Approximability and exact resolution of the Multidimensional Binary Vector Assignment problem

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Abstract. In this paper we consider the multidimensional binary vector assignment problem. An input of this problem is defined by m disjoint sets $V^1, V^2, \ldots, V^m$, each composed of n binary vectors of size p. An output is a set of n disjoint m-tuples of vectors, where each m-tuple is obtained by picking one vector from each set V^i. To each m-tuple we associate a p dimensional vector by applying the bit-wise AND operation on the m vectors of the tuple. The objective is to minimize the total number of zeros in these n vectors. We denote this problem by $\min \sum 0$, and the restriction of this problem where every vector has at most c zeros by $(\min \sum 0)_{\#0 \leq c}$. $(\min \sum 0)_{\#0 \leq 2}$ was only known to be **APX**-complete, even for $m = 3$ [5]. We show that, assuming the unique games conjecture, it is **NP**-hard to $(n - \varepsilon)$-approximate $(\min \sum 0)_{\#0 \leq 1}$ for any fixed n and ε. This result is tight as any solution is a n-approximation. We also prove without assuming UGC that $(\min \sum 0)_{\#0 \leq 1}$ is **APX**-complete even for $n = 2$, and we provide an example of $n - f(n, m)$-approximation algorithm for $\min \sum 0$. Finally, we show that $(\min \sum 0)_{\#0 \leq 1}$ is polynomial-time solvable for fixed m (which cannot be extended to $(\min \sum 0)_{\#0 \leq 2}$ according to [5]).

1 Introduction

1.1 Problem definition

In this paper we consider the multidimensional binary vector assignment problem denoted by $\min \sum 0$. An input of this problem (see Figure 1) is described by m disjoint sets $V^1, \ldots, V^m$, each set V^i containing n binary p-dimensional vectors. For any $j \in [n]$[1], and any $i \in [m]$, the j^{th} vector of set V^i is denoted v^i_j, and for any $k \in [p]$, the k^{th} coordinate of v^i_j is denoted $v^i_j[k]$.

The output of the problem consists in a set S of n disjoint stacks. A stack $s = (v^s_1, \ldots, v^s_m)$ is an $m - tuple$ of vectors such that $v^s_i \in V^i$, for any $i \in [m]$. Two stacks s_1 and s_2 are disjoint if and only if no vector belongs to s_1 and s_2.

We now introduce the operator $\wedge$ which assigns to a pair of vectors (u, v) the vector given by $u \wedge v = (u[1] \wedge v[1], u[2] \wedge v[2], \ldots, u[p] \wedge v[p])$. We associate to each stack s a unique vector given by $v_s = \bigwedge_{i \in [m]} v^s_i$.

[1] Note that $[n]$ stands for $\{1, 2, \ldots, n\}$.

The cost of a vector v is defined as the number of zeros in it. More formally if v is p-dimensional, $c(v) = p - \sum_{k \in [p]} v[k]$. We extend this definition to a set of stacks $S = \{s_1, \dots, s_n\}$ as follows : $c(S) = \sum_{s \in S} c(v_s)$.

The objective is then to find a set S of n disjoint stacks minimizing the total number of zeros. This leads us to the following definition of the problem:

Optimization Problem 1 $\min \sum 0$

Input *m sets of n p-dimensional binary vectors.*

Output *A set S of n disjoint stacks minimizing $c(S)$.*

Throughout this paper, we denote $(\min \sum 0)_{\#0 \leq c}$ the restriction of $\min \sum 0$ where the number of zeros per vector is upper bounded by c.

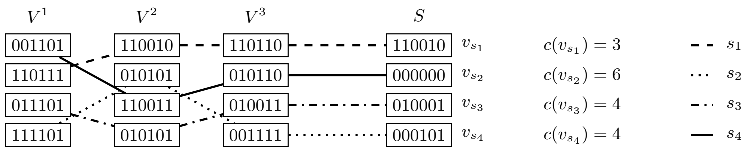

Fig. 1: Example of $\min \sum 0$ instance with $m = 3, n = 4, p = 6$ and of a feasible solution S of cost $c(S) = 17$.

1.2 Related work

The dual version of the problem called $\max \sum 1$ (where the objective is to maximize the total number of 1 in the created stacks) has been introduced by Reda et al. in [8] as the "yield maximization problem in Wafer-to-Wafer 3-D Integration technology". They prove the **NP**-completeness of $\max \sum 1$ and provide heuristics without approximation guarantee. In [6] we proved that, even for $n = 2$, for any $\varepsilon > 0$, $\max \sum 1$ is $\mathcal{O}(m^{1-\varepsilon})$ and $\mathcal{O}(p^{1-\varepsilon})$ inapproximable unless $\mathbf{P} = \mathbf{NP}$. We also provide an ILP formulation proving that $\max \sum 1$ (and thus $\min \sum 0$) is **FPT**[2] when parameterized by p.

We introduced $\min \sum 0$ in [4] where we provide in particular $\frac{4}{3}$-approximation algorithm for $m = 3$. In [5], authors focus on a generalization of $\min \sum 0$, called MULTI DIMENSIONAL VECTOR ASSIGNMENT, where vectors are not necessary binary vectors. They extend the approximation algorithm of [4] to get a $f(m)$-approximation algorithm for arbitrary m. They also prove the **APX**-completeness of the $(\min \sum 0)_{\#0 \leq 2}$ for $m = 3$. This result was the only known inapproximability result for $\min \sum 0$.

1.3 Contribution

In section 2 we study the approximability of $\min \sum 0$. Our main result in this section is to prove that assuming UGC, it is **NP**-hard to $(n - \varepsilon)$-approximate

[2] *i.e.* admits an algorithm in $f(p)poly(|I|)$ for an arbitrary function f.

$(\min\sum 0)_{\#0\leq 1}$ (and thus $\min\sum 0$) for any fixed $n \geq 2$, $\forall\varepsilon > 0$. This result is tight as any solution is a n-approximation.

Notice that this improves the only existing negative result for $\min\sum 0$, which was the **APX**-hardness of [5] (implying only no-**PTAS**).

We also show how this reduction can be used to obtain the **APX**-hardness for $(\min\sum 0)_{\#0\leq 1}$ for $n = 2$ unless **P** = **NP**, which is weaker negative result, but does not require UGC. We then give an example $n - f(n,m)$ approximation algorithm for the general problem $\min\sum 0$.

In section 3, we consider the exact resolution of $\min\sum 0$ (and $\max\sum 1$). We only focus on what we will call sparse instances, *i.e.* instances of $(\min\sum 0)_{\#0\leq 1}$. Indeed, recall that authors of [5] show that $(\min\sum 0)_{\#0\leq 2}$ is **APX**-complete even for $m = 3$, implying that $(\min\sum 0)_{\#0\leq 2}$ cannot be polynomial-time solvable for fixed m unless **P** = **NP**. Thus, it was natural to ask if $(\min\sum 0)_{\#0\leq 1}$ was polynomial-time solvable for fixed m. Section 3 is devoted to answer positively to this question. Notice that the question of determining if $(\min\sum 0)_{\#0\leq 1}$ is **FPT** when parameterized by m remains open. Due to space constraints, results marked with a $\star$ are proved in the appendix.

2 Approximability of $\min\sum 0$

Let us first recall definitions of reductions we use in this paper.

2.1 Definitions

***L*-reduction** The L-reduction has been introduced by Papadimitriou et al. in [7] as follows:

Definition 1. *Let Π_1 and Π_2 be two optimization problems with objective functions m_1 and m_2. Let f be a polynomial-time computable function that given any instance x of Π_1 associates an instance $f(x)$ of Π_2. Let g be another polynomial-time computable function that given any instance x of Π_1, and feasible solution S of $f(x)$, associates a feasible solution $g(x,S)$ of Π_1. If f and g verify the two following conditions:*

1. *$\exists\alpha$ such that $Opt(f(x)) \leq \alpha Opt(x)$*
2. *$\exists\beta$ such that for each solution S of Π_2, $|Opt(x) - m_1(g(x,S))| \leq \beta|Opt(f(x)) - m_2(S)|$*

then (f,g) is an L-reduction.

In following, Π_1 L-reduces to Π_2 is noted $\Pi_1 <_L \Pi_2$.

Gap reduction We briefly recall the definition of such a reduction, as presented in [2] by Ausiello et al.

Definition 2. *Let Π_{dec} be a decision problem and Π_{opt} a minimization problem. Let f be a polynomial-time computable function that given any instance x of Π_{dec} associates an instance $f(x)$ of Π_{opt}. If there exists two function a and r such that:*

1. x is a YES-instance $\Rightarrow Opt(f(x)) \leq a(x)$
2. x is a NO-instance $\Rightarrow Opt(f(x)) \geq r(x)a(x)$

then f is a $r(x)$-Gap reduction.

2.2 Inapproximability results for $(\min \sum 0)_{\#0\leq 1}$

From now we suppose that $\forall k \in [p]$, $\exists i$, $\exists j$ such that $v_j^i[k] = 0$. In other words, for any solution S and $\forall k$, there exists a stack s such that $v_s[k] = 0$. Otherwise, we simply remove such a coordinate from every vector of every set, and decrease p by one. Since this coordinate would be set to 1 in all the stacks of all solutions, such a preprocessing preserves approximation ratios and exact results.

In a first time, we define the following polynomial-time computable function f which associates an instance of $(\min \sum 0)_{\#0\leq 1}$ to any k-uniform hypergraph, *i.e.* an hypergraph $G = (U, E)$ such that every hyperedges of E contains exactly k distinct elements of U.

Definition of f We consider a k-uniform hypergraph $G = (U, E)$. We call f the polynomial-time computable function that creates an instance of $(\min \sum 0)_{\#0\leq 1}$ from a G as follows.

1. We set $m = |E|$, $n = k$ and $p = |U|$.
2. For each hyperedge $e = \{u_1, u_2, \ldots, u_k\} \in E$, we create the set V^e containing k vectors $\{v_j^e, j \in [k]\}$, where for all $j \in [k]$, $v_j^e[u_j] = 0$ and $v_j^e[l] = 1$ for $l \neq u_j$. We say that a vector v **represents** $u \in U$ iff $v[u] = 0$ and $v[l \neq u] = 1$ (and thus vector v_j^e represents u_j).

An example of this construction is given in Figure 2.

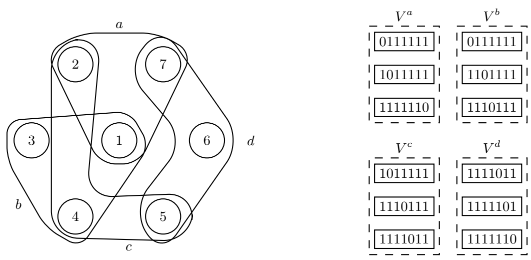

Fig. 2: Illustration of the reduction from an hypergraph $G = (U = \{1, 2, 3, 4, 5, 6, 7\}, E = \{\{1, 2, 7\}, \{1, 3, 4\}, \{2, 4, 5\}, \{5, 6, 7\}\})$ to an instance $(\min \sum 0)_{\#0\leq 1}$

Negative results assuming UGC We consider the following problem. Notice that what we call a vertex cover in a k-regular hypergraph $G = (U, E)$ is a set $U' \subseteq U$ such that for any hyperedge $e \in E$, $U' \cap e \neq \emptyset$.

Decision Problem 1 ALMOST Ek VERTEX COVER

Input *We are given an integer $k \geq 2$, two arbitrary positive constants ε and δ and a k-uniform hypergraph $G = (U, E)$.*

Output *Distinguish between the following cases:*

YES Case *there exist k disjoint subsets $U^1, U^2, \dots, U^k \subseteq U$, satisfying $|U^i| \geq \frac{1-\varepsilon}{k}|U|$ and such that every hyperedge contains at most one vertex from each U^i.*

NO Case *every vertex cover has size at least $(1-\delta)|U|$.*

It is shown in [3] that, assuming UGC, this problem is **NP**-complete.

Theorem 1. *For any fixed $n \geq 2$, for any constants $\varepsilon, \delta > 0$, there exists a $\frac{n-n\delta}{1+n\varepsilon}$-Gap reduction from* ALMOST Ek VERTEX COVER *to $(\min \sum 0)_{\#0 \leq 1}$. Consequently, under UGC, for any fixed n $(\min \sum 0)_{\#0 \leq 1}$ is* **NP***-hard to approximate within a factor $(n - \varepsilon')$ for any $\varepsilon' > 0$.*

Proof. We consider an instance I of ALMOST Ek VERTEX COVER defined by two positive constants δ and ϵ, an integer k and a k-regular hypergraph $G = (U, E)$.

We use the function f previously defined to construct an instance $f(I)$ of $\min \sum 0$. Let us now prove that if I is a positive instance, $f(I)$ admits a solution S of cost $c(S) < (1 + n\varepsilon)|U|$, and otherwise any solution S of $f(I)$ has cost $c(S) \geq n(1-\delta)|U|$.

NO Case Let S be a solution of $f(I)$. Let us first remark that for any stack $s \in S$, the set $\{k : v_s[k] = 0\}$ defines a vertex cover in G. Indeed, s contains exactly one vector per set, and thus by construction s selects one vertex per hyperedge in G. Remark also that the cost of s is equal to the size of the corresponding vertex cover.
Now, suppose that I is a negative instance. Hence each vertex cover has a size at least equal to $(1-\delta)|U|$, and any solution S of $f(I)$, composed of exactly n stacks, verifies $c(S) \geq n(1-\delta)|U|$.

YES Case If I is a positive instance, there exists k disjoint sets $U^1, U^2, \dots, U^k \subseteq U$ such that $\forall i = 1, \dots, k$, $|U^i| \geq \frac{1-\varepsilon}{k}|U|$ and such that every hyperedge contains at most one vertex from each U^i.
We introduce the subset $X = U \setminus \bigcup_{i=1}^{k} U^i$. By definition $\{U^1, U^2, \dots, U^k, X\}$ is a partition of U and $X \leq \varepsilon|U|$. Furthermore, $U^i \cup X$ is a vertex cover $\forall i = 1, \dots, k$. Indeed, each hyperedge $e \in E$ that contains no vertex of U^i, contains at least one vertex of X since e contains k vertices.
We now construct a solution S of $f(I)$. Our objective is to construct stacks $\{s_i\}$ such that for any i, the zeros of s_i are included in $U_i \cup X$ (*i.e.* $\{l : v_{s_i}[l] = 0\} \subseteq U_i \cup X$). For each $e = \{u_1, \dots, u_k\} \in E$, we show how to assign exactly one vector of V^e to each stack $s_1, \dots, s_k$. For all $i \in [k]$, if v^e_j represents a vertex u with $u \in U^i$, then we assign v^e_j to s_i. W.l.o.g., let

$S'_e = \{s_1, \dots, s_{k'}\}$ (for $k' \leq k$) be the set of stacks that received a vertex during this process. Notice that as every hyperedge contains at most one vertex from each U^i, we only assigned one vector to each stack of S'_e. After this, every unassigned vector $v \in V^e$ represents a vertex of X (otherwise, such a vector v would belong to a set U^i, $i \in k'$, a contradiction). We assign arbitrarily these vectors to the remaining stacks that are not in S'_e. As by construction $\forall i \in [k]$, $v_s i$ contains only vectors representing vertices from $U^i \cup X$, we get $c(s_i) \leq |U^i| + |X|$.

Thus, we obtain a feasible solution S of cost $c(S) = \sum_{i=1}^{k} c(s_i) \leq k|X| + \sum_{i=1}^{k} |U^i|$. As by definition we have $|X| + \sum_{i=1}^{k} |U^i| = |U|$, it follows that $c(S) \leq |U| + (k-1)\varepsilon|U|$ and since $k = n$, $c(S) < |U|(1 + n\varepsilon)$.

If we define $a(n) = (1 + n\varepsilon)|U|$ and $r(n) = \frac{n(1-\delta)}{(1+n\varepsilon)}$, the previous reduction is a $r(n)$-Gap reduction. Furthermore, $\lim_{\delta,\varepsilon \to 0} r(n) = n$, thus it is **NP**-hard to approximate $(\min \sum 0)_{\#0 \leq 1}$ within a ratio $(n - \varepsilon')$ for any $\varepsilon' > 0$.

□

Notice that, as a function of n, this inapproximability result is optimal. Indeed, we observe that any feasible solution S is an n-approximation as, for any instance I of $\min \sum 0$[3], $Opt(I) \geq p$ and for any solution S, $c(S) \leq pn$.

Negative results without assuming UGC Let us now study the negative results we can get when only assuming $\mathbf{P} \neq \mathbf{NP}$. Our objective is to prove that $(\min \sum 0)_{\#0 \leq 1}$ is **APX**-hard, even for $n = 2$. To do so, we present a reduction from Odd Cycle Transversal, which is defined as follows. Given an input graph $G = (U, E)$, the objective is to find an odd cycle transversal of minimum size, *i.e.* a subset $T \subseteq U$ of minimum size such that $G[U \setminus T]$ is bipartite.

For any integer $\gamma \geq 2$, we denote $\mathcal{G}_\gamma$ the class of graphs $G = (U, E)$ such that any optimal odd cycle transversal T has size $|T| \geq \frac{|U|}{\gamma}$. Given $\mathcal{G}$ a class of graphs, we denote $OCT_\mathcal{G}$ the Odd Cycle Transversal problem restricted to $\mathcal{G}$.

Lemma 1. *For any constant $\gamma \geq 2$, there exists an L-reduction from $OCT_{\mathcal{G}_\gamma}$ to $(\min \sum 0)_{\#0 \leq 1}$ with $n = 2$.*

Proof. Let us consider an integer γ, an instance I of $OCT_{\mathcal{G}_\gamma}$, defined by a graph $G = (V, E)$ such that $G \in \mathcal{G}_\gamma$. W.l.o.g., we can consider that G contains no isolated vertex.

Remark that any graph can be seen as a 2-uniform hypergraph. Thus, we use the function f previously defined to construct an instance $f(I)$ of $(\min \sum 0)_{\#0 \leq 1}$ such that $n = 2$. Since, G contains no isolated vertex, $f(I)$ contains no position k such that $\forall i \in [m],\ \forall j \in [n],\ v_j^i[k] = 1$.

Let us now prove that I admits an odd cycle transversal of size t if and only if $f(I)$ admits a solution of cost $p + t$.

[3] Recall that we assume $\forall k \in [p], \exists i, \exists j$ such that $v_j^i[k] = 0$

$\Leftarrow$ We consider an instance $f(I)$ of $(\min \sum 0)_{\#0 \leq 1}$ with $n = 2$ admitting a solution $S = \{s_A, s_B\}$ with cost $c(S) = p + t$. Let us specify a function g which produces from S a solution $T = g(I, S)$ of $OCT_{\mathcal{G}_\gamma}$, *i.e.* a set of vertices of U such that $G[U \backslash T]$ is bipartite.

We define $T = \{u \in U : v_{s_A}[u] = v_{s_B}[u] = 0\}$, the set of coordinates equal to zero in both s_A and s_B. We also define $A = \{u \in V : v_{s_A}[u] = 0 \text{ and } v_{s_B}[u] = 1\}$ (resp. $B = \{u \in V : v_{s_B}[u] = 0 \text{ and } v_{s_A}[u] = 1\}$), the set of coordinates set to zero only in s_A (resp. s_B). Notice that $\{T, A, B\}$ is a partition of U.

Remark that A and B are independent sets. Indeed, suppose that $\exists \{u, v\} \in E$ such that $u, v \in A$. As $\{u, v\} \in E$ there exists a set $V^{(u,v)}$ containing a vector that represents u and another vector that represents v, and thus these vectors are assigned to different stacks. This leads to a contradiction. It follows that $G[U \backslash T]$ is bipartite and T is an odd cycle transversal.

Since $c(S) = |A| + |B| + 2|T| = p + |T| = p + t$, we get $|T| = t$.

$\Rightarrow$ We consider an instance I of $OCT_{\mathcal{G}_\gamma}$ and a solution T of size t. We now construct a solution $S = \{s_A, s_B\}$ of $f(I)$ from T.

By definition, $G[U \backslash T]$ is a bipartite graph, thus the vertices in $U \backslash T$ may be split into two disjoint independent sets A and B. For each edge $e \in E$, the following cases can occur:

- if $\exists u \in e$ such that $u \in A$, then the vector corresponding to u is assigned to s_A, and the vector corresponding to $e \setminus \{u\}$ is assigned to s_B (and the same rule holds by exchanging A and B)
- otherwise, u and $v \in T$, and we assign arbitrarily v_u^e to s_A and the other to s_B.

We claim that the stacks s_A and s_B describe a feasible solution S of cost at most $p + t$.

Since, for each set, only one vector is assigned to s_A and the other to s_B, the two stacks s_A and s_B are disjoint and contain exactly m vectors. S is therefore a feasible solution.

Remark that v_{s_A} (resp. v_{s_B}) contains only vectors v such that $v[k] = 0 \implies k \in A \cup T$ (resp. $k \in B \cup T$), and thus $c(v_A) \leq |A| + |T|$ (resp. $c(v_B) \leq |B| + |T|$). Hence $c(S) \leq |A| + |B| + 2|T| = p + t$.

Let us now prove that this reduction is an L-reduction.

1. By definition, any instance I of $OCT_{\mathcal{G}_\gamma}$ verifies $|Opt(I)| \geq |U|/\gamma$. Thus,
$$Opt(f(I)) \leq |U| + Opt(I) \leq (\gamma + 1)Opt(I)$$
2. We consider an arbitrary instance I of $OCT_{\mathcal{G}_\gamma}$, $f(I)$ the corresponding instance of $(\min \sum 0)_{\#0 \leq 1}$, S a solution of $f(I)$ and $T = g(I), S$ the corresponding solution of I.
We proved $|T| - Opt(I) = c(S) - |U| - (Opt(f(I)) - |U|) = c(S) - Opt(f(I))$.

Therefore, we get an L-reduction for $\alpha = \gamma + 1$ and $\beta = 1$. □

Lemma 2. *There exist a constant γ and $\mathcal{G} \subset \mathcal{G}_\gamma$ such that $OCT_{\mathcal{G}}$ is* **APX**-*hard.*

Proof. We present an L-reduction from VC-3, the vertex cover problem in graph with maximum degree 3, to $OCT_{\mathcal{G}_{\mathcal{VC}}}$ for an appropriate $\mathcal{G}_{\mathcal{VC}}$. VC-3 is known to be **APX**-complete [1].

Given an instance $G = (U, E)$ of VC-3, we construct an instance $f(G) = (U', E')$ as follows:

1. For each $(u, v) \in E$, create a vertex $z_{u,v}$. These z-vertices form the set Z.
2. $U' = U \cup Z$.
3. $E' = E \cup \{(u, z_{u,v}), (v, z_{u,v}) : (u, v) \in E\}$. In other words, for each $(u, v) \in E$, we create the triangle $\{u, v, z_{u,v}\}$.

Let us prove that $G = (U, E)$ admits a solution VC of size $|VC| = t$ if and only if $f(G)$ admits a solution T of size $|T| = t$.

- $\Rightarrow$ Consider a vertex cover VC of size $|VC| = t$, for each $u \in VC$, we add the vertex u' to T. By definition, VC covers all the edges of G and then all its (odd) cycles. Furthermore, it also covers all the created triangles in $f(G)$ since each of these cycles contains exactly one edge in common with $f(G)[U' \backslash Z]$. Thus T is an odd cycle transversal and $|T| = |VC|$.
- $\Leftarrow$ Let us construct a function g that, given any solution T of $f(G)$, computes a solution $VC = g(G, T)$ of G. Notice first that we can suppose that T contains no z-vertex. Otherwise every triangle $\{u, v, z_{u,v}\}$ covered by a $z_{u,v} \in T$, can instead be covered by either u or v without increasing the size of T. Thus, we set $VC = T$.

  By definition of an odd cycle transversal, T covers all the odd cycles of $f(G)$ and especially the created triangles. Thus, the triangle $\{u, v, z_{u,v}\}$ corresponding to any edge $\{u, v\} \in E$ is covered by VC. As $VC \cap Z = \emptyset$, VC is a vertex cover of G.

The previous reduction is an L-reduction for $\alpha = \beta = 1$. Let us call $\mathcal{G}_{VC}$ the class of graph generated in this reduction. The previous reduction shows that $OCT_{\mathcal{G}_{VC}}$ is **APX**-hard. It remains to check that $\mathcal{G}_{VC} \subseteq \mathcal{G}_\gamma$ for a constant γ.

Remark that VC-3 is only defined on 3-regular graphs, it implies that for any instance $G = (U, E)$ of VC-3, $Opt(G) \geq \frac{|U|}{3}$. As $|U'| = |U| + |E| \leq \frac{5|U|}{2}$, it follows that $Opt(f(G)) = Opt(G) \geq \frac{|U|}{3} \geq \frac{2|U'|}{15}$. Hence, $\mathcal{G}_{VC} \subset \mathcal{G}_\gamma$ with $\gamma = \frac{15}{2}$. □

The following result is now immediate.

Theorem 2. $(\min \sum 0)_{\#0 \leq 1}$ *is* **APX**-*hard, even for $n = 2$.*

2.3 Approximation algorithm for $\min \sum 0$

Let us now show an example of algorithm achieving a $n - f(n,m)$ ratio. Notice that the $(n-\epsilon)$ inapproximability result holds for fixed n and $\#0 = 1$, while the following algorithm is polynomial-time computable when n is part of the input and $\#0$ is arbitrary.

Proposition 1. *There is a polynomial-time $n - \frac{n-1}{n\rho(n,m)}$ approximation algorithm for* $\min \sum 0$*, where $\rho(n,m) > 1$ is the approximation ratio for independent set in graphs that are the union of m complete n-partite graphs.*

Proof. Let I be an instance of $\min \sum 0$. Let us now consider an optimal solution $S^* = \{s^*_1, \dots, s^*_n\}$ of I. For any $i \in [n]$, let $Z^*_i = \{l \in [p] : v_{s^*_i}[l] = 0 \text{ and } v_{s^*_t}[l] = 1, \forall t \neq i\}$ be the set of coordinates equal to zero only in stack s^*_i. Let $\Delta = \sum_{i=1}^{n} |Z^*_i|$. Notice that we have $c(S^*) \geq \Delta + 2(p - \Delta)$, as for any coordinate l outside $\bigcup_i Z^*_i$, there are at least two stacks with a zero at coordinate l. W.l.o.g., let us suppose that Z^*_1 is the largest set among $\{Z^*_i\}$, implying $|Z^*_1| \geq \frac{\Delta}{n}$.

Given a subset $Z \subset [p]$, we will construct a solution $S = \{s_1, \dots, s_n\}$ such that for any $l \in Z$, $v_{s_1}[l] = 0$, and for any $i \neq 1$, $v_{s_i}[l] = 1$. Informally, the zero at coordinates Z will appear only in s_1, which behaves as a "trash" stack. The cost of such a solution is $c(S) \leq c(s_1) + \sum_{i=2}^{n} c(s_i) \leq p + (n-1)(p - |Z|)$. Our objective is now to compute such a set Z, and to lower bound $|Z|$ according to $|Z^*_1|$.

Let us now define how we compute Z. Let $P = \{l \in [p] : \forall i \in [m], |\{j : v^i_j[l] = 0\}| \leq 1\}$ be the subset of coordinates that are never nullified in two different vectors of the same set. We will construct a simple undirected graph $G = (P, E)$, and thus it remains to define E. For vector v^i_j, let $Z^i_j = Z(v^i_j) \cap P$, where $Z(v) \subseteq [p]$ denotes the set of null coordinates of vector v. For any $i \in [m]$, we add to G the edges of the complete n-partite graph $G^i = (\{Z^i_1 \times \dots \times Z^i_n\})$ (*i.e.* for any j_1, j_2, $v_1 \in Z^i_{j_1}$, $v_2 \in Z^i_{j_2}$, we add edge $\{v_1, v_2\}$ to G). This concludes the description of G, which can be seen as the union of m complete n-partite graphs.

Let us now see the link between independent set in G and our problem. Let us first see why Z^*_1 is a independent set in G. Recall that by definition of Z^*_1, for any $l \in Z^*_1$, $v_{s^*_1}[k] = 0$, but $v_{s^*_j}[k] = 1$, $j \geq 2$. Thus, it is immediate that $Z^*_1 \subseteq P$. Moreover, assume by contradiction that there exists an edge in G between to vertices l_1 and l_2 of Z^*_1. This implies that there exists $i \in [m]$, j_1 and $j_2 \neq j_1$ such that $v^i_{j_1}[l_1] = 0$ and $v^i_{j_2}[l_2] = 0$. As by definition of Z^*_1 we must have $v_{s^*_j}[k_1] = 1$ and $v_{s^*_j}[k_2] = 1$ for $j \geq 2$, this implies that s^*_1 must contains both $v^i_{j_1}$ and $v^i_{j_2}$, a contradiction. Thus, we get $OPT(G) \geq |Z^*_1|$, where $OPT(G)$ is the size of a maximum independent set in G.

Now, let us check that for any independent set $Z \subseteq P$ in G, we can construct a solution $S = \{s_1, \dots, s_n\}$ such that for any $l \in Z$, $v_{s_1}[l] = 0$, and for any $i \neq 1$, $v_{s_i}[l] = 1$. To construct such a solution, we have to prove that we can add in s_1 all the vectors v such that $\exists l \in Z$ such that $v[l] = 0$. However, this last statement is clearly true as for any $i \in [m]$, there is at most one vector v_j^i with $Z(v_j^i) \subseteq Z$.

Thus, any $\rho(n,m)$ approximation algorithm gives us a set Z with $|Z| \geq \frac{|Z_1^*|}{\rho(n,m)} \geq \frac{\Delta}{n\rho(n,m)}$, and we get a ratio of $\frac{p+(n-1)(p-\frac{\Delta}{n\rho(n,m)})}{2p-\Delta} \leq n - \frac{n-1}{n\rho(n,m)}$ for $\Delta = p$.

□

Remark 1. We can get, for example, $\rho(n,m) = mn^{m-1}$ using the following algorithm. For any $i \in [m]$, let $G^i = (A_1^i, \ldots, A_n^i)$ be the i-th complete n-partite graph. W.l.o.g., suppose that A_1^1 is the largest set among $\{A_j^i\}$. Notice that $|A_1^1| \geq \frac{OPT}{m}$. The algorithm starts by setting $S_1 = A_1^1$ (S_1 may not be an independent set). Then, for any i from 2 to m, the algorithm set $S_i = S_{i-1} \setminus (\cup_{j \neq j_0} A_j^i)$, where $j_0 = \arg\max_j \{|S_{i-1} \cap A_j^i|\}$. Thus, for any i we have $|S_i| \geq \frac{|S_{i-1}|}{n}$, and S_i is an independent set when considering only edges from $\cup_{l=1}^i G^l$. Finally, we get an independent set of G of size $|S_m| \geq \frac{S_1}{n^{m-1}} \geq \frac{OPT}{mn^{m-1}}$.

3 Exact resolution of sparse instances

The section is devoted to the exact resolution of $\min \sum 0$ for sparse instances where each vector has at most one zero ($\#0 \leq 1$). As we have seen in Section 2, $(\min \sum 0)_{\#0 \leq 1}$ remains **NP**-hard (even for $n = 2$). Thus it is natural to ask if $(\min \sum 0)_{\#0 \leq 1}$ is polynomial-time solvable for fixed m (for general n). This section is devoted to answer positively to this question. Notice that we cannot extend this result to a more general notion of sparsity as $(\min \sum 0)_{\#0 \leq 2}$ is **APX**-complete for $m = 3$ [5]. However, the question if $(\min \sum 0)_{\#0 \leq 1}$ is fixed parameter tractable when parameterized by m is left open.

We first need some definitions, and refer the reader to Figure 3 where an example is depicted.

Definition 3.

- *For any $l \in [p], i \in [m]$, we define $B^{(l,i)} = \{v_j^i : v_j^i[l] = 0\}$ to be the set of vectors of set i that have their (unique) zero at position l. For the sake of homogeneous notation, we define $B^{(p+1,i)} = \{v_j^i : v_j^i$ is a 1 vector$\}$. Notice that the $B^{(l,i)}$ form a partition of all the vectors of the input, and thus an input of $(\min \sum 0)_{\#0 \leq 1}$ is completely characterized by the $B^{(l,i)}$.*
- *For any $l \in [p+1]$, the **block** $B^l = \bigcup_{i \in [m]} B^{(l,i)}$.*

Informally, the idea to solve $(\min \sum 0)_{\#0 \leq 1}$ in polynomial time for fixed m is to parse the input block after block using a dynamic programming algorithm. When arriving at block B^l we only need to remember for each $c \subseteq [m]$ the number x_c of "partial stacks" that have only one vector for each $V^i, i \in c$. Indeed, we do not need to remember what is "inside" these partial stacks as all the remaining vectors from $B^{l'}, l' \geq l$ cannot "match" (*i.e.* have their zero in the same position) the vectors in these partial stacks.

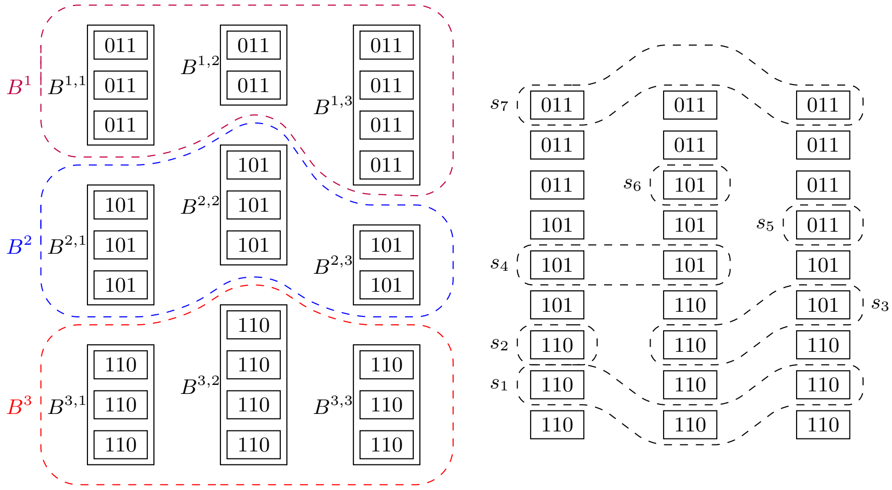

Fig. 3: Left: instance I of $(\min \sum 0)_{\#0 \leq 1}$ partitionned into blocks. Right: A profile $P = \{x_{\{\emptyset\}} = 2, x_{\{1\}} = 1, x_{\{2\}} = 1, x_{\{3\}} = 1, x_{\{1,2\}} = 1, x_{\{1,3\}} = 1, x_{\{2,3\}} = 1, x_{\{1,2,3\}} = 1\}$ encoding a set S of partial stacks of I containing two empty stacks. The support of s_7 is $sup(s_7) = \{1, 3\}$ and has cost $c(s_7) = 1$.

Definition 4.

- *A* ***partial stack*** $s = \{v^s_{i_1}, \ldots, v^s_{i_k}\}$ *of* I *is such that* $\{i_x \in [m], x \in [k]\}$ *are pairwise disjoints, and for any* $x \in [k]$, $v^s_{i_x} \in V^{i_x}$. *The* ***support*** *of a partial stack* s *is* $sup(s) = \{i_x, x \in [k]\}$. *Notice that a stack* s *(i.e. non partial) has* $sup(s) = [m]$.
- *The cost is extended in the natural way: the cost of a partial stack* $c(s) = c(\bigwedge_{x \in [k]} v^s_{i_x})$ *is the number of zeros of the bitwise AND of the vectors of* s.

We define the notion of profile as follows:

Definition 5. *A* ***profile*** $P = \{x_c, c \subseteq [m]\}$ *is a set of* 2^m *positive integers such that* $\sum_{c \subseteq [m]} x_c = n$.

In the following, a profile will be used to encode a set S of n partial stacks by keeping a record of their support. In other words, $x_c, c \subseteq [m]$ will denote the number of partial stacks in S of support c. This leads us to introduce the notion of reachable profile as follows:

Definition 6. *Given two profiles* $P = \{x_c : c \subseteq [m]\}$ *and* $P' = \{x'_{c'} : c' \subseteq [m]\}$ *and a set* $S = \{s_1, \ldots, s_n\}$ *of* n *partial stacks,* P' *is said reachable from* P *through* S iff *there exist* n *couples* $(s_1, c_1), (s_2, c_2), \ldots, (s_n, c_n)$ *such that:*

- *For each couple* (s, c), $sup(s) \cap c = \emptyset$.
- *For each* $c \subseteq [m]$, $|\{(s_j, c_j) : c_j = c, j = 1, \ldots, n\}| = x_c$. *Intuitively, the configuration* c *appears in exactly* x_c *couples.*

– *For each* $c' \subseteq [m], |\{(s_j, c_j) : sup(s_j) \cup c_j = c', j = 1, \ldots, n\}| = x'_{c'}$*. Intuitively, there exist exactly* $x'_{c'}$ *couples that, when associated, create a partial of profile* c'*.*

Given two profiles P *and* P'*,* P *is said reachable from* P'*, if there exists a set* S *of* n *partial stacks such that* P' *is reachable from* P *through* S*.*

Intuitively, a profile P' is reachable from P through S if every partial stack of the set encoded by P can be assigned to a unique partial stack from S to obtain a set of new partial stacks encoded by P'.

Remark that, given a set of partial stacks S only their profile is used to determine whether a profile is reachable or not. An example of a reachable profile is given on Figure 4.

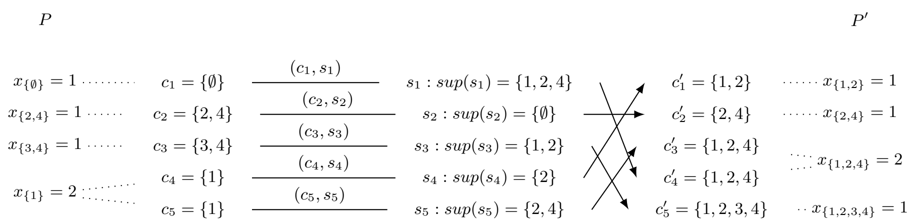

Fig. 4: Example of a profile $P' = \{x_{\{1,2\}} = 1, x_{\{2,4\}} = 1, x_{\{1,2,4\}} = 2, x_{\{1,2,3,4\}} = 1\}$ that is reachable from $P = \{x_{\{\emptyset\}} = 1, x_1 = 2, x_{\{2,4\}} = 1, x_{\{3,4\}} = 1\}$ reachable through $S = \{s_1 : sup(s_1) = \{1,2,4\}, s_2 : sup(s_2) = \{\emptyset\}, s_3 : sup(s_3) = \{1,2\}, s_4 : sup(s_4) = \{2\}, s_5 : sup(s_5) = \{2,4\}\}$.

We introduce now the following problem Π. We then show that this problem can be used to solve $(\min \sum 0)_{\#0 \leq 1}$ problem, and we present a dynamic programming algorithm that solves Π in polynomial time when m is fixed.

Optimization Problem 2 Π

Input (l, P) *with* $l \in [p+1]$*,* P *a profile.*

Output *A set of* n *partial stacks* $S = \{s_1, s_2, \ldots, s_n\}$ *such that* S *is a partition of* $\mathcal{B} = \bigcup_{l' \geq l} B^{l'}$ *and for every* $c \subseteq [m]$*,* $|\{s \in S | sup(s) = [m] \setminus c\}| = x_c$ *and such that* $c(S) = \sum_{j=1}^{n} c(s_j)$ *is minimum.*

Remark that an instance I of $(\min \sum 0)_{\#0 \leq 1}$ can be solved optimally by solving optimally the instance $I' = (1, P = \{x_\emptyset = n, x_c = 0, \forall c \neq \emptyset\})$ of Π. The optimal solution of I' is indeed a set of n partial disjoint stacks of support $[m]$ of minimum cost.

We are now ready to define the following dynamic programming algorithm that solves any instance (l, P) of Π by parsing the instance block after block and branching for each of these blocks on every reachable profile.

Function MinSumZeroDP(l, P)

if $k == p+1$ **then**
    return 0;
return $\min(c(S')+$MinSumZeroDP$(l+1, P'))$, with P' reachable from P through S', where S' partition of B^l;

Note that this dynamic programming assumes the existence of a procedure that enumerates *efficiently* all the profiles P' that are reachable from P. The existence of such a procedure will be shown thereafter.

Lemma 3. *For any instance of Π (l, P), MinSumZeroDP$(l, P) =$ Opt(l, P).*

Proof. Lemma 3 is true as in a given block l, the algorithm tries every reachable profile, and the zeros of vectors in blocks $\mathcal{B} = \bigcup_{l'<l} B^{l'}$ cannot be matched with those of vectors in block $\mathcal{B}' = \bigcup_{l' \geq l} B^{l'}$. This is the reason why the support of the already created partial stacks (stored in profile P) is sufficient to keep a record of what have been done (the positions of the zeros in the partial stacks corresponding to P is not relevant). □

Let us focus now on the procedure in charge of the enumeration of the reachable profile. A first and intuitive way to perform this operation is by guessing, for all $c, c' \subseteq [m]$, $y_{c,c'}$ the number of partial stacks in configuration c that will be turned into configuration c' with vectors of current block B^l. For each such guess it is possible to greedily verify that each $y_{c,c'}$ can be satisfied with the vectors of the current block. As each of the $y_{c,c'}$ can take values from 0 to n and c and c' can be both enumerated in $\mathcal{O}^*(n^{2^m})$, the previous algorithm runs in $\mathcal{O}^*(n^{2^{2m}})$.

This complexity can be improved as follows. The idea is to enumerate every possible profile P' and to verify using another dynamic programming algorithm if such a P' is reachable from P. We define $Aux_{P'}(P, X)$, that verifies if P' is reachable from P by using all vectors of X. If $X = \emptyset$, then the algorithm returns whether P is equal to P' or not. Otherwise, we consider the first vector v of X (we fix any arbitrary order) for which a branching is done on every possible assignment of v. More formally, the algorithm returns $\bigvee_{c \subseteq [m], x_c > 0, c \cap sup(v) = \emptyset} Aux_{P'}(P_2 = \{x'_l\}, X \setminus \{v\})$, where $x'_l = x_l - 1$ if $l = c$, $x'_l = x_l + 1$ if $l = c \cup sup(v)$, and $x'_l = x_l$ otherwise.

Using Aux in MinSumZeroDP, we get the following theorem.

Theorem 3. $(\min \sum 0)_{\#0 \leq 1}$ *can be solved in* $\mathcal{O}^*(n^{2^{m+2}})$.

We compute the overall complexity as follows: for each of the pn^{2^m} possible values of the parameters of MinSumZeroDP, the algorithm tries the n^{2^m} profiles P', and run for each one $Aux_{P'}$ in $\mathcal{O}^*(n^{2^m} nm)$ (the first parameter of Aux can take n^{2^m} values, and the second nm as we just encode how many vectors left in X).

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