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# Approximability and exact resolution of the Multidimensional Binary Vector Assignment problem

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**Abstract.** In this paper we consider the multidimensional binary vector assignment problem. An input of this problem is defined by  $m$  disjoint sets  $V^1, V^2, \dots, V^m$ , each composed of  $n$  binary vectors of size  $p$ . An output is a set of  $n$  disjoint  $m$ -tuples of vectors, where each  $m$ -tuple is obtained by picking one vector from each set  $V^i$ . To each  $m$ -tuple we associate a  $p$  dimensional vector by applying the bit-wise AND operation on the  $m$  vectors of the tuple. The objective is to minimize the total number of zeros in these  $n$  vectors. We denote this problem by  $\min \sum 0$ , and the restriction of this problem where every vector has at most  $c$  zeros by  $(\min \sum 0)_{\#0 \leq c}$ .  $(\min \sum 0)_{\#0 \leq 2}$  was only known to be **APX**-complete, even for  $m = 3$  [5]. We show that, assuming the unique games conjecture, it is **NP**-hard to  $(n - \varepsilon)$ -approximate  $(\min \sum 0)_{\#0 \leq 1}$  for any fixed  $n$  and  $\varepsilon$ . This result is tight as any solution is a  $n$ -approximation. We also prove without assuming UGC that  $(\min \sum 0)_{\#0 \leq 1}$  is **APX**-complete even for  $n = 2$ , and we provide an example of  $n - f(n, m)$ -approximation algorithm for  $\min \sum 0$ . Finally, we show that  $(\min \sum 0)_{\#0 \leq 1}$  is polynomial-time solvable for fixed  $m$  (which cannot be extended to  $(\min \sum 0)_{\#0 \leq 2}$  according to [5]).

## 1 Introduction

### 1.1 Problem definition

In this paper we consider the multidimensional binary vector assignment problem denoted by  $\min \sum 0$ . An input of this problem (see Figure 1) is described by  $m$  disjoint sets  $V^1, \dots, V^m$ , each set  $V^i$  containing  $n$  binary  $p$ -dimensional vectors. For any  $j \in [n]^1$ , and any  $i \in [m]$ , the  $j^{\text{th}}$  vector of set  $V^i$  is denoted  $v_j^i$ , and for any  $k \in [p]$ , the  $k^{\text{th}}$  coordinate of  $v_j^i$  is denoted  $v_j^i[k]$ .

The output of the problem consists in a set  $S$  of  $n$  disjoint stacks. A stack  $s = (v_1^s, \dots, v_m^s)$  is an  $m$ -tuple of vectors such that  $v_i^s \in V^i$ , for any  $i \in [m]$ . Two stacks  $s_1$  and  $s_2$  are disjoint if and only if no vector belongs to  $s_1$  and  $s_2$ .

We now introduce the operator  $\wedge$  which assigns to a pair of vectors  $(u, v)$  the vector given by  $u \wedge v = (u[1] \wedge v[1], u[2] \wedge v[2], \dots, u[p] \wedge v[p])$ . We associate to each stack  $s$  a unique vector given by  $v_s = \bigwedge_{i \in [m]} v_i^s$ .

<sup>1</sup> Note that  $[n]$  stands for  $\{1, 2, \dots, n\}$ .

The cost of a vector  $v$  is defined as the number of zeros in it. More formally if  $v$  is  $p$ -dimensional,  $c(v) = p - \sum_{k \in [p]} v[k]$ . We extend this definition to a set of stacks  $S = \{s_1, \dots, s_n\}$  as follows :  $c(S) = \sum_{s \in S} c(v_s)$ .

The objective is then to find a set  $S$  of  $n$  disjoint stacks minimizing the total number of zeros. This leads us to the following definition of the problem:

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**Optimization Problem 1**  $\min \sum 0$

**Input**  $m$  sets of  $n$   $p$ -dimensional binary vectors.

**Output** A set  $S$  of  $n$  disjoint stacks minimizing  $c(S)$ .

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Throughout this paper, we denote  $(\min \sum 0)_{\#0 \leq c}$  the restriction of  $\min \sum 0$  where the number of zeros per vector is upper bounded by  $c$ .

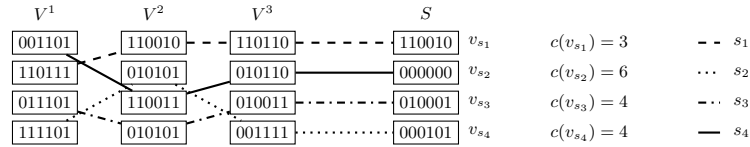


Fig. 1: Example of  $\min \sum 0$  instance with  $m = 3, n = 4, p = 6$  and of a feasible solution  $S$  of cost  $c(S) = 17$ .

## 1.2 Related work

The dual version of the problem called  $\max \sum 1$  (where the objective is to maximize the total number of 1 in the created stacks) has been introduced by Reda et al. in [8] as the “yield maximization problem in Wafer-to-Wafer 3-D Integration technology”. They prove the **NP**-completeness of  $\max \sum 1$  and provide heuristics without approximation guarantee. In [6] we proved that, even for  $n = 2$ , for any  $\varepsilon > 0$ ,  $\max \sum 1$  is  $\mathcal{O}(m^{1-\varepsilon})$  and  $\mathcal{O}(p^{1-\varepsilon})$  inapproximable unless **P** = **NP**. We also provide an ILP formulation proving that  $\max \sum 1$  (and thus  $\min \sum 0$ ) is **FPT**<sup>2</sup> when parameterized by  $p$ .

We introduced  $\min \sum 0$  in [4] where we provide in particular  $\frac{4}{3}$ -approximation algorithm for  $m = 3$ . In [5], authors focus on a generalization of  $\min \sum 0$ , called MULTI DIMENSIONAL VECTOR ASSIGNMENT, where vectors are not necessary binary vectors. They extend the approximation algorithm of [4] to get a  $f(m)$ -approximation algorithm for arbitrary  $m$ . They also prove the **APX**-completeness of the  $(\min \sum 0)_{\#0 \leq 2}$  for  $m = 3$ . This result was the only known inapproximability result for  $\min \sum 0$ .

## 1.3 Contribution

In section 2 we study the approximability of  $\min \sum 0$ . Our main result in this section is to prove that assuming UGC, it is **NP**-hard to  $(n - \varepsilon)$ -approximate

<sup>2</sup> *i.e.* admits an algorithm in  $f(p)poly(|I|)$  for an arbitrary function  $f$ .

$(\min \sum 0)_{\#0 \leq 1}$  (and thus  $\min \sum 0$ ) for any fixed  $n \geq 2$ ,  $\forall \varepsilon > 0$ . This result is tight as any solution is a  $n$ -approximation.

Notice that this improves the only existing negative result for  $\min \sum 0$ , which was the **APX**-hardness of [5] (implying only no-**PTAS**).

We also show how this reduction can be used to obtain the **APX**-hardness for  $(\min \sum 0)_{\#0 \leq 1}$  for  $n = 2$  unless  $\mathbf{P} = \mathbf{NP}$ , which is weaker negative result, but does not require UGC. We then give an example  $n - f(n, m)$  approximation algorithm for the general problem  $\min \sum 0$ .

In section 3, we consider the exact resolution of  $\min \sum 0$  (and  $\max \sum 1$ ). We only focus on what we will call sparse instances, *i.e.* instances of  $(\min \sum 0)_{\#0 \leq 1}$ . Indeed, recall that authors of [5] show that  $(\min \sum 0)_{\#0 \leq 2}$  is **APX**-complete even for  $m = 3$ , implying that  $(\min \sum 0)_{\#0 \leq 2}$  cannot be polynomial-time solvable for fixed  $m$  unless  $\mathbf{P} = \mathbf{NP}$ . Thus, it was natural to ask if  $(\min \sum 0)_{\#0 \leq 1}$  was polynomial-time solvable for fixed  $m$ . Section 3 is devoted to answer positively to this question. Notice that the question of determining if  $(\min \sum 0)_{\#0 \leq 1}$  is **FPT** when parameterized by  $m$  remains open. Due to space constraints, results marked with a  $\star$  are proved in the appendix.

## 2 Approximability of $\min \sum 0$

Let us first recall definitions of reductions we use in this paper.

### 2.1 Definitions

**L-reduction** The  $L$ -reduction has been introduced by Papadimitriou et al. in [7] as follows:

**Definition 1.** Let  $\Pi_1$  and  $\Pi_2$  be two optimization problems with objective functions  $m_1$  and  $m_2$ . Let  $f$  be a polynomial-time computable function that given any instance  $x$  of  $\Pi_1$  associates an instance  $f(x)$  of  $\Pi_2$ . Let  $g$  be another polynomial-time computable function that given any instance  $x$  of  $\Pi_1$ , and feasible solution  $S$  of  $f(x)$ , associates a feasible solution  $g(x, S)$  of  $\Pi_1$ . If  $f$  and  $g$  verify the two following conditions:

1.  $\exists \alpha$  such that  $\text{Opt}(f(x)) \leq \alpha \text{Opt}(x)$
2.  $\exists \beta$  such that for each solution  $S$  of  $\Pi_2$ ,  $|\text{Opt}(x) - m_1(g(x, S))| \leq \beta |\text{Opt}(f(x)) - m_2(S)|$

then  $(f, g)$  is an  $L$ -reduction.

In following,  $\Pi_1$   $L$ -reduces to  $\Pi_2$  is noted  $\Pi_1 <_L \Pi_2$ .

**Gap reduction** We briefly recall the definition of such a reduction, as presented in [2] by Ausiello et al.

**Definition 2.** Let  $\Pi_{dec}$  be a decision problem and  $\Pi_{opt}$  a minimization problem. Let  $f$  be a polynomial-time computable function that given any instance  $x$  of  $\Pi_{dec}$  associates an instance  $f(x)$  of  $\Pi_{opt}$ . If there exists two function  $a$  and  $r$  such that:

1.  $x$  is a YES-instance  $\Rightarrow \text{Opt}(f(x)) \leq a(x)$
2.  $x$  is a NO-instance  $\Rightarrow \text{Opt}(f(x)) \geq r(x)a(x)$

then  $f$  is a  $r(x)$ -Gap reduction.

## 2.2 Inapproximability results for $(\min \sum 0)_{\#0 \leq 1}$

From now we suppose that  $\forall k \in [p], \exists i, \exists j$  such that  $v_j^i[k] = 0$ . In other words, for any solution  $S$  and  $\forall k$ , there exists a stack  $s$  such that  $v_s[k] = 0$ . Otherwise, we simply remove such a coordinate from every vector of every set, and decrease  $p$  by one. Since this coordinate would be set to 1 in all the stacks of all solutions, such a preprocessing preserves approximation ratios and exact results.

In a first time, we define the following polynomial-time computable function  $f$  which associates an instance of  $(\min \sum 0)_{\#0 \leq 1}$  to any  $k$ -uniform hypergraph, *i.e.* an hypergraph  $G = (U, E)$  such that every hyperedges of  $E$  contains exactly  $k$  distinct elements of  $U$ .

**Definition of  $f$**  We consider a  $k$ -uniform hypergraph  $G = (U, E)$ . We call  $f$  the polynomial-time computable function that creates an instance of  $(\min \sum 0)_{\#0 \leq 1}$  from a  $G$  as follows.

1. We set  $m = |E|$ ,  $n = k$  and  $p = |U|$ .
2. For each hyperedge  $e = \{u_1, u_2, \dots, u_k\} \in E$ , we create the set  $V^e$  containing  $k$  vectors  $\{v_j^e, j \in [k]\}$ , where for all  $j \in [k]$ ,  $v_j^e[u_j] = 0$  and  $v_j^e[l] = 1$  for  $l \neq u_j$ . We say that a vector  $v$  **represents**  $u \in U$  iff  $v[u] = 0$  and  $v[l \neq u] = 1$  (and thus vector  $v_j^e$  represents  $u_j$ ).

An example of this construction is given in Figure 2.

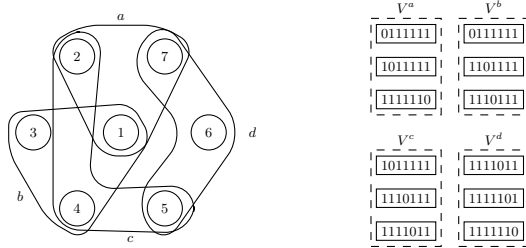


Fig. 2: Illustration of the reduction from an hypergraph  $G = (U = \{1, 2, 3, 4, 5, 6, 7\}, E = \{\{1, 2, 7\}, \{1, 3, 4\}, \{2, 4, 5\}, \{5, 6, 7\}\})$  to an instance  $(\min \sum 0)_{\#0 \leq 1}$

**Negative results assuming UGC** We consider the following problem. Notice that what we call a vertex cover in a  $k$ -regular hypergraph  $G = (U, E)$  is a set  $U' \subseteq U$  such that for any hyperedge  $e \in E$ ,  $U' \cap e \neq \emptyset$ .

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**Decision Problem 1** ALMOST  $Ek$  VERTEX COVER

**Input** We are given an integer  $k \geq 2$ , two arbitrary positive constants  $\varepsilon$  and  $\delta$  and a  $k$ -uniform hypergraph  $G = (U, E)$ .

**Output** Distinguish between the following cases:

**YES Case** there exist  $k$  disjoint subsets  $U^1, U^2, \dots, U^k \subseteq U$ , satisfying  $|U^i| \geq \frac{1-\varepsilon}{k}|U|$  and such that every hyperedge contains at most one vertex from each  $U^i$ .

**NO Case** every vertex cover has size at least  $(1 - \delta)|U|$ .

---

It is shown in [3] that, assuming UGC, this problem is **NP**-complete.

**Theorem 1.** For any fixed  $n \geq 2$ , for any constants  $\varepsilon, \delta > 0$ , there exists a  $\frac{n-n\delta}{1+n\varepsilon}$ -Gap reduction from ALMOST  $Ek$  VERTEX COVER to  $(\min \sum 0)_{\#0 \leq 1}$ . Consequently, under UGC, for any fixed  $n$   $(\min \sum 0)_{\#0 \leq 1}$  is **NP**-hard to approximate within a factor  $(n - \varepsilon')$  for any  $\varepsilon' > 0$ .

*Proof.* We consider an instance  $I$  of ALMOST  $Ek$  VERTEX COVER defined by two positive constants  $\delta$  and  $\varepsilon$ , an integer  $k$  and a  $k$ -regular hypergraph  $G = (U, E)$ .

We use the function  $f$  previously defined to construct an instance  $f(I)$  of  $\min \sum 0$ . Let us now prove that if  $I$  is a positive instance,  $f(I)$  admits a solution  $S$  of cost  $c(S) < (1 + n\varepsilon)|U|$ , and otherwise any solution  $S$  of  $f(I)$  has cost  $c(S) \geq n(1 - \delta)|U|$ .

**NO Case** Let  $S$  be a solution of  $f(I)$ . Let us first remark that for any stack  $s \in S$ , the set  $\{k : v_s[k] = 0\}$  defines a vertex cover in  $G$ . Indeed,  $s$  contains exactly one vector per set, and thus by construction  $s$  selects one vertex per hyperedge in  $G$ . Remark also that the cost of  $s$  is equal to the size of the corresponding vertex cover.

Now, suppose that  $I$  is a negative instance. Hence each vertex cover has a size at least equal to  $(1 - \delta)|U|$ , and any solution  $S$  of  $f(I)$ , composed of exactly  $n$  stacks, verifies  $c(S) \geq n(1 - \delta)|U|$ .

**YES Case** If  $I$  is a positive instance, there exists  $k$  disjoint sets  $U^1, U^2, \dots, U^k \subseteq U$  such that  $\forall i = 1, \dots, k, |U^i| \geq \frac{1-\varepsilon}{k}|U|$  and such that every hyperedge contains at most one vertex from each  $U^i$ .

We introduce the subset  $X = U \setminus \bigcup_{i=1}^k U^i$ . By definition  $\{U^1, U^2, \dots, U^k, X\}$  is a partition of  $U$  and  $|X| \leq \varepsilon|U|$ . Furthermore,  $U^i \cup X$  is a vertex cover  $\forall i = 1, \dots, k$ . Indeed, each hyperedge  $e \in E$  that contains no vertex of  $U^i$ , contains at least one vertex of  $X$  since  $e$  contains  $k$  vertices.

We now construct a solution  $S$  of  $f(I)$ . Our objective is to construct stacks  $\{s_i\}$  such that for any  $i$ , the zeros of  $s_i$  are included in  $U^i \cup X$  (i.e.  $\{l : v_{s_i}[l] = 0\} \subseteq U^i \cup X$ ). For each  $e = \{u_1, \dots, u_k\} \in E$ , we show how to assign exactly one vector of  $V^e$  to each stack  $s_1, \dots, s_k$ . For all  $i \in [k]$ , if  $v_j^e$  represents a vertex  $u$  with  $u \in U^i$ , then we assign  $v_j^e$  to  $s_i$ . W.l.o.g., let

$S'_e = \{s_1, \dots, s_{k'}\}$  (for  $k' \leq k$ ) be the set of stacks that received a vertex during this process. Notice that as every hyperedge contains at most one vertex from each  $U^i$ , we only assigned one vector to each stack of  $S'_e$ . After this, every unassigned vector  $v \in V^e$  represents a vertex of  $X$  (otherwise, such a vector  $v$  would belong to a set  $U^i$ ,  $i \in k'$ , a contradiction). We assign arbitrarily these vectors to the remaining stacks that are not in  $S'_e$ . As by construction  $\forall i \in [k]$ ,  $v_s i$  contains only vectors representing vertices from  $U^i \cup X$ , we get  $c(s_i) \leq |U^i| + |X|$ .

Thus, we obtain a feasible solution  $S$  of cost  $c(S) = \sum_{i=1}^k c(s_i) \leq k|X| + \sum_{i=1}^k |U^i|$ . As by definition we have  $|X| + \sum_{i=1}^k |U^i| = |U|$ , it follows that  $c(S) \leq |U| + (k-1)|U|$  and since  $k = n$ ,  $c(S) < |U|(1+n\varepsilon)$ .

If we define  $a(n) = (1+n\varepsilon)|U|$  and  $r(n) = \frac{n(1-\delta)}{(1+n\varepsilon)}$ , the previous reduction is a  $r(n)$ -Gap reduction. Furthermore,  $\lim_{\delta, \varepsilon \rightarrow 0} r(n) = n$ , thus it is **NP**-hard to approximate  $(\min \sum 0)_{\#0 \leq 1}$  within a ratio  $(n - \varepsilon')$  for any  $\varepsilon' > 0$ .  $\square$

Notice that, as a function of  $n$ , this inapproximability result is optimal. Indeed, we observe that any feasible solution  $S$  is an  $n$ -approximation as, for any instance  $I$  of  $\min \sum 0^3$ ,  $Opt(I) \geq p$  and for any solution  $S$ ,  $c(S) \leq pn$ .

**Negative results without assuming UGC** Let us now study the negative results we can get when only assuming  $\mathbf{P} \neq \mathbf{NP}$ . Our objective is to prove that  $(\min \sum 0)_{\#0 \leq 1}$  is **APX**-hard, even for  $n = 2$ . To do so, we present a reduction from **ODD CYCLE TRANSVERSAL**, which is defined as follows. Given an input graph  $G = (U, E)$ , the objective is to find an odd cycle transversal of minimum size, *i.e.* a subset  $T \subseteq U$  of minimum size such that  $G[U \setminus T]$  is bipartite.

For any integer  $\gamma \geq 2$ , we denote  $\mathcal{G}_\gamma$  the class of graphs  $G = (U, E)$  such that any optimal odd cycle transversal  $T$  has size  $|T| \geq \frac{|U|}{\gamma}$ . Given  $\mathcal{G}$  a class of graphs, we denote  $OCT_{\mathcal{G}}$  the **ODD CYCLE TRANSVERSAL** problem restricted to  $\mathcal{G}$ .

**Lemma 1.** *For any constant  $\gamma \geq 2$ , there exists an L-reduction from  $OCT_{\mathcal{G}_\gamma}$  to  $(\min \sum 0)_{\#0 \leq 1}$  with  $n = 2$ .*

*Proof.* Let us consider an integer  $\gamma$ , an instance  $I$  of  $OCT_{\mathcal{G}_\gamma}$ , defined by a graph  $G = (V, E)$  such that  $G \in \mathcal{G}_\gamma$ . W.l.o.g., we can consider that  $G$  contains no isolated vertex.

Remark that any graph can be seen as a 2-uniform hypergraph. Thus, we use the function  $f$  previously defined to construct an instance  $f(I)$  of  $(\min \sum 0)_{\#0 \leq 1}$  such that  $n = 2$ . Since,  $G$  contains no isolated vertex,  $f(I)$  contains no position  $k$  such that  $\forall i \in [m]$ ,  $\forall j \in [n]$ ,  $v_j^i[k] = 1$ .

Let us now prove that  $I$  admits an odd cycle transversal of size  $t$  if and only if  $f(I)$  admits a solution of cost  $p + t$ .

<sup>3</sup> Recall that we assume  $\forall k \in [p], \exists i, \exists j$  such that  $v_j^i[k] = 0$

$\Leftarrow$  We consider an instance  $f(I)$  of  $(\min \sum 0)_{\#0 \leq 1}$  with  $n = 2$  admitting a solution  $S = \{s_A, s_B\}$  with cost  $c(S) = p + t$ . Let us specify a function  $g$  which produces from  $S$  a solution  $T = g(I, S)$  of  $OCT_{\mathcal{G}_\gamma}$ , *i.e.* a set of vertices of  $U$  such that  $G[U \setminus T]$  is bipartite.

We define  $T = \{u \in U : v_{s_A}[u] = v_{s_B}[u] = 0\}$ , the set of coordinates equal to zero in both  $s_A$  and  $s_B$ . We also define  $A = \{u \in V : v_{s_A}[u] = 0 \text{ and } v_{s_B}[u] = 1\}$  (resp.  $B = \{u \in V : v_{s_B}[u] = 0 \text{ and } v_{s_A}[u] = 1\}$ ), the set of coordinates set to zero only in  $s_A$  (resp.  $s_B$ ). Notice that  $\{T, A, B\}$  is a partition of  $U$ .

Remark that  $A$  and  $B$  are independent sets. Indeed, suppose that  $\exists \{u, v\} \in E$  such that  $u, v \in A$ . As  $\{u, v\} \in E$  there exists a set  $V^{(u,v)}$  containing a vector that represents  $u$  and another vector that represents  $v$ , and thus these vectors are assigned to different stacks. This leads to a contradiction. It follows that  $G[U \setminus T]$  is bipartite and  $T$  is an odd cycle transversal.

Since  $c(S) = |A| + |B| + 2|T| = p + |T| = p + t$ , we get  $|T| = t$ .

$\Rightarrow$  We consider an instance  $I$  of  $OCT_{\mathcal{G}_\gamma}$  and a solution  $T$  of size  $t$ . We now construct a solution  $S = \{s_A, s_B\}$  of  $f(I)$  from  $T$ .

By definition,  $G[U \setminus T]$  is a bipartite graph, thus the vertices in  $U \setminus T$  may be split into two disjoint independent sets  $A$  and  $B$ . For each edge  $e \in E$ , the following cases can occur:

- if  $\exists u \in e$  such that  $u \in A$ , then the vector corresponding to  $u$  is assigned to  $s_A$ , and the vector corresponding to  $e \setminus \{u\}$  is assigned to  $s_B$  (and the same rule holds by exchanging  $A$  and  $B$ )
- otherwise,  $u$  and  $v \in T$ , and we assign arbitrarily  $v_u^e$  to  $s_A$  and the other to  $s_B$ .

We claim that the stacks  $s_A$  and  $s_B$  describe a feasible solution  $S$  of cost at most  $p + t$ .

Since, for each set, only one vector is assigned to  $s_A$  and the other to  $s_B$ , the two stacks  $s_A$  and  $s_B$  are disjoint and contain exactly  $m$  vectors.  $S$  is therefore a feasible solution.

Remark that  $v_{s_A}$  (resp.  $v_{s_B}$ ) contains only vectors  $v$  such that  $v[k] = 0 \implies k \in A \cup T$  (resp.  $k \in B \cup T$ ), and thus  $c(v_A) \leq |A| + |T|$  (resp.  $c(v_B) \leq |B| + |T|$ ). Hence  $c(S) \leq |A| + |B| + 2|T| = p + t$ .

Let us now prove that this reduction is an  $L$ -reduction.

1. By definition, any instance  $I$  of  $OCT_{\mathcal{G}_\gamma}$  verifies  $|Opt(I)| \geq |U|/\gamma$ . Thus,

$$Opt(f(I)) \leq |U| + Opt(I) \leq (\gamma + 1)Opt(I)$$

2. We consider an arbitrary instance  $I$  of  $OCT_{\mathcal{G}_\gamma}$ ,  $f(I)$  the corresponding instance of  $(\min \sum 0)_{\#0 \leq 1}$ ,  $S$  a solution of  $f(I)$  and  $T = g(I, S)$  the corresponding solution of  $I$ .

We proved  $|T| - Opt(I) = c(S) - |U| - (Opt(f(I)) - |U|) = c(S) - Opt(f(I))$ .

Therefore, we get an  $L$ -reduction for  $\alpha = \gamma + 1$  and  $\beta = 1$ . □



**Lemma 2.** *There exist a constant  $\gamma$  and  $\mathcal{G} \subset \mathcal{G}_\gamma$  such that  $OCT_{\mathcal{G}}$  is **APX-hard**.*

*Proof.* We present an  $L$ -reduction from VC-3, the vertex cover problem in graph with maximum degree 3, to  $OCT_{\mathcal{G}_{VC}}$  for an appropriate  $\mathcal{G}_{VC}$ . VC-3 is known to be **APX-complete** [1].

Given an instance  $G = (U, E)$  of VC-3, we construct an instance  $f(G) = (U', E')$  as follows:

1. For each  $(u, v) \in E$ , create a vertex  $z_{u,v}$ . These  $z$ -vertices form the set  $Z$ .
2.  $U' = U \cup Z$ .
3.  $E' = E \cup \{(u, z_{u,v}), (v, z_{u,v}) : (u, v) \in E\}$ . In other words, for each  $(u, v) \in E$ , we create the triangle  $\{u, v, z_{u,v}\}$ .

Let us prove that  $G = (U, E)$  admits a solution  $VC$  of size  $|VC| = t$  if and only if  $f(G)$  admits a solution  $T$  of size  $|T| = t$ .

$\Rightarrow$  Consider a vertex cover  $VC$  of size  $|VC| = t$ , for each  $u \in VC$ , we add the vertex  $u'$  to  $T$ . By definition,  $VC$  covers all the edges of  $G$  and then all its (odd) cycles. Furthermore, it also covers all the created triangles in  $f(G)$  since each of these cycles contains exactly one edge in common with  $f(G)[U' \setminus Z]$ . Thus  $T$  is an odd cycle transversal and  $|T| = |VC|$ .

$\Leftarrow$  Let us construct a function  $g$  that, given any solution  $T$  of  $f(G)$ , computes a solution  $VC = g(G, T)$  of  $G$ . Notice first that we can suppose that  $T$  contains no  $z$ -vertex. Otherwise every triangle  $\{u, v, z_{u,v}\}$  covered by a  $z_{u,v} \in T$ , can instead be covered by either  $u$  or  $v$  without increasing the size of  $T$ . Thus, we set  $VC = T$ .

By definition of an odd cycle transversal,  $T$  covers all the odd cycles of  $f(G)$  and especially the created triangles. Thus, the triangle  $\{u, v, z_{u,v}\}$  corresponding to any edge  $\{u, v\} \in E$  is covered by  $VC$ . As  $VC \cap Z = \emptyset$ ,  $VC$  is a vertex cover of  $G$ .

The previous reduction is an  $L$ -reduction for  $\alpha = \beta = 1$ . Let us call  $\mathcal{G}_{VC}$  the class of graph generated in this reduction. The previous reduction shows that  $OCT_{\mathcal{G}_{VC}}$  is **APX-hard**. It remains to check that  $\mathcal{G}_{VC} \subseteq \mathcal{G}_\gamma$  for a constant  $\gamma$ .

Remark that VC-3 is only defined on 3-regular graphs, it implies that for any instance  $G = (U, E)$  of VC-3,  $Opt(G) \geq \frac{|U|}{3}$ . As  $|U'| = |U| + |E| \leq \frac{5|U|}{2}$ , it follows that  $Opt(f(G)) = Opt(G) \geq \frac{|U|}{3} \geq \frac{2|U'|}{15}$ . Hence,  $\mathcal{G}_{VC} \subset \mathcal{G}_\gamma$  with  $\gamma = \frac{15}{2}$ .  $\square$

The following result is now immediate.

**Theorem 2.**  $(\min \sum_{\#0 \leq 1} 0)$  is **APX-hard**, even for  $n = 2$ .

### 2.3 Approximation algorithm for $\min \sum 0$

Let us now show an example of algorithm achieving a  $n - f(n, m)$  ratio. Notice that the  $(n - \epsilon)$  inapproximability result holds for fixed  $n$  and  $\#0 = 1$ , while the following algorithm is polynomial-time computable when  $n$  is part of the input and  $\#0$  is arbitrary.

**Proposition 1.** *There is a polynomial-time  $n - \frac{n-1}{n\rho(n,m)}$  approximation algorithm for  $\min \sum 0$ , where  $\rho(n, m) > 1$  is the approximation ratio for independent set in graphs that are the union of  $m$  complete  $n$ -partite graphs.*

*Proof.* Let  $I$  be an instance of  $\min \sum 0$ . Let us now consider an optimal solution  $S^* = \{s_1^*, \dots, s_n^*\}$  of  $I$ . For any  $i \in [n]$ , let  $Z_i^* = \{l \in [p] : v_{s_i^*}[l] = 0 \text{ and } v_{s_t^*}[l] = 1, \forall t \neq i\}$  be the set of coordinates equal to zero only in stack  $s_i^*$ . Let  $\Delta = \sum_{i=1}^n |Z_i^*|$ . Notice that we have  $c(S^*) \geq \Delta + 2(p - \Delta)$ , as for any coordinate  $l$  outside  $\bigcup_i Z_i^*$ , there are at least two stacks with a zero at coordinate  $l$ . W.l.o.g., let us suppose that  $Z_1^*$  is the largest set among  $\{Z_i^*\}$ , implying  $|Z_1^*| \geq \frac{\Delta}{n}$ .

Given a subset  $Z \subset [p]$ , we will construct a solution  $S = \{s_1, \dots, s_n\}$  such that for any  $l \in Z$ ,  $v_{s_1}[l] = 0$ , and for any  $i \neq 1$ ,  $v_{s_i}[l] = 1$ . Informally, the zero at coordinates  $Z$  will appear only in  $s_1$ , which behaves as a "trash" stack. The cost of such a solution is  $c(S) \leq c(s_1) + \sum_{i=2}^n c(s_i) \leq p + (n - 1)(p - |Z|)$ . Our objective is now to compute such a set  $Z$ , and to lower bound  $|Z|$  according to  $|Z_1^*|$ .

Let us now define how we compute  $Z$ . Let  $P = \{l \in [p] : \forall i \in [m], |\{j : v_j^i[l] = 0\}| \leq 1\}$  be the subset of coordinates that are never nullified in two different vectors of the same set. We will construct a simple undirected graph  $G = (P, E)$ , and thus it remains to define  $E$ . For vector  $v_j^i$ , let  $Z_j^i = Z(v_j^i) \cap P$ , where  $Z(v) \subseteq [p]$  denotes the set of null coordinates of vector  $v$ . For any  $i \in [m]$ , we add to  $G$  the edges of the complete  $n$ -partite graph  $G^i = (\{Z_1^i \times \dots \times Z_n^i\})$  (i.e. for any  $j_1, j_2, v_1 \in Z_{j_1}^i, v_2 \in Z_{j_2}^i$ , we add edge  $\{v_1, v_2\}$  to  $G$ ). This concludes the description of  $G$ , which can be seen as the union of  $m$  complete  $n$ -partite graphs.

Let us now see the link between independent set in  $G$  and our problem. Let us first see why  $Z_1^*$  is a independent set in  $G$ . Recall that by definition of  $Z_1^*$ , for any  $l \in Z_1^*$ ,  $v_{s_1^*}[l] = 0$ , but  $v_{s_j^*}[l] = 1, j \geq 2$ . Thus, it is immediate that  $Z_1^* \subseteq P$ . Moreover, assume by contradiction that there exists an edge in  $G$  between two vertices  $l_1$  and  $l_2$  of  $Z_1^*$ . This implies that there exists  $i \in [m], j_1$  and  $j_2 \neq j_1$  such that  $v_{j_1}^i[l_1] = 0$  and  $v_{j_2}^i[l_2] = 0$ . As by definition of  $Z_1^*$  we must have  $v_{s_j^*}[l_1] = 1$  and  $v_{s_j^*}[l_2] = 1$  for  $j \geq 2$ , this implies that  $s_1^*$  must contains both  $v_{j_1}^i$  and  $v_{j_2}^i$ , a contradiction. Thus, we get  $OPT(G) \geq |Z_1^*|$ , where  $OPT(G)$  is the size of a maximum independent set in  $G$ .

Now, let us check that for any independent set  $Z \subseteq P$  in  $G$ , we can construct a solution  $S = \{s_1, \dots, s_n\}$  such that for any  $l \in Z$ ,  $v_{s_1}[l] = 0$ , and for any  $i \neq 1$ ,  $v_{s_i}[l] = 1$ . To construct such a solution, we have to prove that we can add in  $s_1$  all the vectors  $v$  such that  $\exists l \in Z$  such that  $v[l] = 0$ . However, this last

statement is clearly true as for any  $i \in [m]$ , there is at most one vector  $v_j^i$  with  $Z(v_j^i) \subseteq Z$ .

Thus, any  $\rho(n, m)$  approximation algorithm gives us a set  $Z$  with  $|Z| \geq \frac{|Z_1^*|}{\rho(n, m)} \geq \frac{\Delta}{n\rho(n, m)}$ , and we get a ratio of  $\frac{p+(n-1)(p-\frac{\Delta}{n\rho(n, m)})}{2p-\Delta} \leq n - \frac{n-1}{n\rho(n, m)}$  for  $\Delta = p$ .  $\square$

*Remark 1.* We can get, for example,  $\rho(n, m) = mn^{m-1}$  using the following algorithm. For any  $i \in [m]$ , let  $G^i = (A_1^i, \dots, A_n^i)$  be the  $i$ -th complete  $n$ -partite graph. W.l.o.g., suppose that  $A_1^1$  is the largest set among  $\{A_j^i\}$ . Notice that  $|A_1^1| \geq \frac{OPT}{m}$ . The algorithm starts by setting  $S_1 = A_1^1$  ( $S_1$  may not be an independent set). Then, for any  $i$  from 2 to  $m$ , the algorithm set  $S_i = S_{i-1} \setminus (\cup_{j \neq j_0} A_j^i)$ , where  $j_0 = \arg \max_j \{|S_{i-1} \cap A_j^i|\}$ . Thus, for any  $i$  we have  $|S_i| \geq \frac{|S_{i-1}|}{n}$ , and  $S_i$  is an independent set when considering only edges from  $\cup_{l=1}^i G^l$ . Finally, we get an independent set of  $G$  of size  $|S_m| \geq \frac{|S_1|}{n^{m-1}} \geq \frac{OPT}{mn^{m-1}}$ .

### 3 Exact resolution of sparse instances

The section is devoted to the exact resolution of  $\min \sum 0$  for sparse instances where each vector has at most one zero ( $\#0 \leq 1$ ). As we have seen in Section 2,  $(\min \sum 0)_{\#0 \leq 1}$  remains **NP**-hard (even for  $n = 2$ ). Thus it is natural to ask if  $(\min \sum 0)_{\#0 \leq 1}$  is polynomial-time solvable for fixed  $m$  (for general  $n$ ). This section is devoted to answer positively to this question. Notice that we cannot extend this result to a more general notion of sparsity as  $(\min \sum 0)_{\#0 \leq 2}$  is **APX**-complete for  $m = 3$  [5]. However, the question if  $(\min \sum 0)_{\#0 \leq 1}$  is fixed parameter tractable when parameterized by  $m$  is left open.

We first need some definitions, and refer the reader to Figure 3 where an example is depicted.

#### Definition 3.

- For any  $l \in [p], i \in [m]$ , we define  $B^{(l,i)} = \{v_j^i : v_j^i[l] = 0\}$  to be the set of vectors of set  $i$  that have their (unique) zero at position  $l$ . For the sake of homogeneous notation, we define  $B^{(p+1,i)} = \{v_j^i : v_j^i \text{ is a } 1 \text{ vector}\}$ . Notice that the  $B^{(l,i)}$  form a partition of all the vectors of the input, and thus an input of  $(\min \sum 0)_{\#0 \leq 1}$  is completely characterized by the  $B^{(l,i)}$ .
- For any  $l \in [p+1]$ , the **block**  $B^l = \cup_{i \in [m]} B^{(l,i)}$ .

Informally, the idea to solve  $(\min \sum 0)_{\#0 \leq 1}$  in polynomial time for fixed  $m$  is to parse the input block after block using a dynamic programming algorithm. When arriving at block  $B^l$  we only need to remember for each  $c \subseteq [m]$  the number  $x_c$  of “partial stacks” that have only one vector for each  $V^i, i \in c$ . Indeed, we do not need to remember what is “inside” these partial stacks as all the remaining vectors from  $B^{l'}, l' \geq l$  cannot “match” (*i.e.* have their zero in the same position) the vectors in these partial stacks.

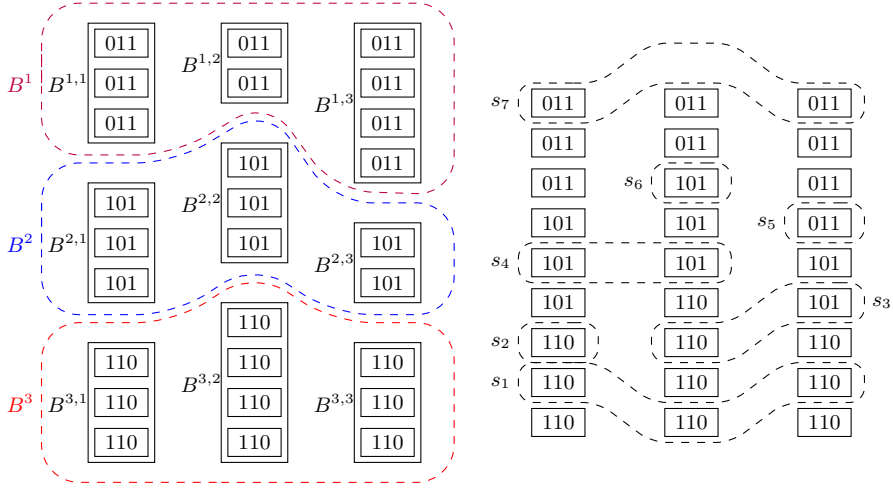


Fig. 3: Left: instance  $I$  of  $(\min \sum 0)_{\#0 \leq 1}$  partitioned into blocks. Right: A profile  $P = \{x_{\{\emptyset\}} = 2, x_{\{1\}} = 1, x_{\{2\}} = 1, x_{\{3\}} = 1, x_{\{1,2\}} = 1, x_{\{1,3\}} = 1, x_{\{2,3\}} = 1, x_{\{1,2,3\}} = 1\}$  encoding a set  $S$  of partial stacks of  $I$  containing two empty stacks. The support of  $s_7$  is  $\text{sup}(s_7) = \{1, 3\}$  and has cost  $c(s_7) = 1$ .

#### Definition 4.

- A **partial stack**  $s = \{v_{i_1}^s, \dots, v_{i_k}^s\}$  of  $I$  is such that  $\{i_x \in [m], x \in [k]\}$  are pairwise disjoint, and for any  $x \in [k]$ ,  $v_{i_x}^s \in V^{i_x}$ . The **support** of a partial stack  $s$  is  $\text{sup}(s) = \{i_x, x \in [k]\}$ . Notice that a stack  $s$  (i.e. non partial) has  $\text{sup}(s) = [m]$ .
- The cost is extended in the natural way: the cost of a partial stack  $c(s) = c(\bigwedge_{x \in [k]} v_{i_x}^s)$  is the number of zeros of the bitwise AND of the vectors of  $s$ .

We define the notion of profile as follows:

**Definition 5.** A **profile**  $P = \{x_c, c \subseteq [m]\}$  is a set of  $2^m$  positive integers such that  $\sum_{c \subseteq [m]} x_c = n$ .

In the following, a profile will be used to encode a set  $S$  of  $n$  partial stacks by keeping a record of their support. In other words,  $x_c, c \subseteq [m]$  will denote the number of partial stacks in  $S$  of support  $c$ . This leads us to introduce the notion of reachable profile as follows:

**Definition 6.** Given two profiles  $P = \{x_c : c \subseteq [m]\}$  and  $P' = \{x'_c : c' \subseteq [m]\}$  and a set  $S = \{s_1, \dots, s_n\}$  of  $n$  partial stacks,  $P'$  is said **reachable** from  $P$  through  $S$  iff there exist  $n$  couples  $(s_1, c_1), (s_2, c_2), \dots, (s_n, c_n)$  such that:

- For each couple  $(s, c)$ ,  $\text{sup}(s) \cap c = \emptyset$ .
- For each  $c \subseteq [m]$ ,  $|\{(s_j, c_j) : c_j = c, j = 1, \dots, n\}| = x_c$ . Intuitively, the configuration  $c$  appears in exactly  $x_c$  couples.

- For each  $c' \subseteq [m]$ ,  $|\{(s'_j, c'_j) : \text{sup}(s'_j) \cup c'_j = c', j = 1, \dots, n\}| = x'_{c'}$ . Intuitively, there exist exactly  $x'_{c'}$  couples that, when associated, create a partial of profile  $c'$ .

Given two profiles  $P$  and  $P'$ ,  $P$  is said reachable from  $P'$ , if there exists a set  $S$  of  $n$  partial stacks such that  $P'$  is reachable from  $P$  through  $S$ .

Intuitively, a profile  $P'$  is reachable from  $P$  through  $S$  if every partial stack of the set encoded by  $P$  can be assigned to a unique partial stack from  $S$  to obtain a set of new partial stacks encoded by  $P'$ .

Remark that, given a set of partial stacks  $S$  only their profile is used to determine whether a profile is reachable or not. An example of a reachable profile is given on Figure 4.

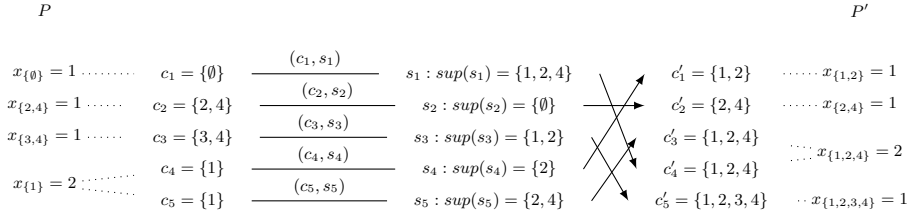


Fig. 4: Example of a profile  $P' = \{x_{\{1,2\}} = 1, x_{\{2,4\}} = 1, x_{\{1,2,4\}} = 2, x_{\{1,2,3,4\}} = 1\}$  that is reachable from  $P = \{x_{\{0\}} = 1, x_1 = 2, x_{\{2,4\}} = 1, x_{\{3,4\}} = 1\}$  reachable through  $S = \{s_1 : \text{sup}(s_1) = \{1, 2, 4\}, s_2 : \text{sup}(s_2) = \{\emptyset\}, s_3 : \text{sup}(s_3) = \{1, 2\}, s_4 : \text{sup}(s_4) = \{2\}, s_5 : \text{sup}(s_5) = \{2, 4\}\}$ .

We introduce now the following problem  $II$ . We then show that this problem can be used to solve  $(\min \sum 0)_{\#0 \leq 1}$  problem, and we present a dynamic programming algorithm that solves  $II$  in polynomial time when  $m$  is fixed.

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### Optimization Problem 2 $II$

**Input**  $(l, P)$  with  $l \in [p + 1]$ ,  $P$  a profile.

**Output** A set of  $n$  partial stacks  $S = \{s_1, s_2, \dots, s_n\}$  such that  $S$  is a partition of  $\mathcal{B} = \bigcup_{l' \geq l} B^{l'}$  and for every  $c \subseteq [m]$ ,  $|\{s \in S \mid \text{sup}(s) = [m] \setminus c\}| = x_c$  and such that  $c(S) = \sum_{j=1}^n c(s_j)$  is minimum.

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Remark that an instance  $I$  of  $(\min \sum 0)_{\#0 \leq 1}$  can be solved optimally by solving optimally the instance  $I' = (1, P = \{x_{\emptyset} = n, x_c = 0, \forall c \neq \emptyset\})$  of  $II$ . The optimal solution of  $I'$  is indeed a set of  $n$  partial disjoint stacks of support  $[m]$  of minimum cost.

We are now ready to define the following dynamic programming algorithm that solves any instance  $(l, P)$  of  $II$  by parsing the instance block after block and branching for each of these blocks on every reachable profile.

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**Function** MinSumZeroDP( $l, P$ )
 

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**if**  $k == p + 1$  **then**  
     **return** 0;  
**return**  $\min(c(S') + \text{MinSumZeroDP}(l + 1, P'))$ , with  $P'$  reachable from  $P$   
 through  $S'$ , where  $S'$  partition of  $B^l$ ;

---

Note that this dynamic programming assumes the existence of a procedure that enumerates *efficiently* all the profiles  $P'$  that are reachable from  $P$ . The existence of such a procedure will be shown thereafter.

**Lemma 3.** *For any instance of  $\Pi$  ( $l, P$ ),  $\text{MinSumZeroDP}(l, P) = \text{Opt}(l, P)$ .*

*Proof.* Lemma 3 is true as in a given block  $l$ , the algorithm tries every reachable profile, and the zeros of vectors in blocks  $\mathcal{B} = \bigcup_{l' < l} B^{l'}$  cannot be matched with those of vectors in block  $\mathcal{B}' = \bigcup_{l' \geq l} B^{l'}$ . This is the reason why the support of the already created partial stacks (stored in profile  $P$ ) is sufficient to keep a record of what have been done (the positions of the zeros in the partial stacks corresponding to  $P$  is not relevant).  $\square$

Let us focus now on the procedure in charge of the enumeration of the reachable profile. A first and intuitive way to perform this operation is by guessing, for all  $c, c' \subseteq [m]$ ,  $y_{c,c'}$  the number of partial stacks in configuration  $c$  that will be turned into configuration  $c'$  with vectors of current block  $B^l$ . For each such guess it is possible to greedily verify that each  $y_{c,c'}$  can be satisfied with the vectors of the current block. As each of the  $y_{c,c'}$  can take values from 0 to  $n$  and  $c$  and  $c'$  can be both enumerated in  $\mathcal{O}^*(n^{2^m})$ , the previous algorithm runs in  $\mathcal{O}^*(n^{2^{2m}})$ .

This complexity can be improved as follows. The idea is to enumerate every possible profile  $P'$  and to verify using another dynamic programming algorithm if such a  $P'$  is reachable from  $P$ . We define  $\text{Aux}_{P'}(P, X)$ , that verifies if  $P'$  is reachable from  $P$  by using all vectors of  $X$ . If  $X = \emptyset$ , then the algorithm returns whether  $P$  is equal to  $P'$  or not. Otherwise, we consider the first vector  $v$  of  $X$  (we fix any arbitrary order) for which a branching is done on every possible assignment of  $v$ . More formally, the algorithm returns  $\bigvee_{c \subseteq [m], x_c > 0, c \cap \text{sup}(v) = \emptyset} \text{Aux}_{P'}(P_2 = \{x'_l\}, X \setminus \{v\})$ , where  $x'_l = x_l - 1$  if  $l = c$ ,  $x'_l = x_l + 1$  if  $l = c \cup \text{sup}(v)$ , and  $x'_l = x_l$  otherwise.

Using  $\text{Aux}$  in  $\text{MinSumZeroDP}$ , we get the following theorem.

**Theorem 3.**  $(\min \sum_0)_{\#0 \leq 1}$  can be solved in  $\mathcal{O}^*(n^{2^{m+2}})$ .

We compute the overall complexity as follows: for each of the  $pn^{2^m}$  possible values of the parameters of  $\text{MinSumZeroDP}$ , the algorithm tries the  $n^{2^m}$  profiles  $P'$ , and run for each one  $\text{Aux}_{P'}$  in  $\mathcal{O}^*(n^{2^m} nm)$  (the first parameter of  $\text{Aux}$  can take  $n^{2^m}$  values, and the second  $nm$  as we just encode how many vectors left in  $X$ ).

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