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Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506
### Which arithmetic for computational tasks?

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**Floating-point computations**

- Easy and fast to implement
- Easily portable
- Requires dedicated hardware
- Slow if emulated in software

**Fixed-point computations**

- Tedious and time consuming to implement
- Relies only on integer instructions
- Efficient

*For embedded systems targets: µ-controllers, DSPs, FPGAs* → have efficient integer instructions

Fixed-point arithmetic is well suited for embedded systems.

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?

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M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

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Floating-point computations are easy and fast to implement, and easily portable, but require dedicated hardware. They can be slow if emulated in software.

Fixed-point computations are tedious and time consuming to implement, but rely only on integer instructions, making them efficient. They are well suited for embedded systems such as µ-controllers, DSPs, and FPGAs, which have efficient integer instructions.

However, making fixed-point arithmetic easy, fast, and numerically safe to use by non-expert programmers remains a challenge.
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- DSPs
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The DEFIS approach

DEFIS (ANR, 2011-2015)

**Goal:** develop techniques and tools to automate fixed-point programming
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- Combines conversion and IP block synthesis
  
  - Ménard *et al.* (CAIRN, Univ. Rennes) [MCCS02]:
    - automatic float-to-fix conversion
  
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    - **Fine grained IP blocks:** dot-products, polynomials, ...
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Our road-map

How to generate certified fixed-point code for matrix inversion?

1. Specify an arithmetic model

   Contributions:
   • formalization of
   •

2. Build a synthesis tool, CGPE, for fine grained IP blocks:

   Contributions:
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   • trade-off implementations for matrix multiplication
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Outline of the talk

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2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
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Fixed-point arithmetic numbers

A fixed-point number $x$ is defined by two integers:

- $X$ the $k$-bit integer representation of $x$
- $f$ the implicit scaling factor of $x$

The value of $x$ is given by

$$x = \frac{X}{2^f} = \sum_{\ell = -f}^{k-1-f} X_{\ell+f} \cdot 2^\ell$$
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An interval arithmetic based model

- For each coefficient or variable $v$, we keep track of 2 intervals $Val(v)$ and $Err(v)$
- Our model assumes a fixed word-length $k$

$Val(v)$ is the range of $v$

$Err(v)$ encloses the rounding error of computing $v$
An arithmetic model for fixed-point code synthesis

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$Val(v)$ is the range of $v$

- the format $Q_{i,f}$ of $v$ is deduced from $Val(v) = [v, \bar{v}]$
  
  - $i = \left\lceil \log_2 (\max(|v|, |\bar{v}|)) \right\rceil + \alpha$
  - $f = k - i$

  $\alpha = \begin{cases} 
  1, & \text{if } \text{mod}(\log_2(\bar{v}), 1) \neq 0, \\
  2, & \text{otherwise} 
  \end{cases}$

$Err(v)$ encloses the rounding error of computing $v$

- a bound $\epsilon$ on rounding errors is deduced from $Err(v) = [e, \bar{e}]$
  
  - $\epsilon = \max(|e|, |\bar{e}|)$
An arithmetic model for fixed-point code synthesis

An interval arithmetic based model

- For each coefficient or variable \( v \), we keep track of 2 intervals \( \text{Val}(v) \) and \( \text{Err}(v) \)
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### \( \text{Val}(v) \) is the range of \( v \)

- the format \( Q_{i,f} \) of \( v \) is deduced from
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  - \( i = \lceil \log_2 \left( \max(|v|, |\overline{v}|) \right) \rceil + \alpha \)
  - \( f = k - i \)

  \[ \alpha = \begin{cases} 
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How to propagate $\text{Val}(v)$ and $\text{Err}(v)$ for $\diamond \in \{+, -, \times, \ll, \gg, \sqrt{}, /\}$?
Fixed-point multiplication

- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1 + i_2.f_1 + f_2}$
Fixed-point multiplication

- The output format of a $\mathbb{Q}^{i_1.f_1} \times \mathbb{Q}^{i_2.f_2}$ is $\mathbb{Q}^{i_1 + i_2.f_1 + f_2}$
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- The output format of a $Q_{i_1.f_1} \times Q_{i_2.f_2}$ is $Q_{i_1+i_2.f_1+f_2}$
- But, doubling the word-length is costly

\[ \text{Err}_x = \left[ 0, 2^{-f_r} - 2^{-(f_1+f_2)} \right] \]
Fixed-point multiplication

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![](image)

$\text{Err}_x = \left[0, 2^{-f_r} - 2^{-(f_1+f_2)}\right]$  

- This multiplication is available on integer processors and DSPs

```c
int32_t mul (int32_t v1, int32_t v2){
    int64_t prod = ((int64_t) v1) * ((int64_t) v2);
    return (int32_t) (prod >> 32);
}
```
Our new fixed-point division

- The output integer part of $Q_{i_1.f_1} / Q_{i_2.f_2}$ may be as large as $i_1 + f_2$
Our new fixed-point division

- The output integer part of \( Q_{i_1.f_1} / Q_{i_2.f_2} \) may be as large as \( i_1 + f_2 \)

\[
\text{Err} / = [-2^{i_2 + f_1}, 2^{i_2 + f_1}]
\]
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- How to obtain sharper error bounds on $\text{Err}$/?

**Err$/ = [-2^{f_r}, 2^{f_r}]$**
-Sharper bound
- Risk of overflow at run-time
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- How to obtain sharper a error bounds on $\text{Err}/$?

\[\text{Err}/ = [-2^{f_r}, 2^{f_r}]\]
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How to decide of the output format of division?

- **A large integer part**
  - **✅** prevents overflow
  - **❌** loose error bounds and loss of precision

- **A small integer part**
  - **❌** may cause overflow
  - **✅** sharp error bounds and more accurate computations
The propagation rule and implementation of division

- Once the output format decided $Q_{ir,fr}$

\[
\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].
\]

\[
\text{Err}(v) = \frac{\text{Val}(v_2) \cdot \text{Err}(v_1) - \text{Val}(v_1) \cdot \text{Err}(v_2)}{\text{Val}(v_2) \cdot (\text{Val}(v_2) + \text{Err}(v_2))} + \text{Err}_/.
\]

- $\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err}_/} \cap \text{Val}(v_2)$ and $\overline{\text{Val}(v)} = [-2^{ir-1}, -2^{fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$
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```c
int32_t div (int32_t V1, int32_t V2, uint16_t eta)
{
    int64_t t1 = ((int64_t)V1) << eta;  
    int64_t V = t1 / V2;

    return (int32_t)V;
}
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Once the output format decided $Q_{ir,fr}$

$$\text{Val}(v) = \text{Range}(Q_{ir,fr}) = [-2^{ir-1}, 2^{ir-1} - 2^{fr}].$$

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$$\text{Val}(v_2) = \frac{\text{Val}(v_1)}{\text{Val}(v) + \text{Err} / \text{Val}(v_2)}$$

and

$$\text{Val}(v) = [-2^{ir-1}, -2^{-fr}] \cup [2^{-fr}, 2^{ir-1} - 2^{fr}]$$

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{
    int64_t t1 = ((int64_t)V1) << eta;
    int64_t V = t1 / V2;
    CGPE_ASSERT(((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
                 || ((V & 0xFFFFFFFF80000000ll) == 0));
    return (int32_t) V;
}
```

Additional code to check for run-time overflows
The division format trade-off: case of inverting $2 \times 2$ matrices

Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbb{Q}_{2.30}$.
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![Division output format diagram]

Maximum experimental error

Overflow rate

Division output format

Maximum error

Overflow rate

0% 20% 40% 60% 80% 100%

-10.42 0.8 4.0 8.24 0.66 0.36 0.24 0.12 0.08 0.04 0.02
The division format trade-off: case of inverting $2 \times 2$ matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $Q_{2.30}$

- Cramer’s rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} \frac{d}{\Delta} & \frac{-b}{\Delta} \\ \frac{-c}{\Delta} & \frac{a}{\Delta} \end{pmatrix}$
The division format trade-off: case of inverting $2 \times 2$ matrices

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- Cramer's rule: if $\Delta = ad - bc \neq 0$ then $A^{-1} = \begin{pmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{pmatrix}$.

![Division output format graph]
Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks
The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
  - synthesizes fixed-point code for polynomial evaluation
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1. Computation step $\leadsto$ front-end
   - computes evaluation schemes $\leadsto$ DAGs
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1. **Computation step ⇝ front-end**
   - computes evaluation schemes ⇝ DAGs

2. **Filtering step ⇝ middle-end**
   - applies the arithmetic model
   - prunes the DAGs that do not satisfy different criteria:
     - latency ⇝ scheduling filter
     - accuracy ⇝ numerical filter
     - ...

3. **Generation step ⇝ back-end**
   - generates C codes and Gappa accuracy certificates
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Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
  <coefficient name="b0" value="0x65718e3b" integer_width="-3" fraction_width="35" width="32"/>
  ...
  <variable name="y3" inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32"/>
</dotproduct>
```
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![Graph showing the comparison between original signal and filtered signals in fixed-point and binary formats.](attachment:graph.png)
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![Graph showing the comparison between the original signal and the filtered signal using fixed-point and binary formats.](image-url)

![Graph showing the certified error bound and the error of the implementations.](image-url)
An implementation of the arithmetic model: the CGPE tool

Code synthesis for an IIR filter using CGPE

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$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```c
int32_t filter(int32_t u0 /*Q5.27*/, int32_t u1 /*Q5.27*/,
                int32_t u2 /*Q5.27*/, int32_t u3 /*Q5.27*/,
                int32_t y1 /*Q6.26*/, int32_t y2 /*Q6.26*/,
                int32_t y3 /*Q6.26*/ )
{
    int32_t r0 = mul(0x4a5cdb26, y1); //Q8.24 [-2^{-24},0]
    int32_t r1 = mul(0xa6eb5908, y2); //Q7.25 [-2^{-25},0]
    int32_t r2 = mul(0x4688a637, y3); //Q5.27 [-2^{-27},0]
    int32_t r3 = mul(0x65718e3b, u0); //Q2.30 [-2^{-30},0]
    int32_t r4 = mul(0x65718e3b, u3); //Q2.30 [-2^{-30},0]
    int32_t r5 = r3 + r4; //Q2.30 [-2^{-29},0]
    int32_t r6 = r5 >> 2; //Q4.28 [-2^{-27.6781},0]
    int32_t r7 = mul(0x4c152aad, u1); //Q4.28 [-2^{-28},0]
    int32_t r8 = mul(0x4c152aad, u2); //Q4.28 [-2^{-28},0]
    int32_t r9 = r7 + r8; //Q4.28 [-2^{-27},0]
    int32_t r10 = r6 + r9; //Q4.28 [-2^{-26.2996},0]
    int32_t r11 = r10 >> 1; //Q5.27 [-2^{-25.9125},0]
    int32_t r12 = r2 + r11; //Q5.27 [-2^{-25.3561},0]
    int32_t r13 = r12 >> 2; //Q7.25 [-2^{-24.3853},0]
    int32_t r14 = r1 + r13; //Q7.25 [-2^{-23.6601},0]
    int32_t r15 = r14 >> 1; //Q8.24 [-2^{-23.1798},0]
    int32_t r16 = r0 + r15; //Q8.24 [-2^{-22.5324},0]
    int32_t r17 = r16 << 2; //Q6.26 [-2^{-22.5324},0]
    return r17;
}
```
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3. Fixed-point code synthesis for linear algebra basic blocks
A strategy to synthesize code for matrix inversion

Let $M$ be a matrix of fixed-point variables, to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute $B$ a lower triangular s.t. $M' = B \cdot B^T$
2. Generate certified code to compute $N = B^{-1}$
3. Generate certified code to compute $M'^{-1} = N^T \cdot N$
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
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The basic blocks we need to include in our tool-chain

- Certified code synthesis for Cholesky decomposition
- Certified code synthesis for triangular matrix inversion
- Certified code synthesis for matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
Linear algebra basic blocks

- Cholesky decomposition
- Triangular matrix inversion
- Matrix multiplication
## Cholesky decomposition and triangular matrix inversion

### Cholesky decomposition

\[ b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \frac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases} \]

with \( c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k} \)

### Triangular matrix inversion

\[ n_{i,j} = \begin{cases} \frac{1}{b_{i,i}} & \text{if } i = j \\ \frac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases} \]

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\]

where \( c_{i,j} = \sum_{k=j}^{i-1} b_{i,k} \cdot n_{k,j} \)

Dependencies of the coefficient \( b_{4,2} \) in the decomposition and inversion of a 6 \( \times \) 6 matrix.
FPLA (Fixed-Point Linear Algebra)

- Problem dispatcher
  - Dot-product solver
  - Matrix multiplication solver
  - Triangular matrix inversion solver
  - Cholesky decomposition solver

- User options
- Coefficients and variables
- Codes
- Certificates
Impact of the output format of division

Different functions to set the output format of division

1. $f_1(i_1, i_2) = t,$
2. $f_2(i_1, i_2) = \min(i_1, i_2) + t,$
3. $f_3(i_1, i_2) = \max(i_1, i_2) + t,$
4. $f_4(i_1, i_2) = \left\lfloor (i_1 + i_2)/2 \right\rfloor + t,$

$i_1$ and $i_2$: integer parts of the numerator and denominator and $t \in [-2, 8]$

Maximum errors with various functions used to determine the output formats of division.

(a) Cholesky $5 \times 5$.

(b) Triangular $10 \times 10$. 

Maximum errors with various functions used to determine the output formats of division.
How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \left\lfloor (i_1 + i_2) / 2 \right\rfloor + 1$ to set the output format of division

Generation time for the inversion of triangular matrices of size 4 to 40.
How fast is generating triangular matrix inversion codes?

- We use \( f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1 \) to set the output format of division.

Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

![Condition number vs Matrix size for different matrices](image)
Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate
Conclusions and perspectives

Contributions

- Formalization and implementation of an arithmetic model
  - allows certification
  - handles $\sqrt{}$ and $/$
## Conclusions and perspectives

### Contributions

- **Formalization and implementation of an arithmetic model**
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  - handles $\sqrt{}$ and $/$

- **Adaptation of the CGPE tool to the model:**
  - generates code for fine grained expressions
  - instruction selection

---

M. A. Najahi (DALI UPVD/LIRMM, UM2, CNRS)

Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks
Conclusions and perspectives

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- Integrate the matrix inversion flow
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Fixed-point code synthesis for linear algebra basic blocks

Fridge: a fixed-point design and simulation environment.

IEEE Standard for Floating-Point Arithmetic.

[MCCS02] Daniel Menard, Daniel Chillet, François Charot, and Olivier Sentieys. 
Automatic floating-point to fixed-point conversion for DSP code generation.

GUSTO: An Automatic Generation and Optimization Tool for Matrix Inversion Architectures.

Sum-of-products evaluation schemes with fixed-point arithmetic, and their application to IIR filter implementation.

Implementation of binary floating-point arithmetic on embedded integer processors - Polynomial evaluation-based algorithms and certified code generation.

Approach based on instruction selection for fast and certified code generation.


[KG08] David R. Koes and Seth C. Goldstein. 
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Automated Synthesis of Target-Dependent Programs for Polynomial Evaluation in Fixed-Point Arithmetic.

Code Size and Accuracy-Aware Synthesis of Fixed-Point Programs for Matrix Multiplication.

Evaluation of static analysis techniques for fixed-point precision optimization.

[LV09] Dong-U Lee and John D. Villasenor. 
Optimized custom precision function evaluation for embedded processors.