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Mohamed Amine Najahi

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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

<table>
<thead>
<tr>
<th>Floating-point computations</th>
<th>Fixed-point computations</th>
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<tbody>
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<td>© Easy and fast to implement © Easily portable</td>
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</tr>
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</tr>
</tbody>
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Fixed-point computations are tedious and time-consuming to implement, with over 50% of design time required [Wil98]. However, they rely only on integer instructions, which are efficient. Embedded systems targets like µ-controllers, DSPs, and FPGAs typically have efficient integer instructions. Fixed-point arithmetic is well-suited for embedded systems. However, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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- Requires dedicated hardware
- Slow if emulated in software

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- Tedious and time consuming to implement
  - > 50% of design time [Wil98]
- Relies only on integer instructions
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Embedded systems targets

- µ-controllers
- DSPs
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But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
The DEFIS approach

- DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
  
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
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      - **Fine grained IP blocks**: dot-products, polynomials, ...
      - **High level IP blocks**: matrix multiplication, triangular matrix inversion, Cholesky decomposition

---

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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1. Specify an arithmetic model
   - Contributions:
     - formalization of $\sqrt{}$ and $/$

2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   - it adheres to the arithmetic model
   - Contributions:

3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   - it generates code using CGPE
   - Contributions:
     - trade-off implementations for matrix multiplication
     - code synthesis for Cholesky decomposition and triangular matrix inversion
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

\[ x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell \]
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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Example:

- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$ $\rightarrow$ $x = (100.11000)_2 = (4.75)_{10}$
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An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \).
- Our model assumes a fixed word-length \( k \).

\[
\text{Val}(v) \text{ is the range of } v
\]

\[
\text{Err}(v) \text{ encloses the rounding error of computing } v
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An arithmetic model for fixed-point code synthesis

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**\( \text{Val}(v) \) is the range of \( v \)**

- the format \( Q_{i,f} \) of \( v \) is deduced from \( \text{Val}(v) = [v, \bar{v}] \)
  
  \( i = \left\lfloor \log_2 \left( \max(|v|, |\bar{v}|) \right) \right\rfloor + \alpha \)

  \[\alpha = \begin{cases} 
  1, & \text{if } \text{mod} \left( \log_2(\bar{v}), 1 \right) \neq 0, \\
  2, & \text{otherwise}
\end{cases}\]

  \( f = k - i \)

**\( \text{Err}(v) \) encloses the rounding error of computing \( v \)**

- a bound \( \epsilon \) on rounding errors is deduced from \( \text{Err}(v) = [e, \bar{e}] \)
  
  \[\epsilon = \max \left( |e|, |\bar{e}| \right)\]
An interval arithmetic based model

For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$.

Our model assumes a fixed word-length $k$.

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  - the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \bar{v}]$
  - $i = \left\lceil \log_2 \left( \max(\lvert v \rvert, \lvert \bar{v} \rvert) \right) \right\rceil + \alpha$
  - $f = k - i$
  - $\alpha = \begin{cases} 1, & \text{if } \text{mod} \left( \log_2(\bar{v}), 1 \right) \neq 0, \\ 2, & \text{otherwise} \end{cases}$

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\[ \epsilon = \max(|e|, |\bar{e}|) \]

How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\} \)?
Fixed-point multiplication

- The output format of a $\text{Q}_{i_1.f_1} \times \text{Q}_{i_2.f_2}$ is $\text{Q}_{i_1+i_2.f_1+f_2}$
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$.
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- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

$$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times$$
$$\text{Err}(v) = \text{Err} \times$$
$$+ \text{Val}(v_1) \times \text{Err}(v_2)$$
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$$+ \text{Err}(v_1) \times \text{Err}(v_2)$$

$$\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]$$
Fixed-point multiplication

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$$+ \text{Err}(v_1) \times \text{Err}(v_2)$$

- $\text{Err}_x = \left[0, 2^{-f_r} - 2^{-(f_1 + f_2)}\right]$  
- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2) {
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

The output integer part of \( Q_{i_1, f_1} / Q_{i_2, f_2} \) may be as large as \( i_1 + f_2 \)
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

\[ \text{Err} = [-2^{i_2 + f_1}, 2^{i_2 + f_1}] \]
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- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $\text{Err}_{/}$?

$\text{Err}_{/} = [-2^{f_r}, 2^{f_r}]$

😊 sharper bound
😊 risk of overflow at run-time
Our new fixed-point division

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- How to obtain sharper error bounds on $\text{Err}/$?

$$\text{Err}/ = [-2^{f_r}, 2^{f_r}]$$

- Sharper bound
- Risk of overflow at run-time

How to decide on the output format of division?

- A large integer part
  - ✔ Prevents overflow
  - ✗ Loose error bounds and loss of precision
- A small integer part
  - ✗ May cause overflow
  - ✔ Sharp error bounds and more accurate computations
The propagation rule and implementation of division

Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \mathcal{R} \text{ange}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}.
\]

\[
\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2) \quad \text{and} \quad \text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]
\]
The propagation rule and implementation of division

- Once the output format decided $Q_{ir.fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir.fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].$$

$$\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \text{Val}(v_1) \cdot \text{Err}(v_2).$$

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err} / \text{Val}(v_2)} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
}
```
The propagation rule and implementation of division

- Once the output format decided $Q_{ir.fr}$

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\text{Val}(v) = \text{Range}(Q_{ir.fr}) = [-2^{ir-1}, 2^{ir-1}-2^{fr}].
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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000001ll) == 0xFFFFFFFF80000001ll) || ((V & 0xFFFFFFFF80000001ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.
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![Diagram of division output format with maximum experimental error and overflow rate]
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

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![Division output format diagram](diagram.png)
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![Division output format diagram]

Maximum experimental error

- Maximum error
- Overflow rate
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2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
The CGPE tool

- **CGPE (Code Generation for Polynomial Evaluation):** initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation

1. **Computation step $$\leadsto$$ front-end**
   - computes evaluation schemes $$\leadsto$$ DAGs
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1. **Computation step** $\mapsto$ front-end
   - computes evaluation schemes $\mapsto$ DAGs

2. **Filtering step** $\mapsto$ middle-end
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency $\mapsto$ scheduling filter
     - accuracy $\mapsto$ numerical filter
     - ...

3. **Generation step** $\mapsto$ back-end
   - generates C codes and Gappa accuracy certificates
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Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$

\[ y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i] \]

```
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
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![Graph showing original signal, filtered signal in fixed-point and binary64, and error bounds.](image)
An implementation of the arithmetic model: the CGPE tool

Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
    int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
    int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
    int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26, y1); //Q8.24 [ -2^{-24} ,0]
    int32_t r1 = mul (0xa6eb5908, y2); //Q7.25 [ -2^{-25} ,0]
    int32_t r2 = mul (0x4688a637, y3); //Q5.27 [ -2^{-27} ,0]
    int32_t r3 = mul (0x65718e3b, u0); //Q2.30 [ -2^{-30} ,0]
    int32_t r4 = mul (0x65718e3b, u3); //Q2.30 [ -2^{-30} ,0]
    int32_t r5 = r3 + r4; //Q2.30 [ -2^{-29} ,0]
    int32_t r6 = r5 >> 2; //Q4.28 [ -2^{-27.6781} ,0]
    int32_t r7 = mul (0x4c152aad, u1); //Q4.28 [ -2^{-28} ,0]
    int32_t r8 = mul (0x4c152aad, u2); //Q4.28 [ -2^{-28} ,0]
    int32_t r9 = r7 + r8; //Q4.28 [ -2^{-27} ,0]
    int32_t r10 = r6 + r9; //Q4.28 [ -2^{-26.2996} ,0]
    int32_t r11 = r10 >> 1; //Q5.27 [ -2^{-25.9125} ,0]
    int32_t r12 = r2 + r11; //Q5.27 [ -2^{-25.3561} ,0]
    int32_t r13 = r12 >> 2; //Q7.25 [ -2^{-24.3853} ,0]
    int32_t r14 = r1 + r13; //Q7.25 [ -2^{-23.6601} ,0]
    int32_t r15 = r14 >> 1; //Q8.24 [ -2^{-23.1798} ,0]
    int32_t r16 = r0 + r15; //Q8.24 [ -2^{-22.5324} ,0]
    int32_t r17 = r16 << 2; //Q6.26 [ -2^{-22.5324} ,0]

    return r17;
}
```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition

Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

### Cholesky decomposition

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
c_{i,j} / b_{j,j} & \text{if } i \neq j 
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

### Triangular matrix inversion

\[
n_{i,j} = \begin{cases} 
1 / b_{i,i} & \text{if } i = j \\
-\frac{c_{i,j}}{b_{i,i}} & \text{if } i \neq j 
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)
Cholesky decomposition and triangular matrix inversion

Cholesky decomposition

\[ b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases} \]

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Triangular matrix inversion

\[ n_{i,j} = \begin{cases} \frac{1}{b_{i,i}} & \text{if } i = j \\ \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases} \]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.
FPLA (Fixed-Point Linear Algebra)

- User options
- Coefficients and variables
- Problem dispatcher
  - Dot-product solver
  - Matrix multiplication solver
  - Triangular matrix inversion solver
  - Cholesky decomposition solver
- FPLA-CGPE interface
- Codes
- Certificates

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \lceil (i_1 + i_2) / 2 \rceil + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 \( \times \) 5.  
(b) Triangular 10 \( \times \) 10.
How fast is generating triangular matrix inversion codes?

We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles √ and /

Adaptation of the CGPE tool to the model:
- generates code for fine grained expressions
- instruction selection

Development of FPLA:
- automated and certified code synthesis for linear algebra basic block
  - Cholesky decomposition and triangular matrix inversion: study of divisions' impact

Perspectives

Integrate the matrix inversion flow
## Conclusions and perspectives

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Fixed-point code synthesis for linear algebra basic blocks

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