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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
Which arithmetic for computational tasks?

<table>
<thead>
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<th>Floating-point computations</th>
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Embedded systems targets

\(\mu\)-controllers DSPs FPGAs

→ have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?
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© Floating-point computations have easy and fast implement and easily portable. 

© Fixed-point computations are tedious and time-consuming to implement, with more than 50% of the design time required according to [Wil98].
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Floating-point computations
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The DEFIS approach

- DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
  
  ▶ Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
    
    • automatic float-to-fix conversion
  
  ▶ Didier *et al.* (PEQUAN, Univ. Paris) [LHD14]:
    
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  - Our approach (DALI, Univ. Perpignan):
    - certified fixed-point synthesis for:
      - **Fine grained IP blocks:** dot-products, polynomials, ...
      - **High level IP blocks:** matrix multiplication, triangular matrix inversion, Cholesky decomposition

---

**Implementation tools**

- Infrastructure for the design of fixed-point systems
- Algorithm level optimization
- IWL Determination
- Dynamic Range evaluation
- FWL Determination
- Back-end transformation
- S2S transformation
- Specific block generation
- Algorithm level optimization
- Floating-point C code
- Parameterized IP blocks

**System level optimization**

- IWL Determination
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- FWL Determination
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**Application description**

- Application description
- Specific block generation
- Floating-point C code
- Parameterized IP blocks

**Infrastructure for the design of fixed-point systems**

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**Architecture model**

- Accuracy constraint
- Validation & Optimization
- Accuracy evaluation
The DEFIS approach

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- **Long term objective**: code synthesis for matrix inversion
Our road-map

How to generate certified fixed-point code for matrix inversion?
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1. Specify an arithmetic model
   - Contributions:
     * formalization of $\sqrt{}$ and $/$
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2. Build a synthesis tool, CGPE, for fine grained IP blocks:
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3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
   ▶ it generates code using CGPE
   ▶ Contributions:
     • trade-off implementations for matrix multiplication
     • code synthesis for Cholesky decomposition and triangular matrix inversion
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by:

$$x = \frac{X}{2^f} = \sum_{\ell = -f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$
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A fixed-point number with $i$ bits of integer part and $f$ bits of fraction part is in the $Q_{i,f}$ format.
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- $x$ in $Q_{3,5}$ and $X = (10011000)_2 = (152)_{10}$ → $x = (100.11000)_2 = (4.75)_{10}$
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How to compute with fixed-point numbers?
An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $\text{Val}(v)$ and $\text{Err}(v)$
- Our model assumes a fixed word-length $k$

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\[
\text{Val}(v) = [v_L, v_U] \quad \text{and} \quad \text{Err}(v) = [e_L, e_U]
\]

\[
\pi_i = \lceil \log_2 (\max( |v_L|, |v_U| )) \rceil + \alpha
\]

\[
\pi_f = k - i \quad \alpha = \begin{cases} 1, & \text{if } \log_2(\text{Val}(v)) \mod 1 \neq 0 \\ 2, & \text{otherwise} \end{cases}
\]

\[
\epsilon = \max(|e_L|, |e_U|)
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How to propagate $Val(v)$ and $Err(v)$ for $\diamond \in \{+,-,\times, \ll, \gg, \sqrt{}, /\}$?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$.
Fixed-point multiplication

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Fixed-point multiplication

- The output format of a $\mathbb{Q}_{i_1.f_1} \times \mathbb{Q}_{i_2.f_2}$ is $\mathbb{Q}_{i_1 + i_2.f_1 + f_2}$
- But, doubling the word-length is costly

\[ \text{Val}(v) = \text{Val}(v_1) \times \text{Val}(v_2) - \text{Err} \times \]
\[ \text{Err}(v) = \text{Err} \times \]
\[ + \text{Val}(v_1) \times \text{Err}(v_2) \]
\[ + \text{Val}(v_2) \times \text{Err}(v_1) \]
\[ + \text{Err}(v_1) \times \text{Err}(v_2) \]

\[ \text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right] \]
Fixed-point multiplication

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- $\text{Err}_{\times} = \left[ 0, 2^{-f_r} - 2^{-(f_1 + f_2)} \right]$
- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2) {
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of \( \frac{Q_{i_1.f_1}}{Q_{i_2.f_2}} \) may be as large as \( i_1 + f_2 \).
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$

$$\text{Err} / = [ -2^{i_2 + f_1}, 2^{i_2 + f_1} ]$$
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- The output integer part of $Q_{i_1.f_1}/Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
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- How to obtain sharper a error bounds on $\text{Err}/$?

\[ \text{Err}/ = [-2^{f_r}, 2^{f_r}] \]

- sharper bound
- risk of overflow at run-time
Our new fixed-point division

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How to decide of the output format of division?

- A large integer part
  - Prevents overflow
  - ✓ Sharp error bounds and more accurate computations
  - ✗ Loose error bounds and loss of precision

- A small integer part
  - ✗ May cause overflow
  - ✓ Sharp error bounds and more accurate computations
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err} / \text{Val}(v_2).
\]

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$
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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t) V1) << eta;
    int64_t V = t1 / V2;

    return (int32_t) V;
}
```
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{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll) || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}^2$.

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![Diagram of division output format with maximum experimental error and overflow rate]
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![Division output format diagram](image)
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- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

- Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{\Delta}{\Delta}$
The division format trade-off: case of inverting $2 \times 2$ matrices

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![Diagram of division output format](image-url)
The division format trade-off: case of inverting $2 \times 2$ matrices

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Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- **CGPE** (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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2. Filtering step \(\rightarrow\) middle-end
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency \(\rightarrow\) scheduling filter
     - accuracy \(\rightarrow\) numerical filter
     - ...

[Diagram showing the front-end, middle-end, and back-end stages of the CGPE tool with XML as output.]
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3. Generation step $\rightarrow$ back-end
   - generates C codes and Gappa accuracy certificates
Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k - i] - \sum_{i=1}^{3} a_i \cdot y[k - i]$$

```xml
<dotproduct inf="0xb1e91685" sup="0x4e1e697b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x6571ae3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e1e697b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
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\[
\begin{align*}
\text{Amplitude} & \quad \text{Time} \\
\text{Original signal} & \quad \text{Filtered in fixed-point using } S_1 \\
& \quad \text{Filtered in binary64}
\end{align*}
\]
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![Graph of original signal and filtered signals](image)

![Graph of certified error bound and error of implementations](image)
Code synthesis for an IIR filter using CGPE

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$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter ( int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
    int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
    int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
    int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul (0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul (0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul (0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul (0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul (0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul (0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul (0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]

    return r17;
}
```
Outline of the talk

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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[
b_{i,j} = \begin{cases} 
\sqrt{c_{i,i}} & \text{if } i = j \\
c_{i,j} / b_{j,j} & \text{if } i \neq j
\end{cases}
\]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

\[
n_{i,j} = \begin{cases} 
1 / b_{i,i} & \text{if } i = j \\
-c_{i,j} / b_{i,i} & \text{if } i \neq j
\end{cases}
\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)
Cholesky decomposition and triangular matrix inversion

**Cholesky decomposition**

\[ b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases} \]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

**Triangular matrix inversion**

\[ n_{i,j} = \begin{cases} \frac{1}{b_{i,i}} & \text{if } i = j \\ \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases} \]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 × 6 matrix.
FPLA (Fixed-Point Linear Algebra)
Impact of the output format of division

Different functions to set the output format of division

1. \( f_1(i_1, i_2) = t, \)
2. \( f_2(i_1, i_2) = \min(i_1, i_2) + t, \)
3. \( f_3(i_1, i_2) = \max(i_1, i_2) + t, \)
4. \( f_4(i_1, i_2) = \left\lfloor (i_1 + i_2)/2 \right\rfloor + t, \)

\( i_1 \) and \( i_2 \): integer parts of the numerator and denominator and \( t \in [-2, 8] \)

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky 5 \( \times \) 5.
(b) Triangular 10 \( \times \) 10.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \left\lfloor (i_1 + i_2)/2 \right\rfloor + 1$ to set the output format of division

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = ⌊(i_1 + i_2)/2⌋ + 1$ to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\times$ and $/$
Conclusions and perspectives

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  - generates code for fine grained expressions
  - instruction selection
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  - automated and certified code synthesis for linear algebra basic block
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- Integrate the matrix inversion flow
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Fixed-point code synthesis for linear algebra basic blocks


[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. Automatic floating-point to fixed-point conversion for DSP code generation.


[FRC03] Claire F. Fang, Rob A. Rutenbar, and Tsuhan Chen. Fast, accurate static analysis for fixed-point finite-precision effects in dsp designs.


[LHD14] Benoit Lopez, Thibault Hilaire, and Laurent-Stéphane Didier. Formatting bits to better implement signal processing algorithms.


