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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

<table>
<thead>
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<th>Fixed-point computations</th>
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Floating-point computations are easy and fast to implement and easily portable, but they require dedicated hardware and can be slow if emulated in software. On the other hand, fixed-point computations might be tedious and time-consuming to implement, with more than 50% of design time required according to [Wil98]. Fixed-point computations rely only on integer instructions and are efficient for embedded systems, such as µ-controllers, DSPs, and FPGAs, which have efficient integer instructions. However, making fixed-point arithmetic easy, fast, and numerically safe to use by non-expert programmers remains a challenge.
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[IEEE754] refers to the Institute of Electrical and Electronics Engineers' standard for floating-point arithmetic. [Wil98] refers to the work of Wilfried.
Which arithmetic for computational tasks?

**Floating-point computations**
- Easy and fast to implement
- Easily portable [IEEE754]
- Requires dedicated hardware
- Slow if emulated in software

**Fixed-point computations**
- Tedious and time consuming to implement
  - > 50% of design time [Wil98]
- Relies only on integer instructions
- Efficient

### Embedded systems targets
- μ-controllers
- DSPs
- FPGAs

→ have efficient integer instructions

- Fixed-point arithmetic is well suited for embedded systems
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#### Embedded systems targets

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But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
The DEFIS approach

DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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  **Goal:** develop techniques and tools to automate fixed-point programming

- Combines conversion and IP block synthesis
  
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
    - automatic float-to-fix conversion
  
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  - Our approach (DALI, Univ. Perpignan):
    - certified fixed-point synthesis for:
      - **Fine grained IP blocks:** dot-products, polynomials, ...
      - **High level IP blocks:** matrix multiplication, triangular matrix inversion, Cholesky decomposition

---

### Implementation tools

**Infrastructure for the design of fixed-point systems**

- **Algorithm level optimization**
  - IWL Determination
  - Dynamic Range evaluation
  - FWL Determination

- **System level optimization**
  - S2S transformation
  - Specific block generation

- **Back-end**
  - Fixed-point C code

- **Application description**
  - Floating-point C code
  - Parameterized IP blocks

**Accuracy constraint**

**Architecture model**

- Validation & Optimization
- Accuracy evaluation

**High level Synthesis**

- Compiler

**Parameterized IP blocks**

**Architecture**
The DEFIS approach

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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
Our road-map

How to generate certified fixed-point code for matrix inversion?

1. Specify an arithmetic model
   ▶ Contributions:
     • formalization of √ and /

2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   ▶ Contributions:
     • implementation of the arithmetic model

3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by:

$$x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$
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A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format
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Example:

- \( x \) in \( Q_{3,5} \) and \( X = (10011000)_2 = (152)_{10} \) \( \rightarrow \) \( x = (100.11000)_2 = (4.75)_{10} \)
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A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.

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- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$ $\rightarrow$ $x = (100.11000)_2 = (4.75)_{10}$

How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

$\text{Val}(v)$ is the range of $v$

$\text{Err}(v)$ encloses the rounding error of computing $v$
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $Val(v)$ and $Err(v)$
- Our model assumes a fixed word-length $k$

$Val(v)$ is the range of $v$

- the format $Q_{i,f}$ of $v$ is deduced from $Val(v) = [v, \bar{v}]$

- $i = \left\lfloor \log_2 (\max(|v|, |\bar{v}|)) \right\rfloor + \alpha$

- $f = k - i$

$Err(v)$ encloses the rounding error of computing $v$

- a bound $\epsilon$ on rounding errors is deduced from $Err(v) = [e, \bar{e}]$

- $\epsilon = \max(|e|, |\bar{e}|)$
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
- Our model assumes a fixed word-length \( k \)

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- the format \( Q_{i,f} \) of \( v \) is deduced from \( \text{Val}(v) = [\underline{v}, \overline{v}] \)

\[ i = \left\lfloor \log_2 \left( \max(|\underline{v}|, |\overline{v}|) \right) \right\rfloor + \alpha \]

\[ f = k - i \]

\[ \alpha = \begin{cases} 
1, & \text{if } \text{mod} \left( \log_2(\overline{v}), 1 \right) \neq 0, \\
2, & \text{otherwise} 
\end{cases} \]

\( \text{Err}(v) \) encloses the rounding error of computing \( v \)

- a bound \( \epsilon \) on rounding errors is deduced from \( \text{Err}(v) = [\underline{e}, \overline{e}] \)

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\text{Err}(v) \) encloses the rounding error of computing \( v \)

- a bound \( \epsilon \) on rounding errors is deduced from
  \[ \text{Err}(v) = [e, \bar{e}] \]
  - \( \epsilon = \max(|e|, |\bar{e}|) \)

How to propagate \( \text{Val}(v) \) and \( \text{Err}(v) \) for \( \diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\} \)?
Fixed-point multiplication

- The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$

\[
Val(v) = Val(v_1) \times Val(v_2) \\
Err(v) = Val(v_1) \times Err(v_2) + Val(v_2) \times Err(v_1) + Err(v_1) \times Err(v_2)
\]
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

\[ \text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right] \]
Fixed-point multiplication

- The output format of a $\mathbb{Q}_{i_1,f_1} \times \mathbb{Q}_{i_2,f_2}$ is $\mathbb{Q}_{i_1 + i_2, f_1 + f_2}$
- But, doubling the word-length is costly

$$
\text{Discarded bits}
$$

$$
\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]
$$

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$. 

\[ \frac{i_1}{i_2} \]
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

$$\text{Err}_/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]$$
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- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $Err/\$?

$$Err/ = [-2^{f_r}, 2^{f_r}]$$

- sharper bound
- risk of overflow at run-time
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $Err_/$?

$Err_/ = [-2^{f_r}, 2^{f_r}]$

- sharper bound
- risk of overflow at run-time

How to decide on the output format of division?

- A large integer part
  - ✓ prevents overflow
  - ☒ loose error bounds and loss of precision

- A small integer part
  - ☒ may cause overflow
  - ✓ sharp error bounds and more accurate computations
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/Val(v_1) \text{Err}(v_1) \text{Val}(v_2) \text{Err}(v_2)
\]

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/} \cap \text{Val}(v_2)$ and $\text{Val}(v) = [-2^{ir-1}, -2^{fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$
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\]

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
}
```
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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t) V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
        || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$

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The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [−1, 1]$ in the format $\mathbb{Q}_{2.30}$.

Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$.
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\[ \begin{array}{c}
[{-1,1}] \\
[{-1,1}] \\
[{-1,1}] \\
[{-1,1}] \\
d \\
- \\
\times \\
\times \\
a & d & b & c
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![Diagram showing division output format with maximum experimental error and overflow rate.](image-url)
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

CGPE (Code Generation for Polynomial Evaluation): initiated by Revy [MR11]
- synthesizes fixed-point code for polynomial evaluation
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Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
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![Graph showing original signal and filtered signals in fixed-point and binary64 representations.](image)

- Amplitude vs. Time for original and filtered signals.
- Certified error bound comparisons between different implementations.
An implementation of the arithmetic model: the CGPE tool

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![Graph showing original signal and filtered signals using different formats](image)

![Graph showing log2 of error bound and error for different implementations](image)
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Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter(int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/, int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/, int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/, int32_t y3 /*Q6.26*/)
{
    // Formats Err
    int32_t r0 = mul(0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul(0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul(0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul(0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul(0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul(0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul(0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
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A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
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The basic blocks we need to include in our tool-chain

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Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
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### Cholesky decomposition and triangular matrix inversion

#### Cholesky decomposition

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
c_{i,j} / b_{j,j} & \text{if } i \neq j
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

#### Triangular matrix inversion

\[
n_{i,j} = \begin{cases} 
1 / b_{i,i} & \text{if } i = j \\
-c_{i,j} / b_{i,i} & \text{if } i \neq j
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\]

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Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.
FPLA (Fixed-Point Linear Algebra)
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \lceil (i_1 + i_2) / 2 \rceil + 1 \) to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

**Graphs:**

- **Condition number**
  - KMS
  - Lehmer
  - Prolate
  - Hilbert
  - Cauchy

- **Maximum error**
  - Hilbert
  - KMS
  - Cauchy
  - Lehmer
  - Prolate

**Points:**

- Ill-conditioned matrices tend to overflow more often
  - Similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

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Development of FPLA:
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  - Cholesky decomposition and triangular matrix inversion: study of divisions' impact

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M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)  Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks 26/25