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# Bounds and approximation results for scheduling coupled-tasks with compatibility constraints

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**Abstract.** This article is devoted to propose some lower and upper bounds for the coupled-tasks scheduling problem in presence of compatibility constraints according to classical complexity hypothesis ( $\mathcal{P} \neq \mathcal{NP}$ ,  $\mathcal{ETH}$ ). Moreover, we develop an efficient polynomial-time approximation algorithm for the specific case for which the topology describing the compatibility constraints is a quasi split-graph.

**Keywords:** coupled-task, compatibility graph, complexity, approximation.

## 1 Introduction, motivations, model

We consider in this paper the coupled-task scheduling problem subject to compatibility constraints. The motivation of this model is related to data acquisition processes using radar sensors: a sensor emits a radio pulse (first sub-task  $a_i$ ), and listen for an echo reply (second sub-task  $b_i$ ). To make the notation less cluttered, the processing time of a sub-task will be denoted by  $a_i$  instead of  $p_{a_i}$  used in the theory of scheduling. Between these two instants (emission and reception), clearly there is an idle time  $L_i$  due to the propagation, in both sides, of the radio pulse. A coupled-task  $(a_i, L_i, b_i)$ , introduced by Shapiro (1980), is a natural way to model such data acquisition. This model has been widely studied in several works, i. e. Blażewicz et al. (2009). Other works proposed a generalization of this model by including compatibility constraints: scheduling a sub-task during the idle time of another requires that both tasks are compatible. The relations of compatibility are modeled by a compatibility graph  $G$ , linking pair of compatible tasks only. This model is detailed in Simonin et al. (2012). In previous works, we studied the complexity of scheduling coupled-tasks with compatible constraints under several parameters like the size of the sub-tasks or the class of the compatibility graph (Simonin et al. 2013).

In this work, we propose original complexity and approximation results for the problem of scheduling *stretched* coupled-task with compatibility constraints. A *stretched* coupled-tasks  $i$  is a coupled-task having both sub-tasks processing time and idle time equal to a triplet  $(\alpha(i), \alpha(i), \alpha(i))$ , where  $\alpha(i)$  is the *stretch factor* of the task  $i$  - one can apply a stretch factor  $\alpha(i)$  to a reference task  $(1, 1, 1)$  to obtain  $i$  -.

The objective is to minimize the makespan  $C_{max}$ . The input of the problem is a collection of coupled-tasks  $\mathcal{T} = \{t_1, t_2, \dots, t_n\}$ , a stretch factor function  $\alpha : \mathcal{T} \rightarrow \mathbb{N}$ , and a compatibility graph  $G_c = (\mathcal{T}, E)$  where edge from  $E$  link pairs of compatible tasks only. When dealing with stretched coupled-tasks only, a edge  $\{x, y\} \in E$  exists if  $\alpha(x) = \alpha(y)$  (then  $x$  and  $y$  can be scheduled together without idle time as the idle time of one task is employed to schedule the sub-task of the other, thus we can schedule sequentially  $a_x, a_y, b_x, b_y$

- or  $a_y, a_x, b_y, b_x$  - in  $\frac{4\alpha(x)}{3}$  time units), or if  $3\alpha(x) \leq \alpha(y)$  (then  $x$  can be entirely executed during the idle time of  $y$  i.e.  $a_y, a_x, b_x, b_y$  and scheduling both tasks requires  $3\alpha(y)$  time units). We note  $\#(X)$  the number of different stretch factors in a set of tasks  $X$ , and we note  $d_G(X)$  the maximum degree of any vertex  $x \in X$  in a graph  $G_c$ .

We use the well-known Graham notation (Graham et al. 1979) to define the problems presented in this paper. In this work, we propose new complexity and inapproximability results when the compatibility graph is a restricted 1-stage bipartite graph  $G = (X, Y, E)$ , i.e. a bipartite graph where edges are oriented from  $X$  to  $Y$  only. Then we show the problem is  $\mathcal{NP}$ -complete on a quasi-split graph  $G = (G_X, G_Y, E)^4$  even if  $\#(V(G_X)) = 1$  and  $\#(V(G_Y)) = 1$ , but is  $5/4$ -approximable.

## 2 Complexity and approximation results

**Theorem 1.** *Deciding whether an instance of  $1|\alpha, G_c = 1$ -stage-bipartite,  $\#(X) = 2, \#(Y) = 1, d_{G_c}(X) \in \{1, 2\}, d_{G_c}(Y) \in \{3, 4\} | C_{max}$  is a problem hard to approximate within  $\frac{21-\rho^{\text{MAX-3DM-2}}}{20} \leq \rho$ , where  $\rho^{\text{MAX-3DM}}$  gives the upper bound for the MAX-3DM. Since  $\rho^{\text{MAX-3DM-2}} \leq \frac{140}{141}$ , we obtain  $1 + \frac{1}{2820}$ .*

*Proof.* We prove first that the problem is  $\mathcal{NP}$ -complete via a polynomial-time reduction. Based on this reduction, we apply the gap-preserving reduction.

The proof is based on a reduction from the maximum 3 DIMENSIONAL MATCHING (MAX-3DM) (Garey & Johnson 1979): let  $A, B$ , and  $C$  be three disjoint sets of equal size, with  $n = |A| = |B| = |C|$ , and a set  $T \subseteq A \times B \times C$  of triplet, with  $|T| = m$ . The aim is to find a matching (set of mutually disjoint triplets)  $T^* \subseteq T$  of maximum size. This problem is well known to be  $\mathcal{NP}$ -complete. The restricted version of this problem in which each element of  $A \cup B \cup C$  appears exactly twice is denoted MAX-3DM-2 and remains  $\mathcal{NP}$ -complete (Chlebik 2003). In this restricted version, we have  $m = 2n$ .

We transform the instance of MAX-3DM-2 to an instance of  $1|\alpha, G_c = 1$ -stage bipartite,  $\#(X) = 2, \#(Y) = 1, d_{G_c}(X) \in \{1, 2\}, d_{G_c}(Y) \in \{3, 4\} | C_{max} = 63n - 3k(1 - \epsilon)$  as follows: we define a set of tasks  $X \cup Y$  and model the compatibility constraint with a graph  $G_c = (X, Y, E)$ . For each element  $x_i \in A \cup B \cup C$ , we add an *item* coupled-task  $x_i$  into  $X$  with  $\alpha(x_i) = 1$ . For each triplet  $t_i \in T$ , we add a *box* coupled-task  $t_i$  to  $Y$  with  $\alpha(t_i) = 9$ , and an *item* coupled-task  $t'_i$  with  $\alpha(t'_i) = 2 + \epsilon$ . For each  $t_i \in T$  and each  $x_i \in t_i$ , we add the compatibility arc  $(x_i, t_i)$  to  $E$ . We also add the compatibility arc  $(t'_i, t_i)$  to  $E$ . So, the set of  $X$ -tasks (resp.  $Y$ -tasks) are constituted by *item* coupled-task  $x_i$  and  $t'_i$  (resp. *box* coupled-task).

Clearly we have  $m$  *box* coupled-tasks (each with an idle time of 9 units) of degree 4 in  $G_c$ ,  $m$  *item* coupled-tasks with stretch factor  $2 + \epsilon$  of degree 1 in  $G_c$ , and  $3n$  *item* coupled-tasks with stretch factor 1 of degree 2 in  $G_c$ . Moreover  $G_c$  is a bipartite graph. The reduction is constructed in polynomial time.

It exists a schedule of length  $63n - 3k(1 - \epsilon)$  iff it exists a matching of size  $k$  for MAX-3DM-2 instance.

Hereafter, we propose some negative results concerning the existence of subexponential-time algorithms under the following complexity-theoretic hypothesis that is known as the Exponential-Time Hypothesis (see (Woeginger 2001) for a survey on exact algorithms for  $\mathcal{NP}$ -hard problems) for stretched coupled-tasks, and other ones previously studied.

Recall first the EXPONENTIAL-TIME HYPOTHESIS ((Impagliazzo & Paturi 2001), and (Impagliazzo et al. 2001)): there exists a constant  $c > 1$  such that there exists no algorithm for 3-Satisfiability that uses only  $O(c^l)$  time where  $l$  denotes the number of variables.

<sup>4</sup> A quasi split graph is a connected graph  $G = (G_X, G_Y, E)$ , with  $G_X$  a connected non-oriented graph (not complete) and  $G_Y$  a independent set. The other arcs are oriented from  $X$  to  $Y$  only.

**Corollary 1.** *Assuming the Exponential-Time Hypothesis, there exists no algorithm with a worst-case running time that is subexponential in  $n$  (the number of vertices), i.e.:*

1. For the  $1|a_i = b_i = p, L_i = 2p, G_c|C_{max}$  problem in  $O(2^{o(n)})$  time
2. For  $1|a_i = a, b_i = b, L_i = a + b, G_c|C_{max}$  in  $O(2^{o(n)})$  time
3.  $1|\alpha, G_c = 1 - \text{bipartite}|C_{max}$  in  $O(2^{O(n)})$ -time algorithm.

*Proof.* 1. For  $1|a_i = b_i = p, L_i = 2p, G_c|C_{max}$ : In (van Rooij et al. 2013), the authors proved that for PARTITION INTO TRIANGLES on graphs of maximum degree four, there is no algorithm with a worst-case running time  $O(2^{o(n)})$  that is subexponential in  $n$ .

Therefore, we transform a PARTITION INTO TRIANGLES instance with  $n$  vertices and  $m$  edges into an equivalent instance  $G_c$  for bounded degree at most four. Since the transformation is linear (see (Simonin et al. 2012)) the result holds.

2. For the problem  $1|a_i = a, b_i = b, L_i = a + b, G_c|C_{max}$ : In (Lokshtanov, Marx & Saurabh 2011) the authors proved that for HAMILTONIAN PATH there is no  $O(2^{o(n)})$ -time algorithm. As the same way as previously the transformation is linear (see (Simonin et al. 2012)).
3.  $1|\alpha, G_c = 1 - \text{bipartite}|C_{max}$ : In (Chen, Jansen & Zhang 2014), the authors proved that for MAX 3DM, there is no  $O(2^{O(n)})$ -time algorithm, therefore this result is transposed to the scheduling problem using the first part of the proof of Theorem 1.

**Theorem 2.** *Scheduling stretched coupled task in presence of a quasi split graph is a  $\mathcal{NP}$ -complete problem even if  $\#(V(G_X)) = 1$  and  $\#(V(G_Y)) = 1$*

*Proof.* The proof is based on a reduction from a variant of the well-know  $\mathcal{NP}$ -complete PARTITION INTO TRIANGLES. This problem consists to ask if the vertices of a graph  $G = (V, E)$ , with  $|V| = 3q, q \in \mathbb{N}$ , can be partitioned into  $q$  disjoint sets  $T_1, T_2, \dots, T_q$ , each containing exactly three vertices, such that for each  $T_i = \{u_i, v_i, w_i\}, 1 \leq i \leq q$ , all three of the edges  $\{u_i, v_i\}, \{u_i, w_i\}, \{w_i, v_i\}$  belong to  $E$ .

The problem PARTITION INTO TRIANGLES remains  $\mathcal{NP}$ -complete even if the graph  $G$  can be partitioned into three sets with the same size,  $A, B$  et  $C$  such that each set is an independent set (Morandini, M. 2004). The polynomial-time transformation is based on this variant. Let  $G = (A \cup B \cup C, E)$  be an instance of the variant of PARTITION INTO TRIANGLES. We consider the split-graph  $G' = (A \cup B, C, E')$  obtained as follows:

$\forall v \in A$  (resp.  $B$ ), we create a vertex  $A_v$  (resp.  $B_v$ ) with processing time  $(1, 1, 1)$ . Moreover,  $\forall v \in C$  we create a task  $C_v$  with processing time  $(4, 4, 4)$ . The edges between  $A$  and  $B$  remained the same as the  $G'$  whereas the edge between  $A \cup B$  and  $C$  are oriented. Finally in order to have a connected graph, we add two news vertices (resp. one)  $z_0$  and  $z_1$  (resp.  $z_2$  with processing time equal to  $(1, 1, 1)$  (resp.  $(4, 4, 4)$ ). We add edges between  $z_0$  to  $A_v$  (resp.  $z_1$  to  $B_v$ ). Lastly, we add the three edges  $(z_0, z_2), (z_1, z_2)$  and  $(z_0, z_1)$ .

Notice that the graph  $A_v \cup B_v$  form a bipartite graph. The problem is clearly in  $\mathcal{NP}$ . It exists a positive solution for the variant of PARTITION INTO TRIANGLES iff a valid schedule of length  $12 \times (|C| + 1)$  exists. It is sufficient to execute the two tasks  $A_v$  and  $B_v$  in four units of time into a task  $C_u$ .

**Theorem 3.** *The problem is  $5/4$ -approximable on quasi split-graph where  $\#(V(G_Y)) = 1$ .*

*Proof.* W.l.o.g., we suppose that the processing time of  $X$ -tasks (resp.  $Y$ -tasks) is  $(1, 1, 1)$  (resp.  $\alpha(y_i)$ ). Indeed, if  $\alpha(x) > 1$ , we put  $\alpha(y_i) = \lfloor \frac{\alpha(y_i)}{\alpha(x)} \rfloor$  and  $\alpha(x) = 1$ .

**Algorithm:** we transform the problem into an oriented maximum flow-problem between  $G_X$  and  $G_Y$  with two sources  $s$  and  $t$ , with  $\omega(s, x) = \omega(x, y) = 1$  and  $\omega(y, t) = \lfloor \frac{\alpha(y_i)}{3\alpha(x)} \rfloor, \forall y_i \in Y, \forall x \in XG_Y$  where  $\omega(i, j)$  is the capacity of an arc  $(i, j)$ . After the computation of a maximum flow  $F$  of value  $f$ , for the uncovered remaining  $X$ -tasks a maximum

$M$ -matching ( $|M| = m$ ) is applied. The schedule consists in processing first, the  $Y$ -tasks with  $X$ -tasks inside. The  $M$ -tasks are executed after. Lastly, we schedule  $s$  isolated-tasks. The length of schedule given by the algorithm is  $C_{max} \leq \sum_{y_i \in Y} 3\alpha(y_i) + 4m + 3s$  with  $2m + s + f = n = |X|$  and  $\sum_{y_i \in Y} 3\alpha(y_i) \geq 9f$ . In similar way, the optimal length is  $C_{max}^* \geq \sum_{y_i \in Y} \alpha(y_i) + 4m^* + 3s^*$ . We suppose that in  $Y$ -tasks where are  $p^*$ -edges processed and  $r^*$  isolated-tasks, then we obtain  $2(p^* + m^*) + r^* + s^* = n$ ,  $p^* + r^* \leq f$ , and  $\sum_{y_i \in Y} \alpha(y_i) \geq 12p^* + 9r^*$ . In the worst-case, the  $p^*$ -edges are split into two tasks (so  $p^*$  news tasks are added to  $s^*$ ), and also the matched-edges are split (for each  $m^*$  edges one task is executed into the  $Y$ -task, instead of one of  $r^*$ -tasks). Therefore,  $2m^*$  tasks are added to the  $s$ -value. In the worst case, we have  $m^* = r^*$ ,  $s = s^* + p^* + 2r^*$  and  $m = 0$ . In such case,  $C_{max} \leq 12p^* + 9r^* + 3s^* + 3p^* + 6r^*$  and  $C_{max}^* = 12p^* + 9r^* + 4r^* + 3s^*$ . Thus  $\rho \leq \frac{15p^* + 15r^* + 3s^*}{12p^* + 13r^* + 3s^*} \leq \max(5/4, 15/13, 1) = 5/4$ .

**Tightness:** it exists an example for the  $C_{max}^* = 36$ , and for the heuristic  $C_{max} = 45$ . Consider the graph: three triangles  $(x_1, x_2, y_1)$ ,  $(x_3, x_4, y_2)$ , and  $(x_5, x_6, y_1)$ . We add the edges  $(x_2, y_3)$ ,  $(x_3, y_1)$  and  $(x_5, y_2)$ . The optimal solution consists in executing the  $X$ -tasks into the  $Y$ -tasks; whereas the heuristic leads the solution in which three  $X$ -tasks are processed after the  $Y$ -tasks.

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