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Bounds and approximation results for scheduling coupled-tasks with compatibility constraints

R. Giroudeau¹, J.C König¹, B. Darties² and G. Simonin³

¹ LIRMM UMR 5506, 161 rue Ada 34392, Montpellier France
   {rgirou,konig}@lirmm.fr
² LE2I UMR6306, Univ. Bourgogne Franche-Comté, F-21000 Dijon, France
   benoit.darties@u-bourgogne.fr
³ Insight Centre for Data Analytics, University College Cork, Ireland
   gilles.simonin@insight-centre.org

Abstract. This article is devoted to propose some lower and upper bounds for the coupled-tasks scheduling problem in presence of compatibility constraints according to classical complexity hypothesis (P ≠ NP, ETH). Moreover, we develop an efficient polynomial-time approximation algorithm for the specific case for which the topology describing the compatibility constraints is a quasi split-graph.

Keywords: coupled-task, compatibility graph, complexity, approximation.

1 Introduction, motivations, model

We consider in this paper the coupled-task scheduling problem subject to compatibility constraints. The motivation of this model is related to data acquisition processes using radar sensors: a sensor emits a radio pulse (first sub-task \(a_i\)), and listen for an echo reply (second sub-task \(b_i\)). To make the notation less cluttered, the processing time of a sub-task will be denoted by \(a_i\) instead of \(p_{a_i}\) used in the theory of scheduling. Between these two instants (emission and reception), clearly there is an idle time \(L_i\) due to the propagation, in both sides, of the radio pulse. A coupled-task \((a_i, L_i, b_i)\), introduced by Shapiro (1980), is a natural way to model such data acquisition. This model has been widely studied in several works, i.e. Blazewicz et al. (2009). Other works proposed a generalization of this model by including compatibility constraints: scheduling a sub-task during the idle time of another requires that both tasks are compatible. The relations of compatibility are modeled by a compatibility graph \(G\), linking pair of compatible tasks only. This model is detailed in Simonin et al. (2012). In previous works, we studied the complexity of scheduling coupled-tasks with compatible constraints under several parameters like the size of the sub-tasks or the class of the compatibility graph (Simonin et al. 2013).

In this work, we propose original complexity and approximation results for the problem of scheduling \textit{stretched} coupled-task with compatibility constraints. A \textit{stretched} coupled-tasks \(i\) is a coupled-task having both sub-tasks processing time and idle time equal to a triplet \((\alpha(i), \alpha(i), \alpha(i))\), where \(\alpha(i)\) is the \textit{stretch factor} of the task \(i\) - one can apply a stretch factor \(\alpha(i)\) to a reference task \((1,1,1)\) to obtain \(i\).

The objective is to minimize the makespan \(C_{\text{max}}\). The input of the problem is a collection of coupled-tasks \(\mathcal{T} = \{t_1, t_2, \ldots, t_n\}\), a stretch factor function \(\alpha : \mathcal{T} \to \mathbb{N}\), and a compatibility graph \(G_c = (\mathcal{T}, E)\) where edge from \(E\) link pairs of compatible tasks only. When dealing with stretched coupled-tasks only, an edge \(\{x, y\} \in E\) exists if \(\alpha(x) = \alpha(y)\) (then \(x\) and \(y\) can be scheduled together without idle time as the idle time of one task is employed to schedule the sub-task of the other, thus we can schedule sequentially \(a_x, a_y, b_x, b_y\).
- or \( a_x, a_y, b_y, b_x \) - in \( \frac{4\alpha(x)}{3} \) time units), or if \( 3\alpha(x) \leq \alpha(y) \) (then \( x \) can be entirely executed during the idle time of \( y \) i.e. \( a_y, a_x, b_x, b_y \) and scheduling both tasks requires \( 3\alpha(y) \) time units). We note \( \#(X) \) the number of different stretch factors in a set of tasks \( X \), and we note \( d_G(x) \) the maximum degree of any vertex \( x \) in a graph \( G_c \).

We use the well-known Graham notation (Graham et al. 1979) to define the problems presented in this paper. In this work, we propose new complexity and inapproximability results when the compatibility graph is a restricted 1-stage bipartite graph \( G = (X, Y, E) \), i.e. a bipartite graph where edges are oriented from \( X \) to \( Y \) only. Then we show the problem is \( \mathcal{NP} \)-complete on a quasi-split graph \( G = (G_X, G_Y, E) \) even if \( \#(V(G_X)) = 1 \) and \( \#(V(G_Y)) = 1 \), but is 5/4-approximable.

## 2 Complexity and approximation results

**Theorem 1.** Deciding whether an instance of \( 1|\alpha, G_c = 1 - \text{stage} - \text{bipartite}, \#(X) = 2, \#(Y) = 1, d_{G_c}(X) \in \{1, 2\}, d_{G_c}(Y) \in \{3, 4\}|C_{\max} \) is a problem hard to approximate within \( 21 - \rho^{\text{Max-3DM}} \), where \( \rho^{\text{Max-3DM}} \) gives the upper bound for the Max-3DM. Since \( \rho^{\text{Max-3DM}} \leq \frac{140}{141} \), we obtain \( 1 + \frac{1}{2520} \).

**Proof.** We prove first that the problem is \( \mathcal{NP} \)-complete via a polynomial-time reduction. Based on this reduction, we apply the gap-preserving reduction.

The proof is based on a reduction from the maximum 3 dimensional matching (Max-3DM) (Garey & Johnson 1979): let \( A, B, \) and \( C \) be three disjoint sets of equal size, with \( n = |A| = |B| = |C| \), and a set \( T \subseteq A \times B \times C \) of triplet, with \( |T| = m \). The aim is to find a matching (set of mutually disjoint triplets) \( T^* \subseteq T \) of maximum size. This problem is well known to be \( \mathcal{NP} \)-complete. The restricted version of this problem in which each element of \( A \cup B \cup C \) appears exactly twice is denoted Max-3DM-2 and remains \( \mathcal{NP} \)-complete (Chlebik 2003). In this restricted version, we have \( m = 2n \).

We transform the instance of Max-3DM-2 to an instance of \( 1|\alpha, G_c = 1 - \text{stage} - \text{bipartite}, \#(X) = 2, \#(Y) = 1, d_{G_c}(X) \in \{1, 2\}, d_{G_c}(Y) \in \{3, 4\}|C_{\max} = 63n - 3k(1 - \epsilon) \) as follows: we define a set of tasks \( X \cup Y \) and model the compatibility constraint with a graph \( G_c = (X, Y, E) \). For each element \( x_i \in A \cup B \cup C \), we add an item coupled-task \( x_i \) into \( X \) with \( \alpha(x_i) = 1 \). For each triplet \( t_i \in T \), we add a box coupled-task \( t_i \) to \( Y \) with \( \alpha(t_i) = 9 \), and an item coupled-task \( t'_i \) with \( \alpha(t'_i) = 2 + \epsilon \). For each \( t_i \in T \), we add the compatibility arc \( (x_i, t_i) \) to \( E \). We also add the compatibility arc \( (t'_i, t_i) \) to \( E \).

So, the set of \( X \)-tasks (resp. \( Y \)-tasks) are constituted by item coupled-task \( x_i \) and \( t'_i \) (resp. box coupled-task).

Clearly we have \( m \) box coupled-tasks (each with an idle time of 9 units) of degree 4 in \( G_c \), \( m \) item coupled-tasks with stretch factor \( 2 + \epsilon \) of degree 1 in \( G_c \), and \( 3n \) item coupled-tasks with stretch factor 1 of degree 2 in \( G_c \). Moreover \( G_c \) is a bipartite graph. The reduction is constructed in polynomial time.

It exists a schedule of length \( 63n - 3k(1 - \epsilon) \) iff it exists a matching of size \( k \) for Max-3DM-2 instance.

Hereafter, we propose some negative results concerning the existence of subexponential-time algorithms under the following complexity-theoretic hypothesis that is known as the Exponential-Time Hypothesis (see (Woeginger 2001) for a survey on exact algorithms for \( \mathcal{NP} \)-hard problems) for stretched coupled-tasks, and other ones previously studied.

Recall first the Exponential-Time Hypothesis ((Impagliazzo & Paturi 2001), and (Impagliazzo et al. 2001)): there exists a constant \( c > 1 \) such that there exists no algorithm for \( 3 \)-Satisfiability that uses only \( O(c^l) \) time where \( l \) denotes the number of variables.
Corollary 1. Assuming the Exponential-Time Hypothesis, there exists no algorithm with a worst-case running time that is subexponential in $n$ (the number of vertices), i.e.:

1. For the problem $1|a_i = b_i = p, L_i = 2p, G_c|C_{max}$ problem in $O(2^{o(n)})$ time
2. For the problem $1|a_i = a, b_i = b, L_i = a + b, G_c|C_{max}$ in $O(2^{o(n)})$ time
3. $1|\alpha, G_c = 1 - \text{bipartite}|C_{max}$ in $O(2^{o(n)})$-time algorithm.

Proof. 1. For the problem $1|a_i = b_i = p, L_i = 2p, G_c|C_{max}$: In (van Rooij et al. 2013), the authors proved that for PARTITION INTO TRIANGLES on graphs of maximum degree four, there is no algorithm with a worst-case running time $O(2^{o(n)})$ that is subexponential in $n$. Therefore, we transform a PARTITION INTO TRIANGLES instance with $n$ vertices and $m$ edges into an equivalent instance $G_c$ for bounded degree at most four. Since the transformation is linear (see (Simonin et al. 2012)) the result holds.

2. For the problem $1|a_i = a, b_i = b, L_i = a + b, G_c|C_{max}$: In (Lokshtanov, Marx & Saurabh 2011) the authors proved that for HAMILTONIAN PATH there is no $O(2^{o(n)})$-time algorithm. As the same way as previously the transformation is linear (see (Simonin et al. 2012)).

3. $1|\alpha, G_c = 1 - \text{bipartite}|C_{max}$: In (Chen, Jansen & Zhang 2014), the authors proved that for MAX 3DM, there is no $O(2^{o(n)})$-time algorithm, therefore this result is transposed to the scheduling problem using the first part of the proof of Theorem 1.

Theorem 2. Scheduling stretched coupled task in presence of a quasi split graph is a $\mathcal{NP}$-complete problem even if $\#(V(G_X)) = 1$ and $\#(V(G_Y)) = 1$

Proof. The proof is based on a reduction from a variant of the well-known $\mathcal{NP}$-complete PARTITION INTO TRIANGLES. This problem consists to ask if the vertices of a graph $G = (V, E)$, with $|V| = 3q, q \in \mathbb{N}$, can be partitioned into $q$ disjoint sets $T_1, T_2, \ldots, T_q$, each containing exactly three vertices, such that for each $T_i = \{u_i, v_i, w_i\}, 1 \leq i \leq q$, all three of the edges $\{u_i, v_i\}, \{u_i, w_i\}, \{w_i, v_i\}$ belong to $E$.

The problem PARTITION INTO TRIANGLES remains $\mathcal{NP}$-complete even if the graph $G$ can be partitioned into three sets with the same size, $A$, $B$ and $C$ such that each set is an independent set (Morandini, M. 2004). The polynomial-time transformation is based on this variant. Let $G = (A \cup B \cup C, E)$ be an instance of the variant of PARTITION INTO TRIANGLES. We consider the split-graph $G' = (A \cup B, C, E')$ obtained as follows:

\[ \forall v \in A \text{ (resp. } B), \text{ we create a vertex } A_v \text{ (resp. } B_v) \text{ with processing time (1, 1, 1). Moreover, } \forall v \in C \text{ we create a task } C_v \text{ with processing time (4, 4, 4). The edges between } A \text{ and } B \text{ remained the same as the } G' \text{ whereas the edge between } A \cup B \text{ and } C \text{ are oriented. Finally in order to have a connected graph, we add two news vertices (resp. one) } z_0 \text{ and } z_1 \text{ (resp. } z_2 \text{ with processing time equal to (1, 1, 1) (resp. (4, 4, 4)). We add edges between } z_0 \text{ to } A_v \text{ (resp. } z_1 \text{ to } B_v). Lastly, we add the three edges } \{z_0, z_2\}, \{z_1, z_2\} \text{ and } \{z_0, z_1\} \text{. Notice that the graph } A_v \cup B_v \text{ form a bipartite graph. The problem is clearly in } \mathcal{NP}. \text{ It exists a positive solution for the variant of PARTITION INTO TRIANGLES a valid schedule of length } 12 \times (|C| + 1) \text{ exists. It is sufficient to execute the two tasks } A_v \text{ and } B_v \text{ in four units of time into a task } C_u. \text{ The problem is 5/4-approximable on quasi split-graph where } \#(V(G_Y)) = 1.

Proof. W.l.o.g., we suppose that the processing time of $X$-tasks (resp. $Y$-tasks) is $(1, 1, 1)$ (resp. $a(y_i)$). Indeed, if $a(x) > 1$, we put $a(y_i) = \lceil \frac{a(y_i)}{a(x)} \rceil$ and $a(x) = 1$.

Algorithm: we transform the problem into an oriented maximum flow problem between $G_X$ and $G_Y$ with two sources $s$ and $t$, with $\omega(s, x) = \omega(x, y) = 1$ and $\omega(y, t) = \lceil \frac{a(y_i)}{a(x)} \rceil, \forall y_i \in Y, \forall x \in XG_Y$ where $\omega(i, j)$ is the capacity of an arc $(i, j)$. After the computation of a maximum flow $F$ of value $f$, for the uncovered remaining $X$-tasks a maximum
M-matching ([|M| = m]) is applied. The schedule consists in processing first, the Y-tasks with X-tasks inside. The M-tasks are executed after. Lastly, we schedule s isolated-tasks.

The length of schedule given by the algorithm is 
\[ C_{\text{max}} \leq \sum_{y_i \in Y} 3\alpha(y_i) + 4m + 3s \] 
with 
\[ 2m + s + f = n = |X| \] 
and 
\[ \sum_{y_i \in Y} 3\alpha(y_i) \geq 9f. \] 
In similar way, the optimal length is 
\[ C^*_{\text{max}} \geq \sum_{y_i \in Y} \alpha(y_i) + 4m^* + 3s^*. \] 
We suppose that in Y-tasks where are p*-edges processed and r*-isolated-tasks, then we obtain 
\[ 2(p^* + m^*) + r^* + s^* = n, \] 
\[ p^* + r^* \leq f, \] 
and 
\[ \sum_{y_i \in Y} \alpha(y_i) \geq 12p^* + 9r^*. \] 
In the worst-case, the p*-edges are split into two tasks (so p* news tasks are added to s*), and also the matched-edges are split (for each m* edges one task is executed into the Y-task, instead of one of r*-tasks). Therefore, 2m* tasks are added to the s-value. In the worst case, we have 
\[ m^* = r^*, \] 
\[ s = s^* + p^* + 2r^* \] 
and 
\[ m = 0. \] 
In such case, 
\[ C_{\text{max}} \leq 12p^* + 9r^* + 3s^* + 3p^* + 6r^* \] 
and 
\[ C^*_{\text{max}} = 12p^* + 9r^* + 4r^* + 3s^*. \] 
Thus 
\[ \rho = \frac{15p^* + 15r^* + 3s^*}{12p^* + 13r^* + 3s^*} \leq \max(\frac{5}{4}, 15/13, 1) = 5/4. \]

**Tightness:** it exists an example for the 
\[ C^*_{\text{max}} = 36, \] 
and for the heuristic 
\[ C_{\text{max}} = 45. \] 
Consider the graph: three triangles \((x_1, x_2, y_1), (x_3, x_4, y_2), \) and \((x_5, x_6, y_1)\). We add the edges \((x_2, y_3), (x_3, y_1)\) and \((x_5, y_2)\). The optimal solution consists in executing the X-tasks into the Y-tasks; whereas the heuristic leads the solution in which three X-tasks are processed after the Y-tasks.

**References**


