Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
Mohamed Amine Najahi

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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

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Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

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<tr>
<th>Floating-point computations</th>
<th>Fixed-point computations</th>
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© Easy and fast to implement
© Easily portable
© Requires dedicated hardware
© Slow if emulated in software

© Tedious and time consuming to implement
• > 50% of design time [Wil98]
© Relies only on integer instructions
© Efficient

Embedded systems targets
µ-controllers DSPs FPGAs → have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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- Combines conversion and IP block synthesis

  - Ménard *et al.* (CAIRN, Univ. Rennes) [*MCCS02*]:
    - automatic float-to-fix conversion
  
  - Didier *et al.* (PEQUAN, Univ. Paris) [*LHD14*]:
    - code generation for the linear filter IP block

Long term objective:

- code synthesis for matrix inversion

Implementation tools

Infrastructure for the design of fixed-point systems

Algorithm level optimization

- IWL Determination
- Dynamic Range evaluation

- FWL Determination
- Validation & Optimization
- Accuracy evaluation

Back-end

- High level Synthesis
- Compiler

Parameterized IP blocks

Floating-point C code

Application description

System level optimization

S2S transformation

Specific block generation

Architecture model

Validation & Optimization

Accuracy constraint

Accuracy evaluation

Fixed-point C code

Architecture
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    - certified fixed-point synthesis for:
      - **Fine grained IP blocks**: dot-products, polynomials, ...
      - **High level IP blocks**: matrix multiplication, triangular matrix inversion, Cholesky decomposition

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Our road-map

How to generate certified fixed-point code for matrix inversion?
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   ▶ Contributions:
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
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   ▶ Contributions:
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by $x = \frac{X}{2^f} = \sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$
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Notation

A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $\mathbb{Q}_{i,f}$ format
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**Example:**

- $x$ in $Q_{3,5}$ and $X = (1001 1000)_2 = (152)_{10}$ \implies $x = (100.11000)_2 = (4.75)_{10}$
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How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
- Our model assumes a fixed word-length \( k \)

\[
\text{Val}(v) \text{ is the range of } v
\]

\[
\text{Err}(v) \text{ encloses the rounding error of computing } v
\]
An arithmetic model for fixed-point code synthesis

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\[ \text{Val}(v) \text{ is the range of } v \]

- the format \( Q_{i,f} \) of \( v \) is deduced from \( \text{Val}(v) = [v, \bar{v}] \)
  \[ i = \left\lceil \log_2 \left( \max(|v|, |\bar{v}|) \right) \right\rceil + \alpha \]
  \[ f = k - i \]

\[ \alpha = \begin{cases} 
1, & \text{if } \text{mod} \left( \log_2(\bar{v}), 1 \right) \neq 0, \\
2, & \text{otherwise} 
\end{cases} \]

\[ \text{Err}(v) \text{ encloses the rounding error of computing } v \]

- a bound \( \epsilon \) on rounding errors is deduced from \( \text{Err}(v) = [e, \bar{e}] \)
  \[ \epsilon = \max(|e|, |\bar{e}|) \]
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
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**Val($v$) is the range of $v**

- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \bar{v}]$
  - $i = \left\lfloor \log_2 \left( \max(|v|, |\bar{v}|) \right) \right\rfloor + \alpha$
  - $f = k - i$

\[ \alpha = \begin{cases} 1, & \text{if mod} \left( \log_2(\bar{v}), 1 \right) \neq 0, \\ 2, & \text{otherwise} \end{cases} \]

**Err($v$) encloses the rounding error of computing $v$**

- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \bar{e}]$
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- the format $Q_{i,f}$ of $v$ is deduced from $\text{Val}(v) = [v, \bar{v}]$
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$\text{Err}(v)$ encloses the rounding error of computing $v$

- a bound $\epsilon$ on rounding errors is deduced from $\text{Err}(v) = [e, \bar{e}]$
  - $\epsilon = \max( |e|, |\bar{e}| )$

How to propagate $\text{Val}(v)$ and $\text{Err}(v)$ for $\odot \in \{+, -, \times, \ll, \gg, \sqrt{}, /\}$?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
**Fixed-point multiplication**

- The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1 + i_2.f_1 + f_2}$

\[
Val(v) = Val(v_1) \times Val(v_2) \\
Err(v) = Val(v_1) \times Err(v_2) + Val(v_2) \times Err(v_1) + Err(v_1) \times Err(v_2)
\]
Fixed-point multiplication

- The output format of a \(Q_{i_1.f_1} \times Q_{i_2.f_2}\) is \(Q_{i_1 + i_2.f_1 + f_2}\)
- But, doubling the word-length is costly

\[
\text{Err}_x = \left[0, 2^{-f_r} - 2^{-(f_1 + f_2)}\right]
\]
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
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$$\text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times$$
$$\text{Err}(v) = \text{Err} \times$$
$$+ \text{Val}(v_1) \times \text{Err}(v_2)$$
$$+ \text{Val}(v_2) \times \text{Err}(v_1)$$
$$+ \text{Err}(v_1) \times \text{Err}(v_2)$$

$\text{Err}_x = \left[0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]$ 

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1,f_1}/Q_{i_2,f_2}$ may be as large as $i_1 + f_2$
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

\[
\text{Err/} = [-2^{i_2+f_1}, 2^{i_2+f_1}]
\]
Our new fixed-point division

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- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper a error bounds on $\text{Err}/$?

\[
\text{Err}/ = [-2^{f_r}, 2^{f_r}]
\]

- sharper bound
- risk of overflow at run-time
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
- But, doubling the word-length is costly
- How to obtain sharper error bounds on $\text{Err}$?

$$\frac{i_1 \rightarrow f_1}{i_2 \rightarrow f_2} \times 2^k$$

$\text{Err} = [-2^{f_r}, 2^{f_r}]$
- ☑ sharper bound
- ☑ risk of overflow at run-time

How to decide of the output format of division?

- A large integer part
  - ✓ prevents overflow
  - ✗ loose error bounds and loss of precision

- A small integer part
  - ✗ may cause overflow
  - ✓ sharp error bounds and more accurate computations
The propagation rule and implementation of division

Once the output format decided $Q_{ir,fr}$

\[
\begin{align*}
\text{Val}(v) &= \text{Range}(Q_{ir,fr}) = \left[-2^{ir-1}, 2^{ir-1} - 2^{fr}\right]. \\
\text{Err}(v) &= \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}/
\end{align*}
\]

\[
\begin{align*}
\text{Val}(v_2) &= \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}/} \cap \text{Val}(v_2) \quad \text{and} \quad \overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}] 
\end{align*}
\]
The propagation rule and implementation of division

Once the output format decided $Q_{ir.fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir.fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

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\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \wedge \text{Val}(v_2) + \text{Err} / \bigcup [2^{-fr}, 2^{ir-1} - 2^{fr}].
\]

\[
\overline{\text{Val}}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err} /} \cap \text{Val}(v_2) \quad \text{and} \quad \overline{\text{Val}}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}].
\]

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
}
```
The propagation rule and implementation of division

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\]

\[
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\]

```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
                  || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

- Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.
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- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

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- Maximum experimental error
- Overflow rate

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![Diagram showing division output format and error rates](image-url)
The division format trade-off: case of inverting $2 \times 2$ matrices

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Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d & -b \\ c & a \end{pmatrix} / \Delta$

Graph and table showing division output format with maximum experimental error and overflow rate.
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[Diagram showing division output format and maximum experimental error vs. overflow rate]
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The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$

![Diagram of division format trade-off](image)
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- **CGPE (Code Generation for Polynomial Evaluation):** initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
The CGPE tool

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1. Computation step $\leadsto$ front-end
   - computes evaluation schemes $\leadsto$ DAGs
The CGPE tool

CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
- synthesizes fixed-point code for polynomial evaluation

1. Computation step ~*~ front-end
   - computes evaluation schemes ~*~ DAGs

2. Filtering step ~*~ middle-end
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency ~*~ scheduling filter
     - accuracy ~*~ numerical filter
     - ...

Front-end

DAG computation

Middle-end

Set of DAGs

Filter 1

...  

Filter n

Decorated DAGs

XML
The CGPE tool

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   - computes evaluation schemes ⇝ **DAGs**

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     - accuracy ⇝ numerical filter
     - ...

3. **Generation step** ⇝ **back-end**
   - generates C codes and Gappa accuracy certificates
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
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\[
y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]
\]

![Graph showing comparison between original signal and filtered signals using CGPE tool.]](image-url)
Code synthesis for an IIR filter using CGPE

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$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

![Graph showing original signal and filtered signals](image)
An implementation of the arithmetic model: the CGPE tool

Code synthesis for an IIR filter using CGPE

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$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```c
int32_t filter( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
                int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
                int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
                int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul(0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul(0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul(0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul(0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul(0x65718e3b, u3); //Q5.27 [-2^{-27},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul(0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul(0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r11 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
Outline of the talk

1. An arithmetic model for fixed-point code synthesis
2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
A strategy to synthesize code for matrix inversion

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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
\begin{align*}
b_{i,j} &= \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
c_{i,j} & \text{if } i \neq j \\
b_{j,j} & \text{if } i = j \\
\end{cases} \\
\text{with } c_{i,j} &= m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}
\end{align*}
\]

**Triangular matrix inversion**

\[
\begin{align*}
n_{i,j} &= \begin{cases} 
1 & \text{if } i = j \\
\frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \\
\end{cases} \\
\text{where } c_{i,j} &= \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j}
\end{align*}
\]
**Cholesky decomposition and triangular matrix inversion**

**Cholesky decomposition**

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
\frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j 
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

\[
n_{i,j} = \begin{cases} 
\frac{1}{b_{i,i}} & \text{if } i = j \\
-\frac{c_{i,j}}{b_{i,i}} & \text{if } i \neq j 
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)
Fixed-point code synthesis for linear algebra basic blocks

FPLA (Fixed-Point Linear Algebra)

Problem dispatcher

- User options
- Coefficients and variables
- Problem dispatcher
- Dot-product solver
- Matrix multiplication solver
- Triangular matrix inversion solver
- Cholesky decomposition solver
- FPLA-CGPE interface
- Codes
- Certificates
Impact of the output format of division

Different functions to set the output format of division

1. $f_1(i_1, i_2) = t$
2. $f_2(i_1, i_2) = \min(i_1, i_2) + t$
3. $f_3(i_1, i_2) = \max(i_1, i_2) + t$
4. $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + t$

$i_1$ and $i_2$: integer parts of the numerator and denominator and $t \in [-2, 8]$

(a) Cholesky 5 × 5.

(b) Triangular 10 × 10.

Maximum errors with various functions used to determine the output formats of division.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division.

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{\quad}$ and $/$

Adaptation of the CGPE tool to the model:
- generates code for fine-grained expressions
- instruction selection

Development of FPLA:
- automated and certified code synthesis for linear algebra basic blocks
  - Cholesky decomposition and triangular matrix inversion: study of divisions' impact

Perspectives

Integrate the matrix inversion flow
Conclusions and perspectives

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\[ \rightarrow \text{Cholesky decomposition and triangular matrix inversion: study of divisions' impact} \]

Perspectives

- Integrate the matrix inversion flow
Fixed-point code synthesis for linear algebra basic blocks


[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.


[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in DSP designs.


[LHD14] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.


