Synthesis of certified programs in fixed-point arithmetic, and its application to linear algebra basic blocks

Mohamed Amine Najahi

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Amine Najahi

Univ. Perpignan Via Domitia, DALI project-team
Univ. Montpellier 2, LIRMM, UMR 5506
CNRS, LIRMM, UMR 5506

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Floating-point computations

Fixed-point computations

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- ☺ Easy and fast to implement
- ☺ Easily portable [IEEE754]

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Embedded systems targets

µ-controllers

DSPs

FPGAs

$\longrightarrow$ have efficient integer instructions

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→ have efficient integer instructions

- Fixed-point arithmetic is well suited for embedded systems

But, how to make it easy, fast, and numerically safe to use by non-expert programmers?

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    - automatic float-to-fix conversion
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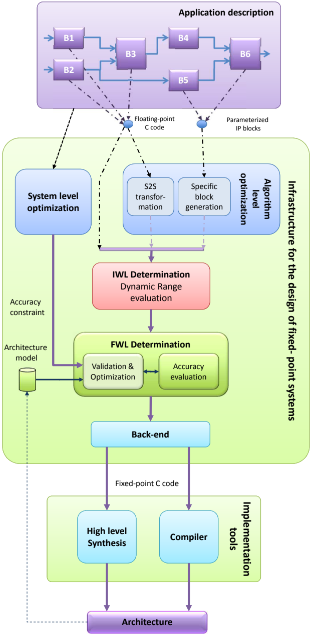

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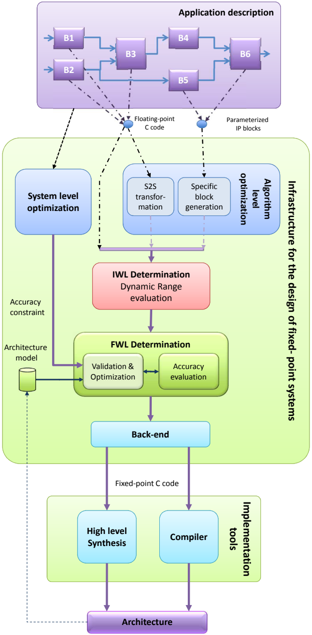
      - **Fine grained IP blocks:** dot-products, polynomials, ...
      - **High level IP blocks:** matrix multiplication, triangular matrix inversion, Cholesky decomposition



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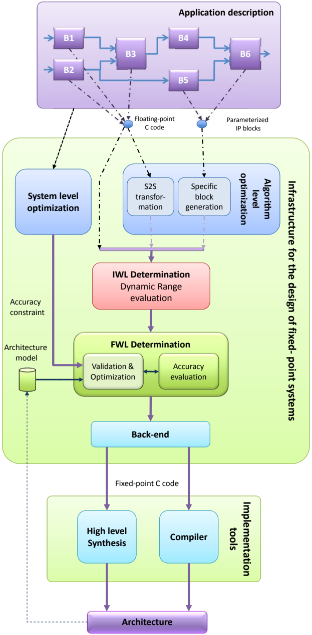
- **Long term objective:** code synthesis for matrix inversion



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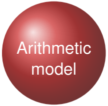

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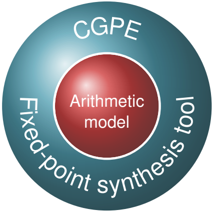

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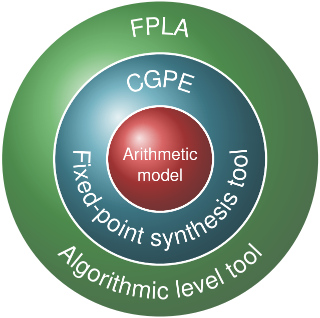
3. Build a second synthesis tool, FPLA, for algorithmic IP blocks:
    - it generates code using CGPE
    - Contributions:
        - trade-off implementations for matrix multiplication
        - code synthesis for Cholesky decomposition and triangular matrix inversion



Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

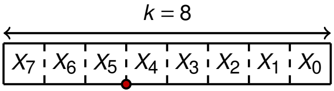
3. Fixed-point code synthesis for linear algebra basic blocks

Outline of the talk

Fixed-point arithmetic numbers

A fixed-point number x is defined by two integers:

- X the k-bit integer representation of x
- f the implicit scaling factor of x



⇝ The value of x is given by $x = \dfrac{X}{2^f} = \displaystyle\sum_{\ell=-f}^{k-1-f} X_{\ell+f} \cdot 2^{\ell}$

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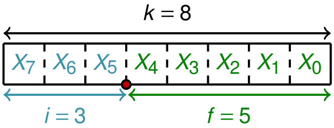

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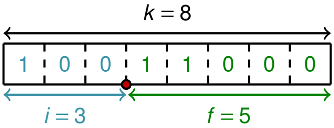

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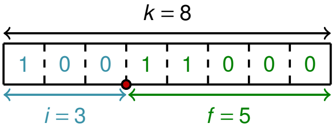

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How to compute with fixed-point numbers?

An interval arithmetic based model

- For each coefficient or variable v, we keep track of 2 intervals $\mathbf{Val}(v)$ and $\mathbf{Err}(v)$
- Our model assumes a fixed word-length k

$\mathbf{Val}(v)$ is the range of v

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- the format $\mathbf{Q}_{i.f}$ of v is deduced from $\mathbf{Val}(v) = [\underline{\mathbf{v}}, \overline{\mathbf{v}}]$
  - $i = \left\lceil \log_2 \left(\max \left(|\underline{\mathbf{v}}|, |\overline{\mathbf{v}}| \right) \right) \right\rceil + \alpha$
  - $f = k - i$

$$\alpha = \begin{cases} 1, & \text{if } \bmod \left(\log_2(\overline{\mathbf{v}}), 1 \right) \neq 0, \\ 2, & \text{otherwise} \end{cases}$$

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- a bound ϵ on rounding errors is deduced from $\mathbf{Err}(v) = [\underline{\mathbf{e}}, \overline{\mathbf{e}}]$
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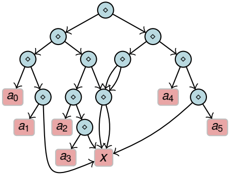

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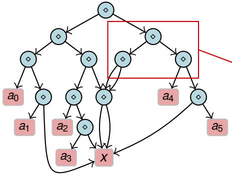
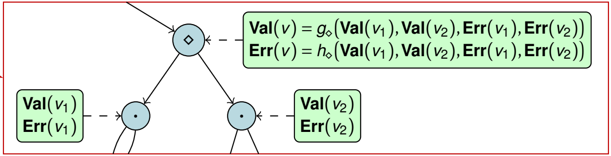
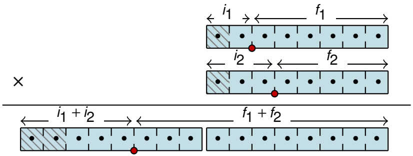




How to propagate $\mathbf{Val}(v)$ and $\mathbf{Err}(v)$ for $\diamond \in \{+, -, \times, \ll, \gg, \sqrt{\ }, /\}$?

Fixed-point multiplication

- The output format of a $\mathbf{Q}_{i_1.f_1} \times \mathbf{Q}_{i_2.f_2}$ is $\mathbf{Q}_{i_1+i_2.f_1+f_2}$



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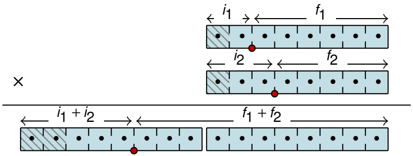

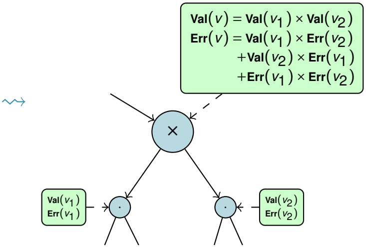

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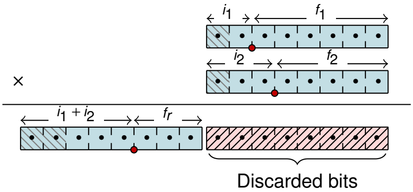

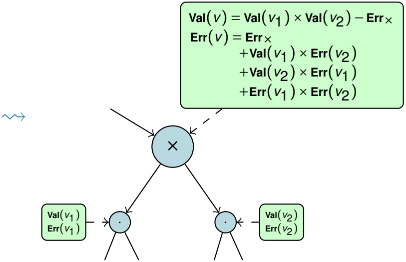

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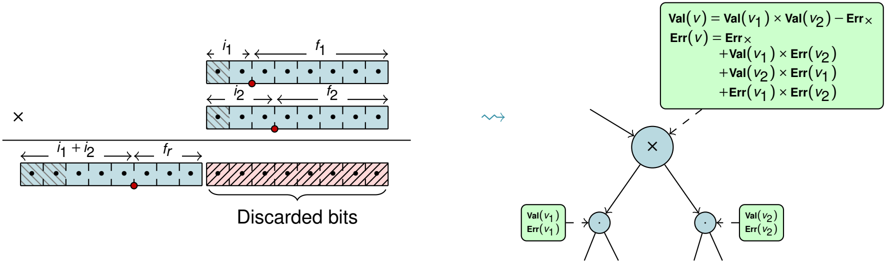
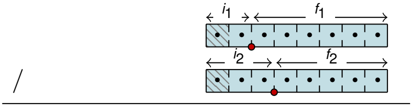


- $\mathbf{Err}_\times = \left[0, 2^{-f_r} - 2^{-(f_1+f_2)}\right]$
- This multiplication is available on integer processors and DSPs

```
int32_t mul (int32_t v1, int32_t v2){
  int64_t prod = ((int64_t) v1) * ((int64_t) v2);
  return (int32_t) (prod >> 32);
}
```

Our new fixed-point division

- The output integer part of $\mathbf{Q}_{i_1.f_1} / \mathbf{Q}_{i_2.f_2}$ may be as large as $i_1 + f_2$



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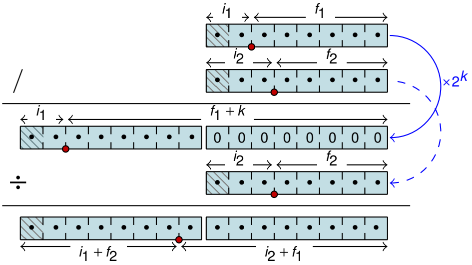

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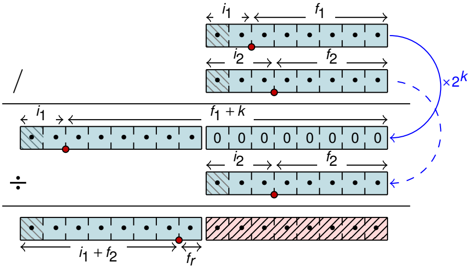

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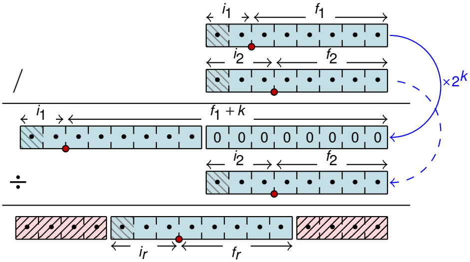
- How to obtain sharper a error bounds on $\mathbf{Err}_{/}$?



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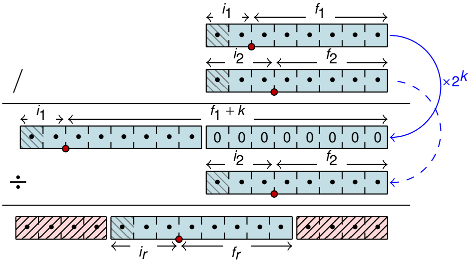

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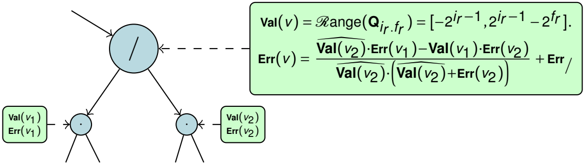
How to decide of the output format of division?

- A large integer part
  - ✓ prevents overflow
  - ✗ loose error bounds and loss of precision
- A small integer part
  - ✗ may cause overflow
  - ✓ sharp error bounds and more accurate computations

The propagation rule and implementation of division

- Once the output format decided $\mathbf{Q}_{i_r.f_r}$



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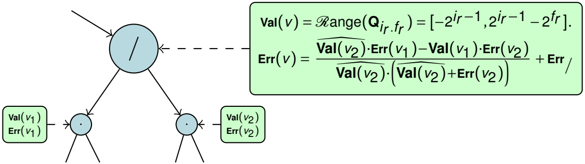

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{
  int64_t t1 = ((int64_t)V1) << eta;
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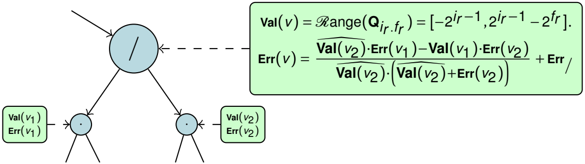

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  int64_t V  = t1 / V2;
  CGPE_ASSERT((((V & 0xFFFFFFFF80000000ll) == 0xFFFFFFFF80000000ll)
             || ((V & 0xFFFFFFFF80000000ll) == 0)));
  return (int32_t) V;
}
```

- Additional code to check for run-time overflows

The division format trade-off: case of inverting 2×2 matrices

- Consider $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in [-1, 1]$ in the format $\mathbf{Q}_{2.30}$

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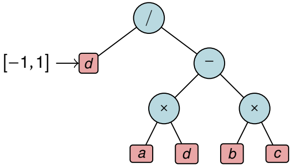

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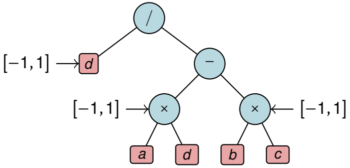

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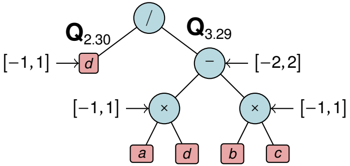

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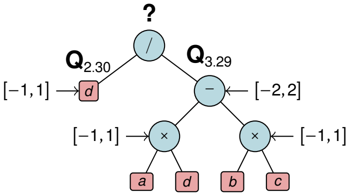

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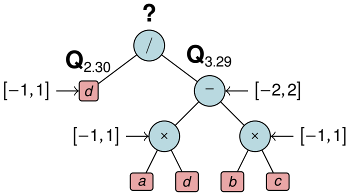

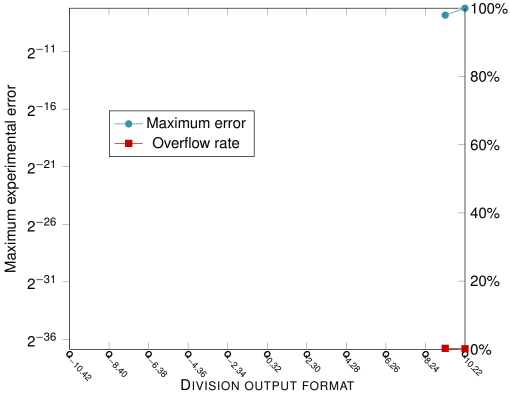

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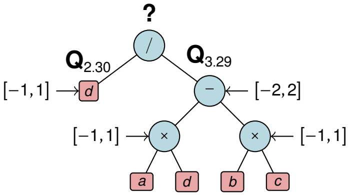

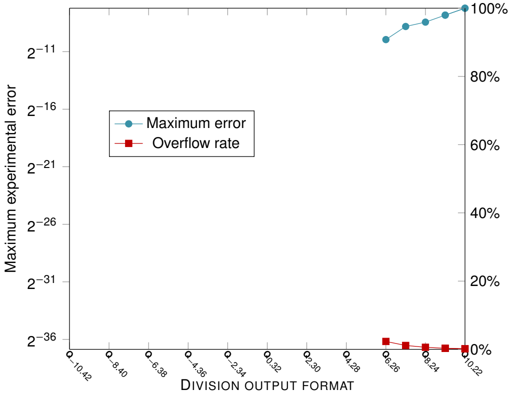

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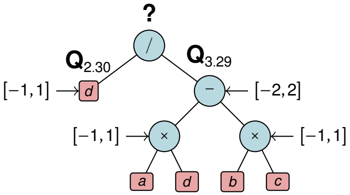

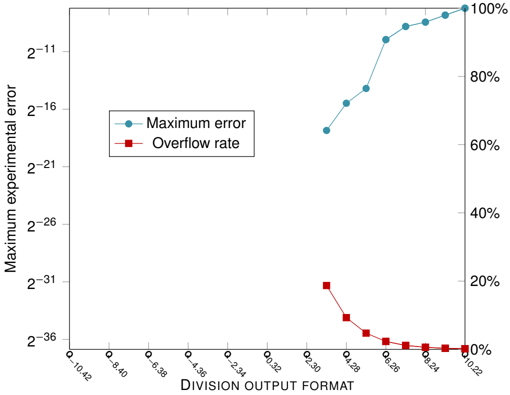

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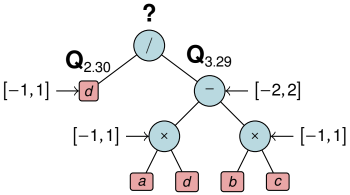

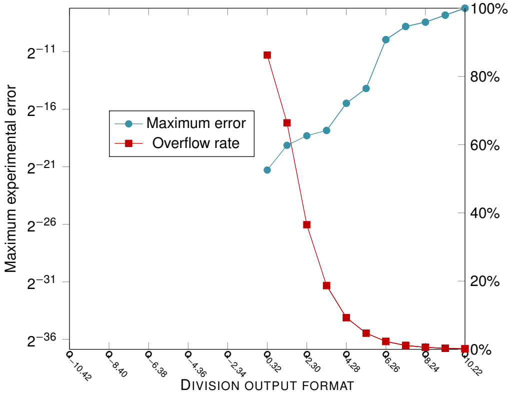

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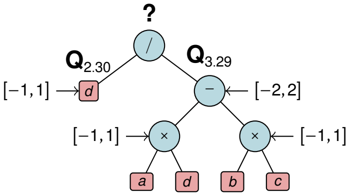

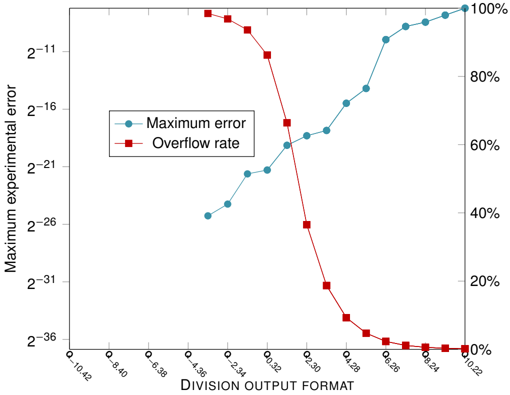

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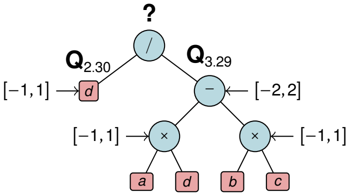

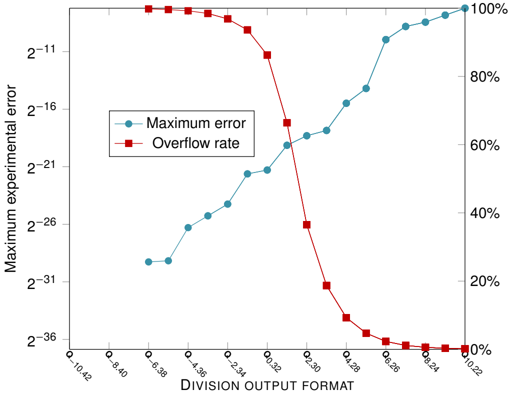

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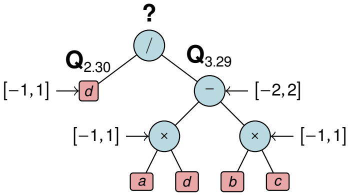

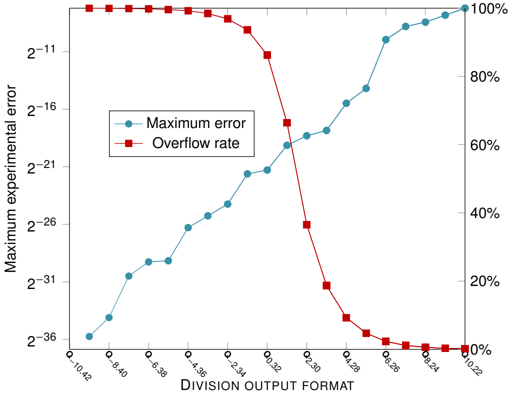

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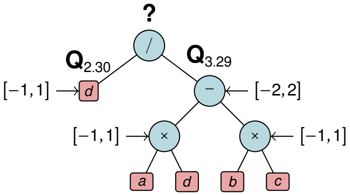

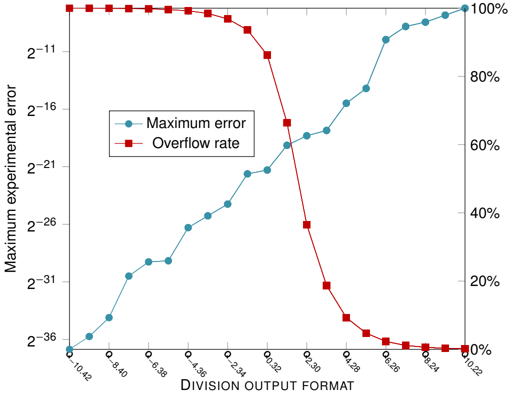

Outline of the talk

1. An arithmetic model for fixed-point code synthesis

2. An implementation of the arithmetic model: the CGPE tool

3. Fixed-point code synthesis for linear algebra basic blocks

The CGPE tool

- CGPE (*Code Generation for Polynomial Evaluation*): initiated by Revy [MR11]
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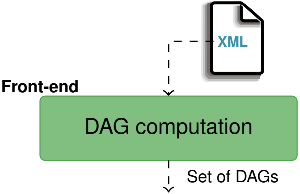

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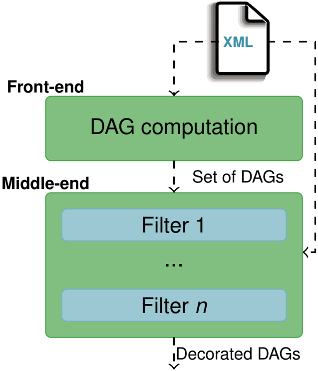

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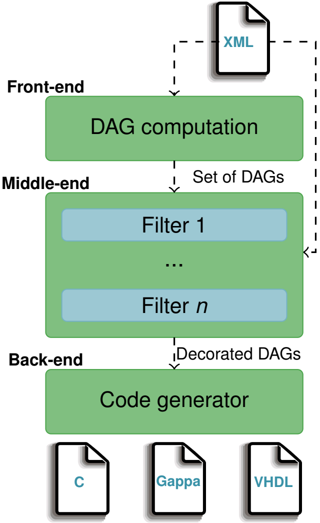

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   - generates C codes and Gappa accuracy certificates

Code synthesis for an IIR filter using CGPE

- Low-pass Butterworth filter with cutoff frequency $0.3 \cdot \pi$:

$$y[k] = \sum_{i=0}^{3} b_i \cdot u[k-i] - \sum_{i=1}^{3} a_i \cdot y[k-i]$$

```
<dotproduct inf="0xb1e91685" sup="0x4e16e97b" integer_width="6" fraction_width="26" width="32">
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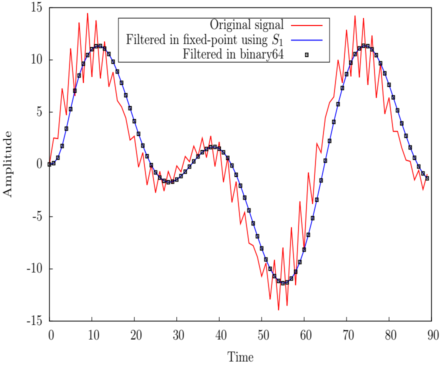

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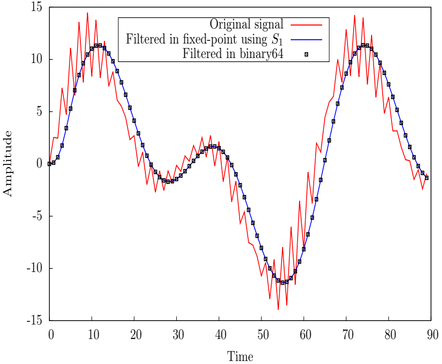

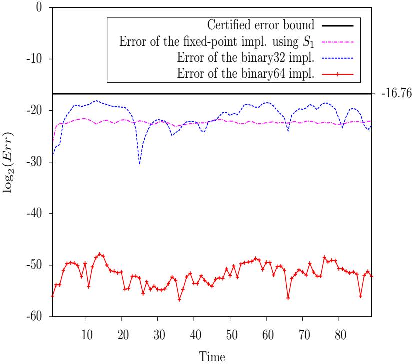

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```
int32_t filter( int32_t u0   /*Q5.27*/ , int32_t u1   /*Q5.27*/ ,
                int32_t u2   /*Q5.27*/ , int32_t u3   /*Q5.27*/ ,
                int32_t y1   /*Q6.26*/ , int32_t y2   /*Q6.26*/ ,
                int32_t y3   /*Q6.26*/ )
{                                           //Formats  Err
  int32_t r0  = mul(0x4a5cdb26, y1);        //Q8.24   [-2^{-24},0]
  int32_t r1  = mul(0xa6eb5908, y2);        //Q7.25   [-2^{-25},0]
  int32_t r2  = mul(0x4688a637, y3);        //Q5.27   [-2^{-27},0]
  int32_t r3  = mul(0x65718e3b, u0);        //Q2.30   [-2^{-30},0]
  int32_t r4  = mul(0x65718e3b, u3);        //Q2.30   [-2^{-30},0]
  int32_t r5  = r3 + r4;                    //Q2.30   [-2^{-29},0]
  int32_t r6  = r5 >> 2;                    //Q4.28   [-2^{-27.6781},0]
  int32_t r7  = mul(0x4c152aad, u1);        //Q4.28   [-2^{-28},0]
  int32_t r8  = mul(0x4c152aad, u2);        //Q4.28   [-2^{-28},0]
  int32_t r9  = r7 + r8;                    //Q4.28   [-2^{-27},0]
  int32_t r10 = r6 + r9;                    //Q4.28   [-2^{-26.2996},0]
  int32_t r11 = r10 >> 1;                   //Q5.27   [-2^{-25.9125},0]
  int32_t r12 = r2 + r11;                   //Q5.27   [-2^{-25.3561},0]
  int32_t r13 = r12 >> 2;                   //Q7.25   [-2^{-24.3853},0]
  int32_t r14 = r1 + r13;                   //Q7.25   [-2^{-23.6601},0]
  int32_t r15 = r14 >> 1;                   //Q8.24   [-2^{-23.1798},0]
  int32_t r16 = r0 + r15;                   //Q8.24   [-2^{-22.5324},0]
  int32_t r17 = r16 << 2;                   //Q6.26   [-2^{-22.5324},0]
  return r17;
}
```

Outline of the talk

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2. An implementation of the arithmetic model: the CGPE tool
3. Fixed-point code synthesis for linear algebra basic blocks

A strategy to synthesize code for matrix inversion

Let M be a matrix of fixed-point variables,

to generate certified code that inverts $M' \in M$ a symmetric positive definite, we need to:

1. Generate certified code to compute B a lower triangular s.t. $M' = B \cdot B^T$
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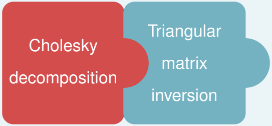

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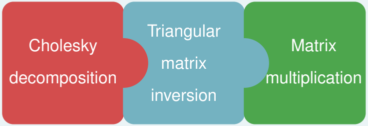
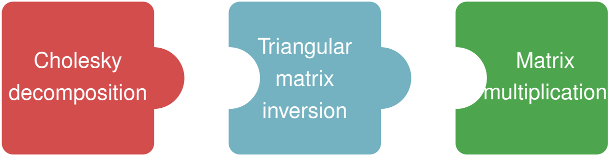
- Certified code synthesis for matrix multiplication



Linear algebra basic blocks



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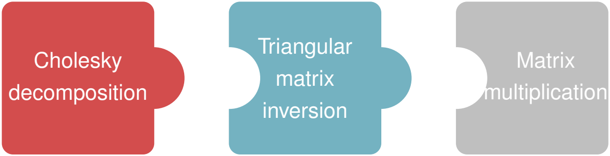

Cholesky decomposition and triangular matrix inversion

Cholesky decomposition

$$b_{i,j} = \begin{cases} \sqrt{c_{i,i}} & \text{if } i = j \\ \dfrac{c_{i,j}}{b_{j,j}} & \text{if } i \neq j \end{cases}$$

$$\text{with } c_{i,j} = m_{i,j} - \sum_{k=0}^{j-1} b_{i,k} \cdot b_{j,k}$$

Triangular matrix inversion

$$n_{i,j} = \begin{cases} \dfrac{1}{b_{i,i}} & \text{if } i = j \\ \dfrac{-c_{i,j}}{b_{i,i}} & \text{if } i \neq j \end{cases}$$

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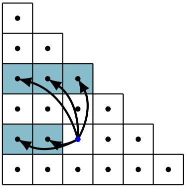
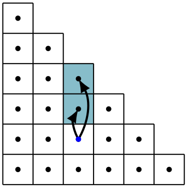
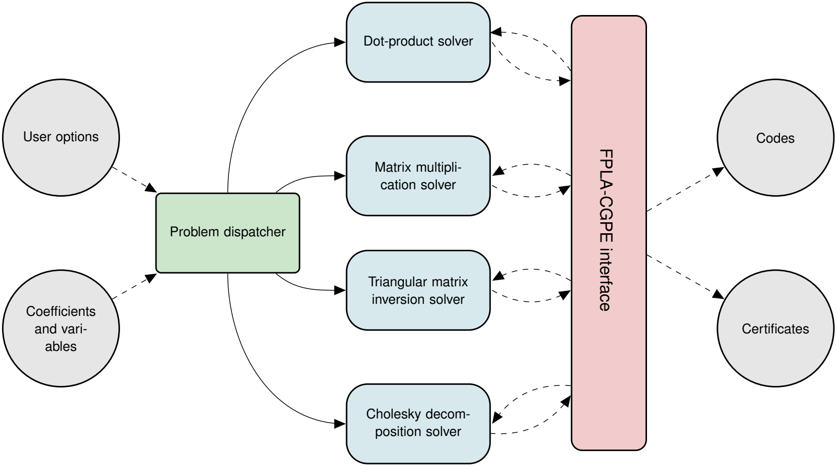




Dependencies of the coefficient $b_{4,2}$ in the decomposition and inversion of a 6×6 matrix.

FPLA (Fixed-Point Linear Algebra)

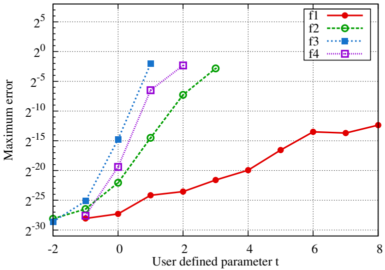


Impact of the output format of division

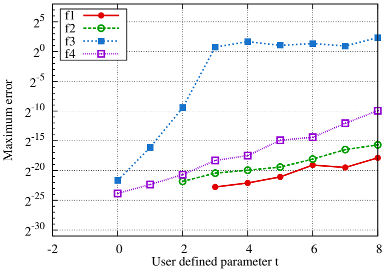
Different functions to set the output format of division

1. $f_1(i_1,i_2) = t$,
2. $f_2(i_1,i_2) = \min(i_1,i_2) + t$,
3. $f_3(i_1,i_2) = \max(i_1,i_2) + t$,
4. $f_4(i_1,i_2) = \lfloor (i_1+i_2)/2 \rfloor + t$,

i_1 and i_2: integer parts of the numerator and denominator and $t \in [-2,8]$



(a) Cholesky 5 × 5.

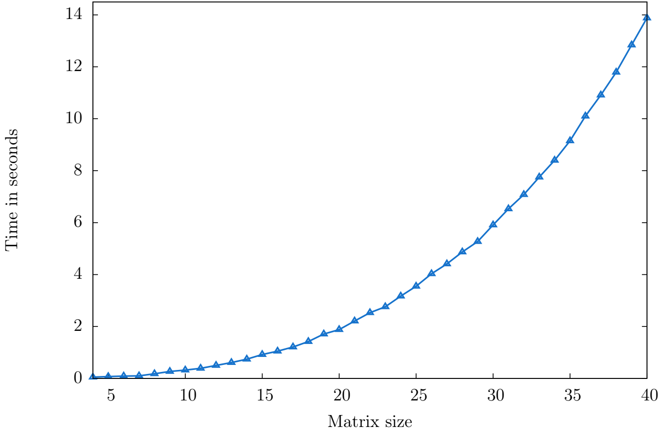


(b) Triangular 10 × 10.

Maximum errors with various functions used to determine the output formats of division.

How fast is generating triangular matrix inversion codes?

- We use $f_4(i_1, i_2) = \lfloor (i_1 + i_2)/2 \rfloor + 1$ to set the output format of division



Generation time for the inversion of triangular matrices of size 4 to 40.

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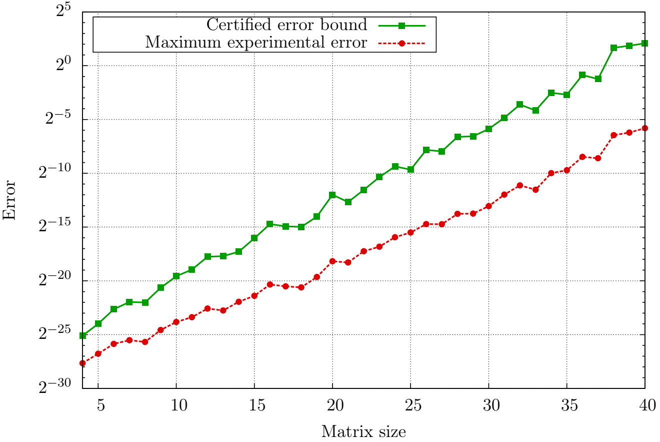
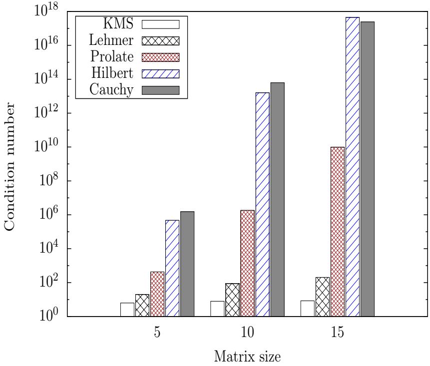


Error bounds and experimental errors for the inversion of triangular matrices of size 4 to 40.

Decomposing some well known matrices

- 2 ill-conditioned matrices: Hilbert and Cauchy
- 2 well-conditioned matrices: KMS and Lehmer



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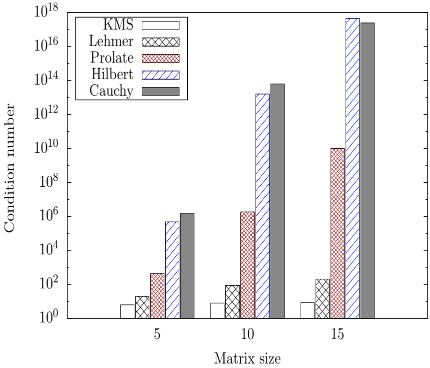

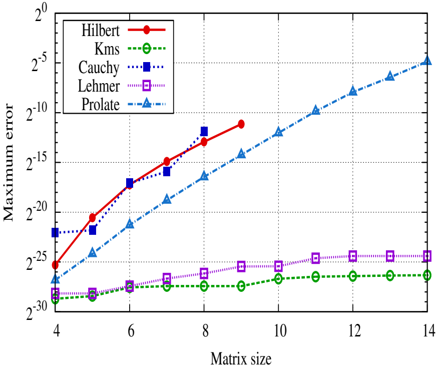

- Ill-conditioned matrices tend to overflow more often
  - similar behaviour in floating-point arithmetic
- The decompositions of KMS and Lehmer are highly accurate

Conclusions and perspectives

Contributions

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  - allows certification
  - handles $\sqrt{}$ and $/$

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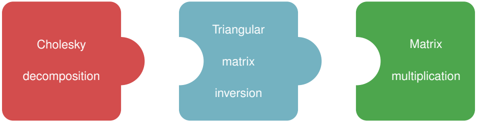
Perspectives

- Integrate the matrix inversion flow



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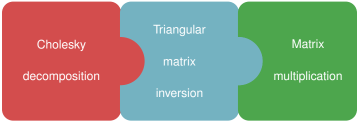

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