || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$.
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$.

- Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$.
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Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$.

![Diagram showing the division format trade-off for inverting $2 \times 2$ matrices with examples for $Q_{2.30}$, $Q_{3.29}$, $[-1, 1]$, $d$, $[-2, 2]$, $[a, d, b, c]$, $[2^{-36}, 2^{-31}, 2^{-26}, 2^{-21}, 2^{-16}, 2^{-11}]$ with maximum experimental error ranging from 0% to 100%.](image)
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The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

- Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$

![Diagram of division format trade-off]

![Graph of division output format]

**Maximum experimental error**

- 0%
- 20%
- 40%
- 60%
- 80%
- 100%

**Overflow rate**

- 0%
- 20%
- 40%
- 60%
- 80%
- 100%

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- **CGPE (Code Generation for Polynomial Evaluation)**: initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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1. **Computation step** $\leadsto$ front-end
   - computes evaluation schemes $\leadsto$ DAGs
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1. **Computation step ⇛ front-end**
   - computes evaluation schemes ⇛ DAGs

2. **Filtering step ⇛ middle-end**
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency ⇛ scheduling filter
     - accuracy ⇛ numerical filter
     - ...

   

   ```
   Front-end
   DAG computation
   Middle-end
   Set of DAGs
   Filter 1
   ...
   Decorated DAGs
   ```
The CGPE tool

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     - ...

3. **Generation step** $\Rightarrow$ back-end
   - generates C codes and Gappa accuracy certificates
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
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![Graph showing original signal and filtered signals with different implementations](image-url)
Code synthesis for an IIR filter using CGPE

Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
                 int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
                 int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
                 int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul (0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul (0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul (0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul (0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul (0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul (0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
A strategy to synthesize code for matrix inversion

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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
A strategy to synthesize code for matrix inversion

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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

Cholesky decomposition

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
c_{i,j} / b_{j,j} & \text{if } i \neq j 
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

Triangular matrix inversion

\[
n_{i,j} = \begin{cases} 
1 / b_{i,i} & \text{if } i = j \\
 -c_{i,j} / b_{i,i} & \text{if } i \neq j 
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)
## Cholesky decomposition and triangular matrix inversion

### Cholesky decomposition

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### Triangular matrix inversion

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where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.
FPLA (Fixed-Point Linear Algebra)

User options

Problem dispatcher

- Dot-product solver
- Matrix multiplication solver
- Triangular matrix inversion solver
- Cholesky decomposition solver

FPLA-CGPE interface

Codes

Certificates

Coefficients and variables
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.

Maximum errors with various functions used to determine the output formats of division.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \left\lfloor \frac{(i_1 + i_2)}{2} \right\rfloor + 1$ to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic

The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

### Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles \( \sqrt{\text{and}} / \)

---

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)  
Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
Conclusions and perspectives

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Perspectives

- Integrate the matrix inversion flow
Conclusions and perspectives

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Fixed-point code synthesis for linear algebra basic blocks

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS) Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

