Control of a Perturbed Under-Actuated Mechanical System

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Abstract— **In this work, the trajectory tracking problem for an under-actuated mechanical system in presence of unknown input perturbation is addressed. The studied inertia wheel inverted pendulum falls in the class of non minimum phase systems. The proposed high order sliding mode structure composed of controller and differentiator allows to track accurately the predefined trajectory and to stabilize the internal dynamics. The robustness of the approach is illustrated through different perturbation and output noise configurations.**

I. Introduction

Under-actuated systems are gaining an increasing interest especially in robotics application, aerospace and marine vehicles [1]. Their distinctive feature is the high number of degrees of freedom compared to the actuated ones, allowing for a high degree of dexterity and a good configuration of the reachable space. Consequently, motion planning and control of robots can be designed more flexibly while guaranteeing certain operational advantages such as minimizing weight and cost.

Nevertheless, the generally nonlinear coupling between the actuated and the non actuated degrees of freedom for under-actuated mechanical systems, may result in some difficulties related to the stabilization of internal dynamics. Consequently, when it comes to robots control, the design of robust control laws is of crucial importance.

Additionally to the proposed variety of control approaches ([2], [3], [1], ...), output feedback sliding mode controllers (SMC) ([4], [5], [6]) are known for their robustness and accuracy and can therefore cope with such problems. Based on exactly maintaining a well-chosen variable (generally the difference between actual and reference outputs) at zero, SMC use high frequency switchings to steer the system such that the desired constraint is satisfied. For the standard SMC, these switchings are at the origin of undesired vibrations, also called chattering effect ([7]), that may damage the system. This issue has been addressed in different ways, including dead zone and smooth approximations of the sign function realizing the switchings. A more interesting perspective was to develop SMC that allow not only to stabilize the sliding variable at zero but also to steer its time derivatives to zero, which results in a smoother dynamics. This class of modern controllers, namely High Order Sliding Mode (HOSM) controllers ([8], [9], [10], [11], ...), are more accurate than the first order SMC in presence of sampling and measurement noise and more robust against modeling uncertainties and disturbances. The implementation of HOSM controllers requires good estimates of the derivatives, which can be achieved using HOSM differentiators ([9]).

The main feature of these observers is to provide high precision real time estimations of a given signal derivatives without the need for direct differentiation of noisy measurements. More precisely, in the noise free case, a HOSM differentiator provides finite time exact estimates of the r first derivatives of a given function provided that its $(r+1)$ derivative is bounded by a known constant. In presence of an additive bounded noise, the estimates converge to the true derivatives values with optimal asymptotics.

In this paper, we implement a HOSM controller together with the underlying HOSM differentiator to achieve periodic trajectory tracking for an example of under-actuated mechanical system, in presence of perturbations. The proposed approach is applied to an inertia wheel inverted pendulum, where the pendulum is subject to torque perturbations. The

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accuracy of trajectory tracking and the robustness of the method against perturbations and measurement noise are illustrated through examples of different torque profiles.
This paper is organized as follows: Section 2 introduces the inertia wheel inverted pendulum and describes its equations of motion. In Section 3, the HOSM controller and differentiator are presented and their application to periodic trajectory tracking of the pendulum in presence of torque perturbations is discussed. Finally, Section 4 illustrates the robustness and accuracy of the developed approach through simulation examples of different configurations of the system and the perturbations.

II. The inertia wheel inverted pendulum

The inertia wheel inverted pendulum is an example of under-actuated systems, representing a classic system used to test different control schemes (see [12] and [13] for example). There are different types of inverted pendulum systems having in common the fact that, unlike normal pendulums that are stable when hanging down, they are intrinsically unstable. Consequently, the unstable vertical equilibrium position needs to be continuously balanced.
The inertia wheel inverted pendulum considered in this work consists of a rotating wheel that is mounted on an inverted pendulum as shown in Figure 1.

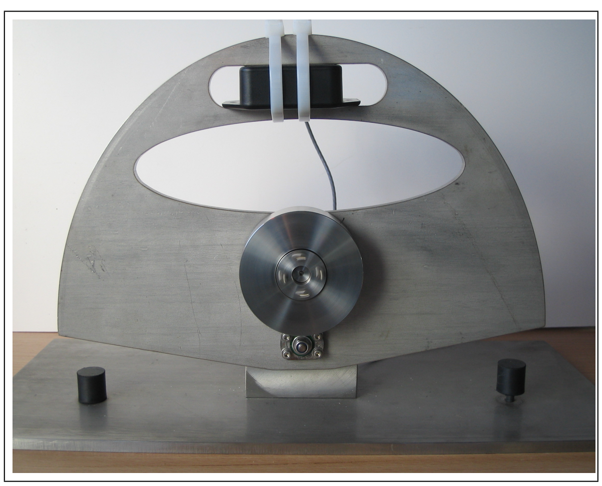

Fig. 1. Inertia wheel inverted pendulum

The angular position of the inertia wheel is actuated through a torque generated by a DC motor. The pendulum is steered as a consequence of the dynamic coupling of the inertia wheel and the pendulum motions as shown in the next subsection.

A. Dynamics equations

Let θ_1, $\dot{\theta}_1$ and $\ddot{\theta}_1$ be respectively the angular position, velocity and acceleration of the pendulum body; θ_2, $\dot{\theta}_2$ and $\ddot{\theta}_2$ the angular position, velocity and acceleration of the inertia wheel as in the schematic description of Figure 2.

If C_1 and C_2 are respectively the perturbation

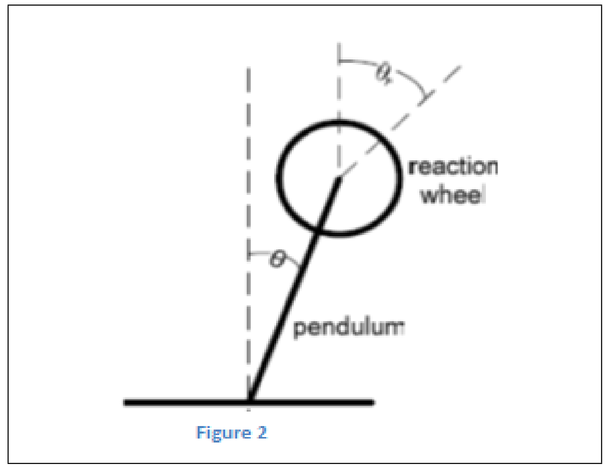

Fig. 2. Schematic representation of the inertia wheel inverted pendulum

torque applied to the pendulum and the torque delivered by the DC motor directly to the inertia wheel, then the equations describing the dynamic behavior of the system are as follows:

$$\begin{cases} (I+i_2)\ddot{\theta}_1 + i_2\ddot{\theta}_2 - \overline{ml}g\sin\theta_1 = C_1 \\ i_2(\ddot{\theta}_1 + \ddot{\theta}_2) = C_2 \end{cases} \quad (1)$$

where i_1 and i_2 are the moment of inertia of the pendulum and the wheel towards the lower extremity of the pendulum (taken as the origin and denoted O), l is the length of the rod and g is the gravity constant. If l_1 and l_2 denote respectively the distances from the centre of gravity of the pendulum and the inertia wheel to O, then the moment of inertia I of the whole system at the origin is given by:

$$I = i_1 + m_1 l_1^2 + m_2 l_2^2$$

where m_1 and m_2 are respectively the masses of the pendulum and the wheel. Finally $\overline{ml} = m_1 l_1 + m_2 l_2$ is the centre of mass of the system. In this study, the main goal is to perform a periodic trajectory tracking based on the available measurement of the pendulum angular position θ_1. Choosing the state vector $X=[x_1,x_2,x_3]^T=[\theta_1,\dot\theta_1,\dot\theta_2]^T$, the system (1) can be written in the following state space representation:

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = \frac{1}{I}\overline{ml}g\sin(x_1)+u+\zeta \\ \dot x_3 = -(\frac{1}{I}\overline{ml}g\sin(x_1)+(1+\frac{I}{i_2})u+\zeta) \\ y = x_1 \end{cases} \tag{2}$$

Where $u=-\frac{C_2}{I}$ is the control input and $\zeta=\frac{C_1}{I}$ is the unknown input perturbation.

B. Control problem statement

It is clear from the second motion equation in system (1) that the inertia wheel angular velocity $\dot\theta_2$ is an internal dynamics of system (3) and thus the system is non minimum phase, which is a common feature of under-actuated mechanical systems.
Due to physical limitations, stabilizing internal dynamics for this class of mechanical systems is a pre-requisite for any control purpose.
Choosing the state vector $x=[x_1,x_2]^T=[\theta_1,\dot\theta_1]^T$, the system (3) can be written in the following state space representation:

$$\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = \frac{1}{I}\overline{ml}g\sin(x_1)+u+\zeta \\ y = x_1 \end{cases} \tag{3}$$

Written in the form of system (3), the inertia wheel inverted pendulum dynamics is condensed in a minimum phase subsystem. This was possible because the dynamics of the internal dynamics does not affect those of the remaining state components. Furthermore, it will be shown in next section that the designed control law allows to stabilize the internal dynamics.

III. High order sliding mode controller

For the inertia wheel inverted pendulum (formulated in system (3)), the purpose is to design a controller, that is robust to the perturbation ζ, that enables the system to track a predefined trajectory (sine shape). The standard sliding mode approach could be used to control the system efficiently if the control input appears explicitly in the first derivative of the output, that is the relative degree of the system equals 1, which is not the case here. HOSM were designed to generalize the notion of sliding mode to higher order derivatives of the output, thus allowing to deal with higher relative degrees and to remove the restriction imposed by standard sliding modes.
Given the sliding surface $\sigma\equiv 0$, which is the difference between the reference and reached outputs, the HOSM approach consists in deriving its first total derivatives that are understood in the sense of Filippov [7]. The number of these first derivatives defines the class of sliding modes, satisfying the following set of equations:

$$\sigma=\dot\sigma=\cdots=\sigma^{(r-1)}=0$$

where r is the degree of smoothness in the neighborhood of the sliding surface, called the order of the HOSM, or simply r-sliding. In the following,we present a general class of HOSM controllers, namely the quasi-continuous ones and their implementation for inertia wheel inverted pendulum problem. Then we describe the corresponding HOSM differentiator and provide an estimation of the unknown perturbation.

A. Quasi-continuous HOSM controllers

Quasi-continuous controllers ([14]) represent a class of the arbitrary order sliding modes. The latter controllers are defined in a recursive way as follows:

$$u=-\alpha\Psi_{r-1,r}(\sigma,\dot\sigma,\cdots,\sigma^{(r-1)})$$

where r is the sliding order and $\alpha>0$ is the unique parameter to be adjusted for the controller.
While a main characteristic of the sliding mode framework is the fact that developed control laws are discontinuous, the quasi-continuous HOSMs provide control laws for which discontinuities occur only when the set of conditions $\sigma=\dot\sigma=\cdots=\sigma^{(r-1)}=0$ are met. It is worth noting that any HOSM controller is at least discontinuous when the previous set of equalities hold. Moreover, in practice these conditions cannot be satisfied simultaneously because of the imperfections and noises, thus the underlying quasi-continuous control law is indeed continuous when the sliding order is greater than 1.

The recursive procedure defining quasi-continuous controllers is the following:

$$\Psi_{i,r} = \frac{\phi_{i,r}}{N_{i,r}}, \quad i = 0, \cdots, r-1$$

where $\phi_{i,r}$ and $N_{i,r}$ are obtained as follows:

$$\begin{cases} \phi_{0,r} = \sigma, \quad N_{0,r} = |\sigma| \\ \phi_{i,r} = \sigma^{(i)} + \gamma_i N_{i-1,r}^{(r-1)/(r-i+1))} \Psi_{i-1,r} \\ N_{i,r} = |\sigma^{(i)}| + \gamma_i N_{i-1,r}^{(r-1)/(r-i+1))} \end{cases} \quad (4)$$

where the coefficients γ_i are predetermined independently from the studied system and have the standard values: $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 2$, ...
For the chattering attenuation purpose, it has been shown that increasing the relative degree by 1 improves significantly the performance of the controller. Therefore, we consider a 3-sliding mode controller instead of a second order one (practical relative degree equals 3).
Denoting the sliding surface and its derivatives as follows:

$$\sigma = \hat{\theta}_1 - \theta_1^{ref}$$
$$\dot{\sigma} = \hat{\dot{\theta}}_1 - \dot{\theta}_1^{ref}$$
$$\ddot{\sigma} = \hat{\ddot{\theta}}_1 - \ddot{\theta}_1^{ref}$$

The proposed 3-sliding controller is obtained from system (4) as follows:

$$\begin{cases} u = 0; 0 \leq t \leq t_0 \\ u = -\alpha \frac{\ddot{\sigma} + 2*(|\dot{\sigma}| + |\sigma|^{2/3})^{-1/2}(\dot{\sigma} + |\sigma|^{2/3} sign(\sigma))}{|\ddot{\sigma}| + 2*(|\dot{\sigma}| + |\sigma|^{2/3})^{1/2}}; t > t_0 \end{cases}$$

The sliding surface derivatives are obtained via the HOSM presented below.

Remark 1: The initial offset applied to the control law is used to allow the differentiator to converge and thus to have accurate derivatives' estimations before applying the control.

B. HOSM differentiators

HOSM differentiators were developed in order to provide the derivatives required to implement HOSM controllers. They are known for their robustness in presence of noise and their exactness in its absence.
Even if, in the case of the inertia wheel inverted pendulum, the variable of interest is provided directly by the measurement, to implement the above controller, we need to estimate the second derivative of θ_1. For this we use the second order high order differentiator introduced by Levant [9], [**?**]:

$$\begin{cases} \dot{z_0} = v_1, v_1 = z_0 - \lambda_2 L^{1/3} |z_0 - y|^{2/3} sign(z_0 - \sigma) \\ \dot{z_1} = v_2, v_2 = z_1 - \lambda_1 L^{1/2} |z_1 - v_0|^{1/2} sign(z_1 - v_0) \\ \dot{z_2} = -\lambda_0 L sign(z_2 - v_1) \end{cases} \quad (5)$$

where y is the measurement (θ_1), z_0 its estimate and z_1 and z_2 represent the estimated first and second derivatives of the output to be used for the controller. A possible choice for the variables λ_i in system (5) is: $\lambda_0 = 1.1$, $\lambda_1 = 1.5$ and $\lambda_2 = 2$.

C. Perturbation estimation

The designed controller allows to track the desired trajectory and to reject the perturbation ζ. So an estimate of the perturbation can be obtained as follows:

$$\hat{\zeta} = z_2 - \frac{1}{I} \overline{ml} g \sin(\hat{x}_1) + u$$

In practice, for the perturbation to be estimated, the second derivative of the state variable θ needs to be considered as an additional state component. A common procedure to obtain z_2 is to artificially increase the differentiator order from a second to a third order, without modifying the control design. Let's just recall that the perturbation estimation was not emphasized in our study but could easily be obtained.

IV. NUMERICAL RESULTS

The HOSM differentiator and the 3-sliding mode controller described in the previous section were applied to the inverted pendulum system with a time step $T_e = 5ms$ for a total simulation duration of $7s$. The reference trajectory has a sine shape with a period of 3 seconds and the system was taken initially to be in a non equilibrium position corresponding to $\theta_1(t=0) = 2^\circ$.
The parameters of the HOSM differentiator were taken as follows: $\lambda_0 = 1.1$, $\lambda_1 = 1.5$ and $L = 100$. The sign function was approximated by a sigmoid $t \mapsto \frac{t}{|t|+\varepsilon}$ where $\varepsilon = 1e-4$.
The parameter of the controller is $\alpha = -2$, the minus sign is added since the control is taken with a minus sign in the system dynamics equations (this

choice is not straightforward, but it corresponds to a physical torque).
The results corresponding to different perturbation signals (zero, constant and sine) are shown below. Figures 3, 4 and 5 illustrate respectively the state vector components, the phase diagram (to show the limit cycle) and the corresponding control input in the ideal case, i.e absence of perturbation and measurement noise.

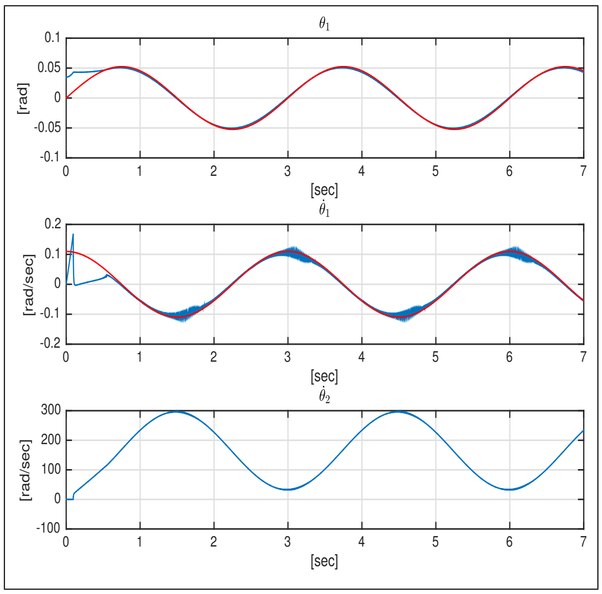

Fig. 3. (1) & (2) State vector: reference (red) and achieved (blue) trajectories. (3) Internal dynamics.

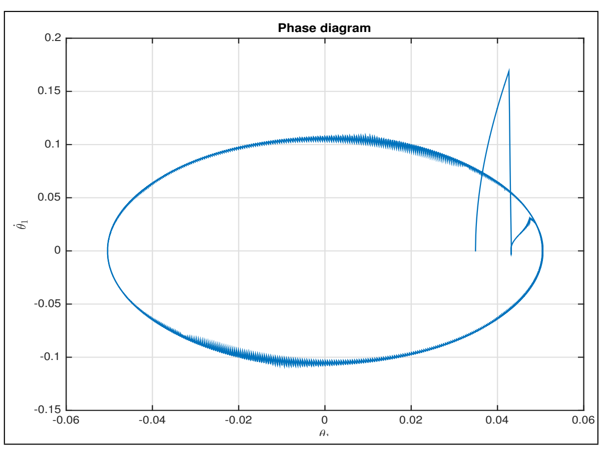

Fig. 4. Phase diagram

Then a white measurement noise, with a standard deviation value corresponding to 3.8% of the sine amplitude, has been added to the measured pendulum angular position. Figure 6 shows the

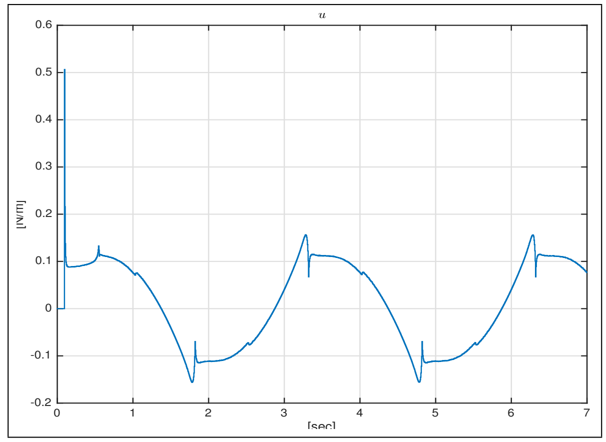

Fig. 5. Control law

results corresponding to a constant perturbation, while Figure 7 illustrates the sine perturbation case.

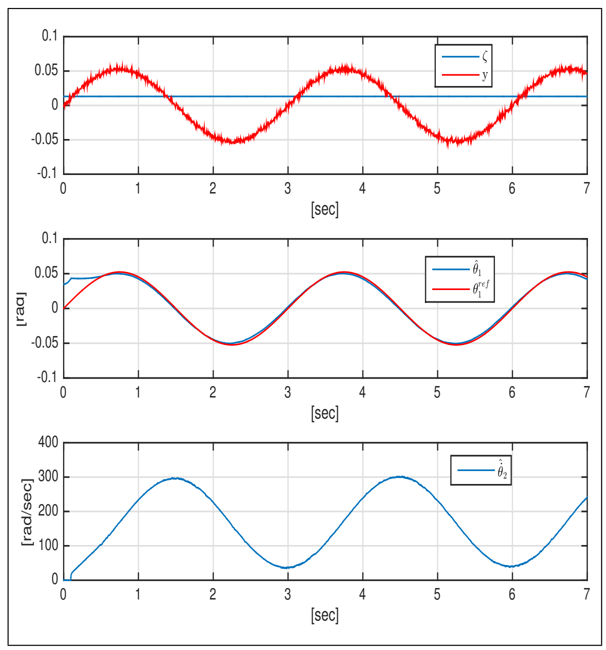

Fig. 6. (1): Perturbation and noisy measurement. (2)Reference (red) and achieved (blue) trajectories. (3) Internal dynamics.

In both figures, the noisy measurements together with the injected perturbation are shown in the first subfigure, the convergence of the pendulum angular position to the reference one in the second subfigure and the corresponding wheel velocity in the last one.
It is important to note that the proposed control law stabilizes the internal dynamics of the system (the inertia wheel angular position) and allows an accurate and fast convergence of controlled

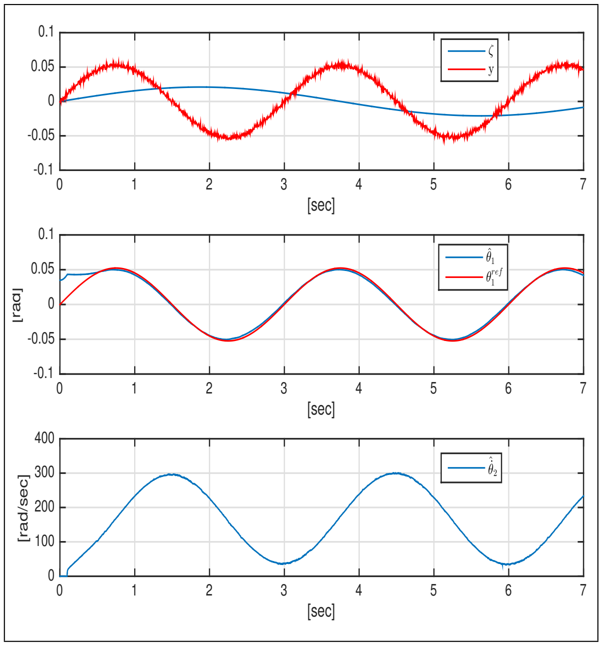

Fig. 7. (1): Perturbation and noisy measurement. (2)Reference (red) and achieved (blue) trajectories. (3) Internal dynamics.

variable to the reference trajectory.

V. CONCLUSIONS

In this paper, a quasi-continuous high-order sliding mode controller has been proposed to perform trajectory tracking for an under-actuated mechanical system with unknown perturbations. The HOSM controller coupled to the corresponding HOSM differentiator, providing the required output derivatives, allow to perform a robust and accurate periodic trajectory tracking for an inertia wheel inverted pendulum. Furthermore, the system was subject to different profiles of torque perturbations, considered as unknown inputs, for which the implemented controller/differentator provided estimations.

The good performance of the studied method was illustrated through numerical simulations of different perturbation profiles in presence of measurement noise. The implementation of the quasi-continuous HOSM control law for an inertia wheel inverted pendulum prototype is under investigation.

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