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Bounds and approximation results for scheduling coupled-tasks with compatibility constraints

R. Giroudeau¹, J.C König¹, B. Darties² and G. Simonin³

¹ LIRMM UMR 5506, 161 rue Ada 34392, Montpellier France
{rgirou,konig}@lirmm.fr
² LE2I UMR6306, Univ. Bourgogne Franche-Comté, F-21000 Dijon, France
benoit.darties@u-bourgogne.fr
³ Insight Centre for Data Analytics, University College Cork, Ireland
gilles.simonin@insight-centre.org

Abstract. This article is devoted to propose some lower and upper bounds for the coupled-tasks scheduling problem in presence of compatibility constraints according to classical complexity hypothesis (\( P \neq NP, ETH \)). Moreover, we develop an efficient polynomial-time approximation algorithm for the specific case for which the topology describing the compatibility constraints is a quasi split-graph.

Keywords: coupled-task, compatibility graph, complexity, approximation.

1 Introduction, motivations, model

We consider in this paper the coupled-task scheduling problem subject to compatibility constraints. The motivation of this model is related to data acquisition processes using radar sensors: a sensor emits a radio pulse (first sub-task \( a_i \)), and listen for an echo reply (second sub-task \( b_i \)). To make the notation less cluttered, the processing time of a sub-task will be denoted by \( a_i \) instead of \( p_{a_i} \) used in the theory of scheduling. Between these two instants (emission and reception), clearly there is an idle time \( L_i \) due to the propagation, in both sides, of the radio pulse. A coupled-task \((a_i, L_i, b_i)\), introduced by Shapiro (1980), is a natural way to model such data acquisition. This model has been widely studied in several works, i.e. Blażewicz et al. (2009). Other works proposed a generalization of this model by including compatibility constraints: scheduling a sub-task during the idle time of another requires that both tasks are compatible. The relations of compatibility are modeled by a compatibility graph \( G \), linking pair of compatible tasks only. This model is detailed in Simonin et al. (2012). In previous works, we studied the complexity of scheduling coupled-tasks with compatible constraints under several parameters like the size of the sub-tasks or the class of the compatibility graph (Simonin et al. 2013).

In this work, we propose original complexity and approximation results for the problem of scheduling stretched coupled-task with compatibility constraints. A stretched coupled-tasks \( i \) is a coupled-task having both sub-tasks processing time and idle time equal to a triplet \( (\alpha(i), \alpha(i), \alpha(i)) \), where \( \alpha(i) \) is the stretch factor of the task \( i \) - one can apply a stretch factor \( \alpha(i) \) to a reference task \((1,1,1)\) to obtain \( i \).

The objective is to minimize the makespan \( C_{max} \). The input of the problem is a collection of coupled-tasks \( \mathcal{T} = \{t_1, t_2, \ldots, t_n\} \), a stretch factor function \( \alpha : \mathcal{T} \rightarrow \mathbb{N} \), and a compatibility graph \( G_c = (\mathcal{T}, E) \) where edge from \( E \) link pairs of compatible tasks only. When dealing with stretched coupled-tasks only, a edge \( \{x, y\} \in E \) exists if \( \alpha(x) = \alpha(y) \) (then \( x \) and \( y \) can be scheduled together without idle time as the idle time of one task is employed to schedule the sub-task of the other, thus we can schedule sequentially \( a_x, a_y, b_x, b_y \).
The reduction is constructed in polynomial time. 

- or \( a_x, b_x, b_y, b_z \) in \( \frac{4\alpha(x)}{3} \) time units), or if \( 3\alpha(x) \leq \alpha(y) \) (then \( x \) can be entirely executed during the idle time of \( y \) i.e. \( a_y, a_x, b_x, b_y \) and scheduling both tasks requires \( 3\alpha(y) \) time units). We note \( \#(X) \) the number of different stretch factors in a set of tasks \( X \), and we note \( d_G(X) \) the maximum degree of any vertex \( x \in X \) in a graph \( G_c \).

We use the well-known Graham notation (Graham et al. 1979) to define the problems presented in this paper. In this work, we propose new complexity and inapproximability results when the compatibility graph is a restricted 1-stage bipartite graph \( G = (X, Y, E) \), i.e. a bipartite graph where edges are oriented from \( X \) to \( Y \) only. Then we show the problem is \( \mathcal{NP} \)-complete on a quasi-split graph \( G = (G_X, G_Y, E) \) even if \( \#(V(G_X)) = 1 \) and \( \#(V(G_Y)) = 1 \), but is 5/4-approximable.

2 Complexity and approximation results

**Theorem 1.** Deciding whether an instance of \( 1|\alpha, G_c = 1 - \text{stage - bipartite}, \#(X) = 2, \#(Y) = 1, d_{G_c}(X) \in \{1, 2\}, d_{G_c}(Y) \in \{3, 4\}|C_{\max} \) is a problem hard to approximate within \( 21 - \rho_{\text{MAX-3DM-2}} \leq \rho \), where \( \rho_{\text{MAX-3DM-2}} \) gives the upper bound for the MAX-3DM. Since \( \rho_{\text{MAX-3DM-2}} \leq \frac{149}{140} \), we obtain \( 1 + \frac{1}{2520} \).

**Proof.** We prove first that the problem is \( \mathcal{NP} \)-complete via a polynomial-time reduction. Based on this reduction, we apply the gap-preserving reduction.

The proof is based on a reduction from the maximum 3 DIMENSIONAL MATCHING (MAX-3DM) (Garey & Johnson 1979): let \( A, B, \) and \( C \) be three disjoint sets of equal size, with \( n = |A| = |B| = |C| \), and a set \( T \subseteq A \times B \times C \) of triplet, with \( |T| = m \). The aim is to find a matching (set of mutually disjoint triplets) \( T^* \subseteq T \) of maximum size. This problem is well known to be \( \mathcal{NP} \)-complete. The restricted version of this problem in which each element of \( A \cup B \cup C \) appears exactly twice is denoted MAX-3DM-2 and remains \( \mathcal{NP} \)-complete (Chlebik 2003). In this restricted version, we have \( m = 2n \).

We transform the instance of MAX-3DM-2 to an instance of \( 1|\alpha, G_c = 1 - \text{stage bipartite}, \#(X) = 2, \#(Y) = 1, d_{G_c}(X) \in \{1, 2\}, d_{G_c}(Y) \in \{3, 4\}|C_{\max} = 63n - 3k(1 - \epsilon) \) as follows: we define a set of tasks \( X \cup Y \) and model the compatibility constraint with a graph \( G_c = (X, Y, E) \). For each element \( x_i \in A \cup B \cup C \), we add an item coupled-task \( x_i \) into \( X \) with \( \alpha(x_i) = 1 \). For each triplet \( t_i \in T \), we add a box coupled-task \( t_i \) to \( Y \) with \( \alpha(t_i) = 9 \), and an item coupled-task \( \prime \) with \( \alpha(\prime) = 2 + \epsilon \). For each \( \prime \in T \) and each \( x_i \in t_i \), we add the compatibility arc \( (x_i, t_i) \) to \( E \). We also add the compatibility arc \( (\prime, t_i) \) to \( E \). So, the set of \( X \)-tasks (resp. \( Y \)-tasks) are constituted by item coupled-task \( x_i \) and \( \prime \) (resp. box coupled-task).

Clearly we have \( m \) box coupled-tasks (each with an idle time of 9 units) of degree 4 in \( G_c \), \( m \) item coupled-tasks with stretch factor 2 + \( \epsilon \) of degree 1 in \( G_c \), and \( 3n \) item coupled-tasks with stretch factor 1 of degree 2 in \( G_c \). Moreover \( G_c \) is a bipartite graph. The reduction is constructed in polynomial time.

It exists a schedule of length \( 63n - 3k(1 - \epsilon) \) if it exists a matching of size \( k \) for MAX-3DM-2 instance.

Hereafter, we propose some negative results concerning the existence of subexponential-time algorithms under the following complexity-theoretic hypothesis that is known as the Exponential-Time Hypothesis (see (Woeginger 2001) for a survey on exact algorithms for \( \mathcal{NP} \)-hard problems) for stretched coupled-tasks, and other ones previously studied.

Recall first the EXPONENTIAL-TIME HYPOTHESIS ((Impagliazzo & Paturi 2001), and (Impagliazzo et al. 2001)): there exists a constant \( c > 1 \) such that there exists no algorithm for 3-Satisfiability that uses only \( O(c^l) \) time where \( l \) denotes the number of variables.

\(^4\) A quasi split graph is a connected graph \( G = (G_X, G_Y, E) \), with \( G_X \) a connected non-oriented graph (not complete) and \( G_Y \) an independent set. The other arcs are oriented from \( X \) to \( Y \) only.
**Corollary 1.** Assuming the Exponential-Time Hypothesis, there exists no algorithm with a worst-case running time that is subexponential in $n$ (the number of vertices), i.e.:

1. For the $1|a_i = b_i = p, L_i = 2p, G_e|C_{max}$ problem in $O(2^{O(n)})$ time.
2. For $1|a_i = a, b_i = b, L_i = a + b, G_e|C_{max}$ in $O(2^{O(n)})$ time.
3. $1|\alpha, G_e = 1 - \text{bipartite}|C_{max}$ in $O(2^{o(n)})$-time algorithm.

**Proof.** 1. For $1|a_i = b_i = p, L_i = 2p, G_e|C_{max}$: In (van Rooij et al. 2013), the authors proved that for PARTITION INTO TRIANGLES on graphs of maximum degree four, there is no algorithm with a worst-case running time $O(2^{o(n)})$ that is subexponential in $n$.

Therefore, we transform a PARTITION INTO TRIANGLES instance with $n$ vertices and $m$ edges into an equivalent instance $G_e$ for bounded degree at most four. Since the transformation is linear (see (Simonin et al. 2012)) the result holds.

2. For the problem $1|a_i = a, b_i = b, L_i = a + b, G_e|C_{max}$: In (Lokshin, Marx & Saurabh 2011) the authors proved that for HAMILTONIAN PATH there is no $O(2^{o(n)})$-time algorithm. As the same way as previously the transformation is linear (see (Simonin et al. 2012)).

3. $1|\alpha, G_e = 1 - \text{bipartite}|C_{max}$: In (Chen, Jansen & Zhang 2014), the authors proved that for MAX 3DM, there is no $O(2^{o(n)})$-time algorithm, therefore this result is transposed to the scheduling problem using the first part of the proof of Theorem 1.

**Theorem 2.** Scheduling stretched coupled task in presence of a quasi split graph is a $\mathcal{NP}$-complete problem even if $\#(V(G_X)) = 1$ and $\#(V(G_Y)) = 1$.

**Proof.** The proof is based on a reduction from a variant of the well-known $\mathcal{NP}$-complete PARTITION INTO TRIANGLES. This problem consists to ask if the vertices of a graph $G = (V, E)$, with $|V| = 3q, q \in \mathbb{N}$, can be partitioned into $q$ disjoint sets $T_1, T_2, \ldots, T_q$, each containing exactly three vertices, such that for each $T_i = \{u_i, v_i, w_i\}, 1 \leq i \leq q$, all three of the edges $\{u_i, v_i\}, \{u_i, w_i\}, \{w_i, v_i\}$ belong to $E$.

The problem PARTITION INTO TRIANGLES remains $\mathcal{NP}$-complete even if the graph $G$ can be partitioned into three sets with the same size, $A, B$ and $C$ such that each set is an independent set (Morandini, M. 2004). The polynomial-time transformation is based on this variant. Let $G = (A \cup B \cup C, E)$ be an instance of the variant of PARTITION INTO TRIANGLES. We consider the split-graph $G' = (A \cup B, C, E')$ obtained as follows:

- $\forall v \in A$ (resp. $B$), we create a vertex $A_v$ (resp. $B_v$) with processing time $(1,1,1)$.
- Moreover, $\forall v \in C$ we create a task $C_v$ with processing time $(4,4,4)$. The edges between $A$ and $B$ remained the same as the $G'$ whereas the edges between $A \cup B$ and $C$ are oriented.
- Finally in order to have a connected graph, we add two new vertices (resp. one) $z_0$ and $z_1$ (resp. $z_2$) with processing time equal to $(1,1,1)$ (resp. $(4,4,4)$). We add edges between $z_0$ to $A_v$ (resp. $z_1$ to $B_v$). Lastly, we add the three edges $(z_0, z_2), (z_1, z_2)$ and $(z_0, z_1)$.

Notice that the graph $A_v \cup B_v$ forms a bipartite graph. The problem is clearly in $\mathcal{NP}$. It exists a positive solution for the variant of PARTITION INTO TRIANGLES if and only if a valid schedule of length $12 \times (|C| + 1)$ exists. It is sufficient to execute the two tasks $A_v$ and $B_v$ in four units of time into a task $C_v$.

**Theorem 3.** The problem is $5/4$-approximable on quasi split-graph where $\#(V(G_Y)) = 1$.

**Proof.** W.l.o.g., we suppose that the processing time of $X$-tasks (resp. $Y$-tasks) is $(1,1,1)$ (resp. $\alpha(y_i)$). Indeed, if $\alpha(x) > 1$, we put $\alpha(y_i) = \left\lfloor \frac{\alpha(y_i)}{\alpha(x)} \right\rfloor$ and $\alpha(x) = 1$.

**Algorithm:** We transform the problem into an oriented maximum flow problem between $G_X$ and $G_Y$ with two sources $s$ and $t$, with $\omega(s, x) = \omega(x, y) = 1$ and $\omega(y, t) = \left\lfloor \frac{\alpha(y)}{\alpha(x)} \right\rfloor$. $v_{yi} \in Y, \forall x \in XGY$ where $\omega(i, j)$ is the capacity of an arc $(i, j)$. After the computation of a maximum flow $F$ of value $f$, for the uncovered remaining $X$-tasks a maximum
The length of schedule given by the algorithm is $C_{\text{max}} \leq \sum_{y_i \in Y} 3\alpha(y_i) + 4m + 3s$ with $2m + s + f = n = |X|$ and $\sum_{y_i \in Y} 3\alpha(y_i) \geq 9f$. In similar way, the optimal length is $C^{\ast}_{\text{max}} \geq \sum_{y_i \in Y} \alpha(y_i) + 4m^{\ast} + 3s^{\ast}$. We suppose that in $Y$-tasks where are $p^{\ast}$-edges processed and $r^{\ast}$ isolated-tasks, then we obtain $2(p^{\ast} + m^{\ast}) + r^{\ast} + s^{\ast} = n$, $p^{\ast} + r^{\ast} \leq f$, and $\sum_{y_i \in Y} \alpha(y_i) \geq 12p^{\ast} + 9r^{\ast}$. In the worst-case, the $p^{\ast}$-edges are split into two tasks (so $p^{\ast}$ news tasks are added to $s^{\ast}$), and also the matched-edges are split (for each $m^{\ast}$ edges one task is executed into the $Y$-task, instead of one of $r^{\ast}$-tasks). Therefore, $2m^{\ast}$ tasks are added to the $s$-value. In the worst case, we have $m^{\ast} = r^{\ast}$, $s = s^{\ast} + p^{\ast} + 2r^{\ast}$ and $m = 0$. In such case, $C_{\text{max}} \leq 12p^{\ast} + 9r^{\ast} + 3s^{\ast} + 3p^{\ast} + 6r^{\ast}$ and $C^{\ast}_{\text{max}} = 12p^{\ast} + 9r^{\ast} + 4r^{\ast} + 3s^{\ast}$. Thus $\rho \leq \frac{15p^{\ast} + 15r^{\ast} + 3s^{\ast}}{12p^{\ast} + 13r^{\ast} + 3s^{\ast}} \leq \max(5/4, 15/13, 1) = 5/4$.

**Tightness:** it exists an example for the $C^{\ast}_{\text{max}} = 36$, and for the heuristic $C_{\text{max}} = 45$. Consider the graph: three triangles $(x_1, x_2, y_1)$, $(x_3, x_4, y_2)$, and $(x_5, x_6, y_1)$. We add the edges $(x_2, y_3)$, $(x_3, y_1)$ and $(x_5, y_2)$. The optimal solution consists in executing the $X$-tasks into the $Y$-tasks; whereas the heuristic leads the solution in which three $X$-tasks are processed after the $Y$-tasks.

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