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# Fractional triangle decompositions in graphs with large minimum degree

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## Abstract

A triangle decomposition of a graph is a partition of its edges into triangles. A fractional triangle decomposition of a graph is an assignment of a non-negative weight to each of its triangles such that the sum of the weights of the triangles containing any given edge is one. We prove that every graph on  $n$  vertices with minimum degree at least  $0.9n$  has a fractional triangle decomposition. This improves a result of Garaschuk that the same conclusion holds for graphs with minimum degree at least  $0.956n$ . Together with a recent result of Barber, Kühn, Lo and Osthus, this implies that for all  $\epsilon > 0$ , every large enough triangle divisible graph on  $n$  vertices with minimum degree at least  $(0.9 + \epsilon)n$  admits a triangle decomposition.

## 1 Introduction

Decomposition and packing problems are central and classical problems in combinatorics, and, in particular, in design theory. Kirkman's theorem [8] from the middle of 19th century gives a necessary and sufficient condition on the existence of a Steiner triple system with a certain number of elements. In the language of graph theory, Kirkman's result asserts that every complete graph with an odd number of vertices and a number of edges divisible by three can be decomposed into triangles. Note that if a graph can be decomposed into triangles, then its vertex degrees are even and the total number of edges is divisible by three. Barber, Kühn, Lo and Osthus [2] showed that the same conclusion is true for large graphs satisfying these divisibility conditions if their minimum degree is not too far from the number of their vertices. In this short paper, we study the fractional variant of the problem and we use it, together with a result of Barber, Kühn, Lo and Osthus [2], to improve the best known bound.

Let us fix the terminology we are going to use. A *graph* is a pair of sets  $(V, E)$  such that elements of  $E$  are unordered pairs of elements of  $V$ . The elements of  $V$  are called *vertices* and the elements of  $E$  are called *edges*. We denote by  $uv$  (or  $vu$ ) the edge with vertices  $u$  and  $v$ . We denote by  $|G|$  the number of vertices of  $G$ . Two distinct vertices contained in the same edge are said to be *adjacent* or to be *neighbours*. Two edges that share a vertex are said to be *adjacent*. The *degree* of a vertex  $v$  is equal to the number

of neighbours of  $v$ . Let  $\gcd(G)$  denote the greatest common divisor of the degrees of the vertices of  $G$ .

Two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are *isomorphic* if there exists a bijection  $b$  from  $V_1$  to  $V_2$  such that  $uv$  is an edge of  $G_1$  if and only if  $b(u)b(v)$  is an edge of  $G_2$  for every two vertices  $u$  and  $v$  of  $G_1$ . The *complete graph*  $K_k$  is the graph with  $k$  vertices all mutually adjacent. The graph  $K_3$  is also called a *triangle*. A graph  $G_1 = (V_1, E_1)$  is a *subgraph* of  $G_2 = (V_2, E_2)$  if  $V_1 \subseteq V_2$  and  $E_1 \subseteq E_2$ . The subgraphs of  $G_2$  isomorphic to  $G_1$  will be referred to as copies of  $G_1$ .

Let  $H$  be a graph. An  $H$ -*decomposition* of a graph  $G$  is a set of subgraphs of  $G$  isomorphic to  $H$  that are edge disjoint such that each edge of  $G$  is contained in one of them. A graph is  $H$ -*decomposable* if it admits an  $H$ -decomposition. A  $K_3$ -decomposition is also called a *triangle decomposition* and a graph is *triangle decomposable* if it is  $K_3$ -decomposable. A graph  $G$  is  $H$ -*divisible* if  $\gcd(G)$  is a multiple of  $\gcd(H)$  and the number of edges of  $G$  is a multiple of the number of edges of  $H$ . It is easy to see that every  $H$ -decomposable graph is  $H$ -divisible. However, the converse is not true. For example a cycle on six vertices is  $K_3$ -divisible, but not  $K_3$  decomposable. As noted previously, Kirkman [8] proved that every  $K_3$ -divisible complete graph is  $K_3$ -decomposable. The fact that, for all  $H$ , every  $H$ -divisible complete graph is  $H$ -decomposable remained an open problem for over one hundred years before it was solved by Wilson [10].

The first generalisation to graphs that are near complete is due to Gustavsson [6]. He proved that for every graph  $H$ , there exist  $n_0(H)$  and  $\epsilon(H)$  such that every  $H$ -divisible graph with  $n \geq n_0(H)$  vertices and minimum degree at least  $(1 - \epsilon(H))n$  is  $H$ -decomposable. This has been generalised to hypergraphs in a recent result of Keevash [7]. The best that is known to date for a general graph  $H$  is due to Barber, Kühn, Lo and Osthus [2], who gave a way to turn a fractional decomposition into an exact one. They proved the following: Let  $F$  be a graph with minimum degree  $\delta(F)$  and let  $H$  be a graph with chromatic number  $\chi(H)$  and  $e(H)$  edges. Let  $C := \min\{9\chi(H)^2(\chi(H) - 1)^2/2, 10^4\chi(H)^{3/2}\}$  and let  $t := \max\{C, 6e(H)\}$ . Then for all  $\epsilon > 0$ , there exists an  $n_0$  such that every  $H$ -divisible graph  $G$  on  $n \geq n_0$  vertices with  $\delta(G) \geq (1 - 1/t + \epsilon)n$  has an  $H$ -decomposition. For some particular classes of graphs, the exact asymptotic minimum degree threshold is known [2][11].

A *fractional  $H$ -decomposition* of a graph  $G$  is an assignment of non-negative weights to the copies of  $H$  in  $G$  such that for an edge  $e$ , the sum of the weights of the copies of  $H$  that contain  $e$  is equal to one. A graph is *fractionally  $H$ -decomposable* if it admits a fractional  $H$ -decomposition. A graph can be fractionally  $H$ -decomposable without being  $H$ -divisible. A fractional  $K_3$ -decomposition is also called a *fractional triangle decomposition* and a graph is *fractionally triangle decomposable* if it is fractionally  $K_3$ -decomposable. For all  $r \geq 2$ , Yuster [12] proved that every graph on  $n$  vertices with minimum degree at least  $(1 - \frac{1}{9r^{10}})n$  is fractionally  $K_r$ -decomposable, and Dukes [3][4] proved that the same result holds for sufficiently large graphs on  $n$  vertices with minimum degree at least  $(1 - \frac{2}{9r^2(r-1)^2})n$ . Our paper already led to further research: Barber, Kühn, Lo, Montgomery and Osthus [1] proved, building on our combinatorial approach, that every graph on  $n \geq 10^4r^3$  vertices with minimum degree at least  $(1 - 1/10^4r^{3/2})n$  has a fractional  $K_r$ -decomposition.

In this paper we will focus on triangle decompositions of graphs with large minimum degree. The following conjecture is due to Nash-Williams [9]:

**Conjecture 1** (Nash-Williams [9]). *Let  $G$  be a  $K_3$ -divisible graph with  $n$  vertices and minimum degree at least  $\frac{3}{4}n$ . If  $n$  is large enough, then  $G$  is  $K_3$ -decomposable.*

The best result towards a proof of Conjecture 1 is due to the combination of results of Garaschuk [5] and Barber, Kühn, Lo and Osthus [2].

**Theorem 2** (Garaschuk [5], Barber, Kühn, Lo and Osthus [2]). *There exists an  $n_0$  such that every  $K_3$ -divisible graph  $G$  on  $n \geq n_0$  vertices with minimum degree at least  $0.956n$  is  $K_3$ -decomposable.*

The proof of Theorem 2 relies on a result on fractional  $K_3$ -decomposability, which we now state. The following appears as a conjecture in [5]. Note that for  $K_3$ -divisible graphs, this is a consequence of Conjecture 1.

**Conjecture 3** (Garaschuk [5]). *Let  $G$  be a graph with  $n$  vertices and minimum degree at least  $\frac{3}{4}n$ . If  $n$  is large enough, then  $G$  is fractionally  $K_3$ -decomposable.*

The best known result towards proving Conjecture 1 was established by Garaschuk [5].

**Theorem 4** (Garaschuk [5]). *Let  $G$  be a graph with  $n$  vertices and minimum degree at least  $0.956n$ . The graph  $G$  admits a fractional triangle decomposition.*

In this paper we use a different method to prove the following.

**Theorem 5.** *Every graph with  $n$  vertices and minimum degree at least  $\frac{9}{10}n$  admits a fractional triangle decomposition.*

In [2], a particular case of Theorem 11.1 and Lemma 12.3 imply the following.

**Theorem 6** (Barber, Kühn, Lo and Osthus [2]). *Suppose there exist  $n_0$  and  $\delta$  such that every graph on  $n \geq n_0$  vertices with minimum degree at least  $\delta n$  is fractionally  $K_3$ -decomposable. For all  $\epsilon > 0$ , there exists  $n_1$  such that every  $K_3$ -divisible graph on  $n \geq n_1$  vertices with minimum degree at least  $(\max(\delta, \frac{3}{4}) + \epsilon)n$  is  $K_3$ -decomposable.*

Together with Theorem 6, our result improves Theorem 2.

**Theorem 7.** *Let  $\epsilon > 0$ . There exists an  $n_0$  such that every  $K_3$ -divisible graph on  $n \geq n_0$  vertices with minimum degree at least  $(\frac{9}{10} + \epsilon)n$  is  $K_3$ -decomposable.*

## 2 Proof of Theorem 5

Let  $\delta = \frac{1}{10}$ . Fix a graph  $G$  with  $n$  vertices and minimum degree at least  $(1 - \delta)n$ . Suppose the graph  $G$  has at least one triangle with three vertices of degree at least  $(1 - \delta)n + 2$ . Let  $G'$  be the graph  $G$  where the edges of one such triangle are removed. Observe that  $G'$  has minimum degree at least  $(1 - \delta)n$  and that if  $G'$  has a fractional triangle decomposition, then  $G$  has one too. By doing this operation several times, we can assume that  $G$  has no triangle with three vertices of degree at least  $(1 - \delta)n + 2$ . Let  $m$  be the number of edges of  $G$ .

Initially, we give the same weight  $w_\Delta$  to every triangle such that the sum of the weights of the triangles is equal to  $\frac{m}{3}$ . We will modify the weights of the triangles to obtain a fractional triangle decomposition. We will do so in a way that the total sum of the weights is preserved.

We define the weight of an edge  $e$  to be the sum of the weights of the triangles that contain  $e$ . Given  $H$  a copy of  $K_4$  in  $G$ , and two non-adjacent edges  $e_1$  and  $e_2$  in  $H$ , let

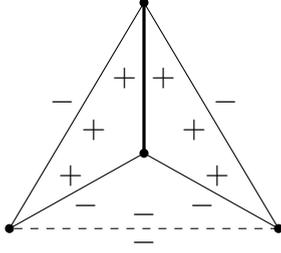


Figure 1: By removing some weight  $w$  from the two triangles containing the thick edge and adding  $w$  to the two triangles containing the dashed edge, we remove  $2w$  from weight of the dashed edge and add  $2w$  to the weight of the thick edge.

us call  $(H, \{e_1, e_2\})$  a *rooted*  $K_4$  of  $G$ . We will use the following procedure to modify the weights of the edges of a rooted  $K_4$  of  $G$ :

*Let  $(H, \{e_1, e_2\})$  be a rooted  $K_4$  of  $G$ . By removing a weight  $w$  from the two triangles of  $H$  that contain  $e_1$  and adding the same weight  $w$  to each of the other two triangles (i.e. those that contain  $e_2$ ), we transfer a weight of  $2w$  from  $e_1$  to  $e_2$ . The weights of all the other edges of the graph remain unchanged (see Figure 1).*

To prevent the weight of any triangle from becoming negative, we have to restrict how much weight we can transfer using the procedure above. If for some  $w$  we use the procedure to transmit a weight of  $2w$  from an edge to another one, then any triangle's weight is lowered by at most  $w$  for triangles that are in the  $K_4$ , and does not change for other triangles. Moreover, since every triangle contains a vertex with degree at most  $(1 - \delta)n + 1$ , any triangle is in at most  $(1 - \delta)n - 1$  copies of  $K_4$ , and thus in at most  $3(1 - \delta)n - 3$  rooted  $K_4$  (since for each  $K_4$  there are three possible choices for the pair of edges). Since each triangle has an initial weight of  $w_\Delta$ , if each rooted  $K_4$  containing that triangle is used to transfer weight of at most  $\frac{2w_\Delta}{3(1-\delta)n-3}$  between its labelled edges, then its final weight will be non-negative.

For each edge  $e$ , let  $T_e$  be the number of triangles of  $G$  containing  $e$ . We express redistributing the weights as a flow problem in an auxiliary graph, which is denoted by  $\widehat{G}$ . The vertices of  $\widehat{G}$  are the edges of  $G$ , plus two special vertices, called the *supersource* and the *supersink*. Two edges of  $G$  are adjacent as vertices in  $\widehat{G}$  if they form a pair in a rooted  $K_4$ . The edge between them is set to have the capacity  $c = \frac{2w_\Delta}{3(1-\delta)n-3}$ . Let  $E_c$  be the set of these edges. If  $T_e w_\Delta > 1$ , then the vertex of  $\widehat{G}$  corresponding to  $e$  is joined to the supersource and the capacity of the corresponding edge of  $\widehat{G}$  is  $T_e w_\Delta - 1$ . Likewise, if  $T_e w_\Delta < 1$ , then the vertex of  $\widehat{G}$  corresponding to  $e$  is joined to the supersink and the capacity of the corresponding edge is  $1 - T_e w_\Delta$ . The vertices of  $G$  adjacent to the supersource are referred to as *sources* and those adjacent to the supersink as *sinks*. Let

$$M = \sum_{e \text{ source}} (T_e w_\Delta - 1) = \sum_{e \text{ sink}} (1 - T_e w_\Delta).$$

We will show that  $\widehat{G}$  has a flow of value  $M$  from the supersource to the supersink.

If  $\widehat{G}$  does not have a flow of value  $M$ , then it has a vertex cut  $(A_0, B_0)$  such that the supersource is contained in  $A_0$ , the supersink in  $B_0$  and the sum of the capacities of the edges from  $A_0$  to  $B_0$  is less than  $M$ . Let  $A$  be the edges of  $G$  corresponding to the vertices of  $A_0$  and  $B$  the edges corresponding to the vertices of  $B_0$ . Note that  $|A| = |A_0| - 1$  and  $|B| = |B_0| - 1$ . Finally, let  $k = |A|$ , and observe that  $|B| = m - k$ .

Let  $T_A$  and  $T_B$  be the average  $T_e$  for  $e$  in  $A$  and in  $B$  respectively. Let  $e = uv$  be an edge of  $G$ . Let  $W_e$  be the set of the vertices  $w$  such that  $uvw$  is a triangle. By the definition of  $T_e$ ,  $|W_e| = T_e$ . Each vertex of  $W_e$  is non-adjacent to at most  $\delta n$  vertices of  $G$ , and thus is non-adjacent to at most  $\delta n$  vertices of  $W_e$ . So each vertex of  $W_e$  is adjacent to at least  $T_e - \delta n$  vertices of  $W_e$ . Therefore  $e$  is in at least  $\frac{T_e(T_e - \delta n)}{2}$  distinct copies of  $K_4$ .

Let  $e$  be a vertex of  $A$ . It is adjacent to at least  $\frac{T_e(T_e - \delta n)}{2} - k$  vertices of  $B$ . Therefore the cut contains at least

$$\sum_{e \in A} \left( \frac{T_e(T_e - \delta n)}{2} - k \right)$$

edges of  $E_c$ . Similarly, it contains at least

$$\sum_{e \in B} \left( \frac{T_e(T_e - \delta n)}{2} - (m - k) \right)$$

edges of  $E_c$ . Moreover, for each source  $e$  that is in  $B$  and each sink  $e$  that is in  $A$ , the cut contains the edge between  $e$  and the supersource or the supersink. Recall that the capacities of the edges of  $E_1$  is  $c = \frac{2w_\Delta}{3(1-\delta)n}$ . Therefore the sum of the capacities of the edges of  $\hat{G}$  is at least

$$\sum_{e \in A} \left( \frac{T_e(T_e - \delta n)}{2} - k \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta).$$

At the same time, it is also at least

$$\sum_{e \in B} \left( \frac{T_e(T_e - \delta n)}{2} - (m - k) \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta).$$

Since the sum of the capacities of the edges in the considered cut is less than  $M$ , we get that

$$\sum_{e \in A} \left( \frac{T_e(T_e - \delta n)}{2} - k \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta) < M \quad (1)$$

and

$$\sum_{e \in B} \left( \frac{T_e(T_e - \delta n)}{2} - (m - k) \right) c + \sum_{e \text{ source} \in B} (T_e w_\Delta - 1) + \sum_{e \text{ sink} \in A} (1 - T_e w_\Delta) < M. \quad (2)$$

The inequalities (1) and (2) can be rewritten using that

$$M = \sum_{e \text{ source}} (T_e w_\Delta - 1)$$

and

$$M = \sum_{e \text{ sink}} (1 - T_e w_\Delta)$$

respectively as follows.

$$\begin{aligned} \sum_{e \in A} (T_e(T_e - \delta n) - 2k) c - 2 \sum_{e \in A} (T_e w_\Delta - 1) &< 0 \\ \sum_{e \in B} (T_e(T_e - \delta n) - 2(m - k)) c - 2 \sum_{e \in B} (1 - T_e w_\Delta) &< 0 \end{aligned}$$

Since the summand is a convex function of  $T_e$ , we obtain the following.

$$(T_A(T_A - \delta n) - 2k)c - 2(T_A w_\Delta - 1) < 0 \quad (3)$$

$$(T_B(T_B - \delta n) - 2(m - k))c - 2(1 - T_B w_\Delta) < 0 \quad (4)$$

The inequality (3) implies that

$$T_A(T_A - \delta n) + \frac{2}{c}(1 - T_A w_\Delta) < 2k. \quad (5)$$

The inequality (4) implies that

$$2k < 2m - T_B(T_B - \delta n) + \frac{2}{c}(1 - T_B w_\Delta). \quad (6)$$

We now combine the inequalities (5) and (6) and we substitute  $c = \frac{2w_\Delta}{3(1-\delta)n-3}$  to get the following.

$$T_A(T_A - \delta n) - (3n(1 - \delta) - 3)T_A < 2m - T_B(T_B - \delta n) - (3n(1 - \delta) - 3)T_B \quad (7)$$

Let  $e$  be an edge of  $G$ . Each end-vertex of  $e$  is non-adjacent to at most  $\delta n$  vertices of  $G$ . Hence, the edge  $e$  is contained in at least  $n - 2\delta n$  triangles. Since  $e$  cannot be contained in more than  $n$  triangles, we get that  $n - 2\delta n \leq T_e \leq n$ . Consequently, we have  $n - 2\delta n \leq T_A, T_B \leq n$ .

A standard analytic argument shows that the left hand side of (7) is minimized when  $T_A = n$  and the right hand side is maximized when  $T_B = n - 2\delta n$ . Consequently, it must hold that

$$n(n - \delta n) - 3(1 - \delta)n^2 + 3n < 2m - (n - 2\delta n)(n - 3\delta n) - 3n(1 - \delta)(n - 2\delta n) + 3n - 6\delta n.$$

Therefore

$$(2 - 12\delta + 12\delta^2)n^2 < 2m - 6\delta n. \quad (8)$$

Recall that we assumed that in  $G$  there is no triangle with three vertices of degree at least  $(1 - \delta)n + 2$ . Let  $V_b$  be the set of vertices of degree at least  $(1 - \delta)n + 2$  in  $G$ , and let  $n_b$  be the number of vertices in  $V_b$ . Let us prove that  $n_b \leq 2\delta n - 4$ . Assume by contradiction that  $n_b \geq 2\delta n - 3$ . Since every vertex in  $V_b$  has at most  $\delta n - 3$  non-neighbours in  $V(G)$ , every vertex in  $V_b$  has at most  $\delta n - 3$  non-neighbours in  $V_b$ , and thus has at least  $n_b - \delta n + 2$  neighbours in  $V_b$ . The graph induced by  $V_b$  is triangle-free, has  $n_b \geq 2\delta n - 3$  vertices, and has minimum degree at least  $n_b - \delta n + 2$ . A triangle-free graph on  $k$  vertices has at most  $\frac{k^2}{4}$  edges, thus it has minimum degree at most  $\frac{k}{2}$ . This implies that  $n_b - \delta n + 2 \leq n_b/2$ , i.e.  $n_b/2 \leq \delta n - 2$ , and thus  $\delta n - 1.5 \leq \delta n - 2$ , a contradiction. Therefore there are at most  $2\delta n - 4$  vertices in  $V_b$ . We have

$$2m \leq (2\delta n - 4)n + ((1 - 2\delta)n + 4)((1 - \delta)n + 1) = (1 - \delta + 2\delta^2)n^2 + 4 + n - 6\delta n$$

and thus, with (8),

$$(2 - 12\delta + 12\delta^2)n^2 < (1 - \delta + 2\delta^2)n^2 + 4 + n - 12\delta n. \quad (9)$$

Assume  $n < 20$ . Since  $G$  has minimum degree at least  $(1 - \delta)n = 0.9n$ , the graph  $G$  is a complete graph. Then giving the same weight to every triangle leads to a fractional triangle decomposition of  $G$ .

Therefore we can assume that  $n \geq 20$ , and thus  $4 + n - 12\delta n \leq 0$ . We get from (9) that  $(1 - 10\delta)(1 - \delta) = 1 - 11\delta + 10\delta^2 < 0$ . Since  $\delta = 0.1$ , this leads to a contradiction. Therefore, there must exist a flow of value  $M$  in  $\widehat{G}$ , and hence, as described previously, the weights of the triangles can be adjusted in such a way that these weights now form a fractional decomposition of  $G$ . This finishes the proof of Theorem 5.

### 3 Conclusion

In this paper we proved that every graph on  $n$  vertices with minimum degree at least  $\frac{9}{10}n$  is fractionally triangle decomposable. Together with a result of Barber, Kühn, Lo and Osthus [2], this implies that, for all  $\epsilon > 0$ , there exists a constant  $n_0$  such that every triangle divisible graph on  $n \geq n_0$  vertices with minimum degree at least  $(0.9 + \epsilon)n$  is triangle decomposable.

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