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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
### Which arithmetic for computational tasks?

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<th>Floating-point computations</th>
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<td>© [IEEE754]</td>
<td>© Embedded systems targets</td>
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<tr>
<td>± §</td>
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- Fixed-point computations: Tedious and time consuming to implement. More than 50% of design time [Wil98]. Relies only on integer instructions. Efficient for embedded systems targets like µ-controllers, DSPs, and FPGAs. Fixed-point arithmetic is well suited for embedded systems.

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS) - Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
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Embedded systems targets

- µ-controllers
- DSPs
- FPGAs

→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems
Which arithmetic for computational tasks?

### Floating-point computations
- 😊 Easy and fast to implement
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**Embedded systems targets**

- µ-controllers
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→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
The DEFIS approach

- DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
The DEFIS approach

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  **Goal:** develop techniques and tools to automate fixed-point programming

- Combines conversion and IP block synthesis
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
    - automatic float-to-fix conversion
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  - Our approach (DALI, Univ. Perpignan):
    - certified fixed-point synthesis for:
      
      - **Fine grained IP blocks:** dot-products, polynomials, ...
      - **High level IP blocks:** matrix multiplication, triangular matrix inversion, Cholesky decomposition

  *Implementation tools*

  *Infrastructure for the design of fixed-point systems*

  *Algorithm level optimization*

  *System level optimization*

  *Application description*

  *Back-end*

  *Architecture model*

  *Validation & Optimization*

  *Accuracy evaluation*

  *S2S transformation*

  *Specific block generation*

  *FWL Determination*

  *Dynamic Range evaluation*

  *IWL Determination*

  *Floating-point C code*

  *Parameterized IP blocks*
The DEFIS approach

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- **Long term objective:** code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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1. Specify an arithmetic model
   ▶ Contributions:
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
   ▶ it adheres to the arithmetic model
   ▶ Contributions:
     • implementation of the arithmetic model

3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

$\leadsto$ The value of $x$ is given by $x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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Example:

- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$ $\Rightarrow$ $x = (100.11000)_2 = (4.75)_{10}$
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A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.

Example:

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- $x = (100.11000)_2 = (4.75)_{10}$

How to compute with fixed-point numbers?
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

$\text{Val}(v)$ is the range of $v$

$\text{Err}(v)$ encloses the rounding error of computing $v$
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$\text{Val}(v)$ is the range of $v$
- the format $Q_{i,f}$ of $v$ is deduced from
  $\text{Val}(v) = [v, \bar{v}]$
  
  $i = \left\lceil \log_2 (\max(|v|, |\bar{v}|)) \right\rceil + \alpha$
  $f = k - i$

  $\alpha = \begin{cases} 1, & \text{if } \text{mod} (\log_2(\bar{v}), 1) \neq 0, \\ 2, & \text{otherwise} \end{cases}$

$\text{Err}(v)$ encloses the rounding error of computing $v$
- A bound $\epsilon$ on rounding errors is deduced from
  $\text{Err}(v) = [e, \bar{e}]$
  
  $\epsilon = \max(|e|, |\bar{e}|)$
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
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  \]

  \[
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An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$.
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  - $i = \lceil \log_2(\max(|v|, |\overline{v}|)) \rceil + \alpha$
  - $f = k - i$

$$\alpha = \begin{cases} 
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**$\text{Err}(v)$** encloses the rounding error of computing $v$

- A bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \overline{e}]$.
  - $\epsilon = \max(|e|, |\overline{e}|)$

How to propagate $\text{Val}(v)$ and $\text{Err}(v)$ for $\circ \in \{+, -, \times, \ll, \gg, \sqrt{}, /\}$?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

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$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2)$
$\text{Err}(v) = \text{Val}(v_1) \times \text{Err}(v_2)$
$+ \text{Val}(v_2) \times \text{Err}(v_1)$
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Fixed-point multiplication

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$$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times$$
$$\text{Err}(v) = \text{Err} \times$$
$$+ \text{Val}(v_1) \times \text{Err}(v_2)$$
$$+ \text{Val}(v_2) \times \text{Err}(v_1)$$
$$+ \text{Err}(v_1) \times \text{Err}(v_2)$$

$$\text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-f_1 - f_2} \right]$$
Fixed-point multiplication

- The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

$$\text{Err}_x = \left[0, 2^{-f_r} - 2^{-(f_1 + f_2)}\right]$$

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = (((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1,f_1} / Q_{i_2,f_2}$ may be as large as $i_1 + f_2$
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

$$\text{Err}/ = [-2^{i_2+f_1}, 2^{i_2+f_1}]$$
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Our new fixed-point division

- The output integer part of $Q_{i_1,f_1}/Q_{i_2,f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $\mathbf{Err}/$?

\[
\mathbf{Err}/ = [-2^{f_r}, 2^{f_r}]
\]

- sharper bound
- risk of overflow at run-time
Our new fixed-point division

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$$\text{Err} = [-2^{f_r}, 2^{f_r}]$$

- sharper bound
- risk of overflow at run-time

How to decide about the output format of division?

- A large integer part
  - prevents overflow
  - ✓ prevents overflow
  - ✓ sharp error bounds and more accurate computations

- A small integer part
  - may cause overflow
  - x may cause overflow
  - lose error bounds and loss of precision
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \text{Val}(v_1).
\]

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$
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\]

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta) {
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
}
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```c
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{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll) || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$.
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$

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![Diagram showing division output format with maximum experimental error and overflow rate graphs.]
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The division format trade-off: case of inverting $2 \times 2$ matrices

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- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$
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Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- **CGPE** (*Code Generation for Polynomial Evaluation*): initiated by Revy \[MR11\]
  - synthesizes fixed-point code for polynomial evaluation
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1. Computation step $\mapsto$ front-end
   - computes evaluation schemes $\mapsto$ DAGs
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2. **Filtering step ⇝ middle-end**
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency ⇝ scheduling filter
     - accuracy ⇝ numerical filter
     - ...

- **Generation step ⇝ back-end**
  - generates C codes and Gappa accuracy certificates
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Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
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![Graph showing original signal and filtered signals using fixed-point and binary formats.](image-url)
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```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/, int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
                 int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/, int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26 , y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul (0xa6eb5908 , y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul (0x4688a637 , y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul (0x65718e3b , u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul (0x65718e3b , u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2;  //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul (0x4c152aad , u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul (0x4c152aad , u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8;  //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1;  //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11;  //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2;  //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r13 >> 1;  //Q8.24 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1;  //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r15 >> 2;  //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2;  //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
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A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
\frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

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n_{i,j} = \begin{cases} 
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---

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a \( 6 \times 6 \) matrix.
FPLA (Fixed-Point Linear Algebra)
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t \),
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t \),
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t \),
4. \( f_4(i_1, i_2) = \lceil (i_1 + i_2)/2 \rceil + t \),

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 \times 5.
(b) Triangular 10 \times 10.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1 \) to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{\text{and}}$ /
## Conclusions and perspectives

### Contributions

- **Formalization and implementation of an arithmetic model**
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  - generates code for fine grained expressions
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- Integrate the matrix inversion flow
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Fixed-point code synthesis for linear algebra basic blocks


[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.


[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.


[LHD14] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.


