# Antistrong digraphs

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April 2, 2015

**Abstract**

An antidirected trail in a digraph is a trail (a walk with no arc repeated) in which the arcs alternate between forward and backward arcs. An antidirected path is an antidirected trail where no vertex is repeated. We show that it is NP-complete to decide whether two vertices $x, y$ in a digraph are connected by an antidirected path, while one can decide in linear time whether they are connected by an antidirected trail. A digraph $D$ is antistrong if it contains an antidirected $(x, y)$-trail starting and ending with a forward arc for every choice of $x, y \in V(D)$. We show that antistrong connectivity can be decided in linear time. We discuss relations between antistrong connectivity and other properties of a digraph and show that the arc-minimal antistrong spanning subgraphs of a digraph are the bases of a matroid on its arc-set. We show that one can determine in polynomial time the minimum number of new arcs whose addition to $D$ makes the resulting digraph the arc-disjoint union of $k$ antistrong digraphs. In particular, we determine the minimum number of new arcs which need to be added to a digraph to make it antistrong. We use results from matroid theory to characterize graphs which have an antistrong orientation and give a polynomial time algorithm for constructing such an orientation when it exists. This immediately gives analogous results for graphs which have a connected bipartite 2-detachment. Finally, we study arc-decompositions of antistrong digraphs and pose several problems and conjectures.

**Keywords:** antidirected path, bipartite representation, matroid, detachment, anticonnected digraph

# 1 Introduction

We refer the reader to [1] for notation and terminology not explicitly defined in this paper. An **antidirected path** in a digraph $D$ is a path in which the arcs alternate between forward and backward arcs. The digraph $D$ is said to be **anticonnected** if it contains an antidirected path between $x$ and $y$ for every pair of distinct vertices $x, y$ of $D$. Anticonnected digraphs were studied in [4], where several properties such as antihamiltonian connectivity have been considered. We will show in Theorem 2.2 below that it is NP-complete to decide whether a given digraph contains an antidirected path between given vertices.

Our main purpose is to introduce a related connectivity property based on the concept of a **forward antidirected trail**, i. e. a walk with no arc repeated which begins and ends with a forward arc and in which the arcs alternate between forward and backward arcs. A digraph $D$ is **antistrong** if it has at least three vertices and contains a forward antidirected $(x, y)$-trail for every pair of distinct vertices $x, y$ of $D$. We say that $D$ is **$k$-arc-antistrong** if it has at least three vertices and contains $k$ arc-disjoint forward antidirected $(x, y)$-trails for all distinct $x, y \in V(D)$.

*This work was initiated while Bang-Jensen and Jackson attended the program "Graphs, hypergraphs and computing" at Institute Mittag-Leffler, spring 2014. Support from IML is gratefully acknowledged

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The paper is organized as follows. First we show that, from an algorithmic point of view, anticonnectivity is not an easy concept to work with, since deciding whether a digraph contains an anticonnected path between a given pair of vertices is NP-complete. Then we move to the main topic of the paper, antistrong connectivity, and show that this relaxed version of anticonnectivity is easy to check algorithmically. In fact, we show in Section 3 that there is a close relation between antistrong connectivity of a digraph $D$ and its so called bipartite representation $B(D)$, namely $D$ is antistrong if and only if $B(D)$ is connected. This allows us in Section 4 to find the minimum number of new arcs we need to add to a digraph which is not antistrong so that the resulting digraph is antistrong. Furthermore, using the bipartite representation we show in Section 5 that the arc-minimal antistrong spanning subdigraphs of a digraph $D$ form the bases of a matroid on the arc-set of $D$. More generally, we show that the subsets of $A$ which contain no closed antidirected trails are the independent sets of a matroid on $A$. In Section 6 we study the problem of deciding whether a given undirected graph has an antistrong orientation. We show how to reduce this problem to a matroid problem and give a characterization of those graphs that have an antistrong orientation. Our proof leads to a polynomial time algorithm which either finds an antistrong orientation of the given input graph $G$ or produces a certificate which shows that $G$ has no such orientation. In Section 7 we show that being orientable as an antistrong digraph can be expressed in terms of connected 2-detachments of graphs (every vertex $v$ is replaced by two copies $v', v''$ and every original edge $uv$ becomes an edge between precisely one of the 4 possible pairs $u'v', u'v'', u''v', u''v''$) with the extra requirement that the 2-detachment is bipartite and contains no edge of the form $u'v'$ or $u''v''$. This imediately leads to a characterization of graphs having such a 2-detachment. Finally, in Section 8 we show that one can decide in polynomial time whether a given digraph $D$ has a spanning antistrong subdigraph $D'$ so that $D - A(D')$ is connected in the underlying sense (while it is NP-hard to decide whether a given digraph contains a non-separating strong spanning subdigraph).
We conclude the paper with some remarks and open problems.

# 2 Anticonnectivity

It was shown in [4] that every connected graph $G$ has an anticonnected orientation. This can be seen by considering a breath first search tree rooted at some vertex $r$. Let $\{r\} = L_0, L_1, L_2, \ldots, L_k$ be the distance classes of $G$. Orient all edges between $r$ and $L_1$ from $r$ to these vertices, orient all edges between $L_1 \cup L_3$ and $L_2$ from $L_2$ to $L_1 \cup L_3$, orient all edges from $L_4$ to $L_3 \cup L_5$ etc. Finally, orient all the remaining, not yet oriented edges arbitrarily.

We will show that it is NP-complete to decide if a digraph has an antidirected path between two given vertices. We need the following result which is not new, as it follows from a result in [8] on the vertex analogue, but we include a new and short proof for completeness.

**Theorem 2.1** *It is NP-complete to decide for a given graph $G = (V, E)$, two specified vertices $x, y \in V$ and pairs of distinct edges $\mathcal{P} = \{(e_1, f_1), (e_2, f_2), \ldots, (e_p, f_p)\}$, all from $E$, whether $G$ has an $(x, y)$-path which avoids at least one edge from each pair in $\mathcal{P}$.*

**Proof:** We first slightly modify a very useful polynomial reduction, used in many papers such as [3], from 3-SAT to a simple path problem and then show how to extend this to a reduction from 3-SAT to the problem above. For simplicity our proof uses multigraphs but it is easy to change to graphs.

Let $W[u, v, p, q]$ be the graph (the variable gadget) with vertices $\{u, v, y_1, y_2, \ldots y_p, z_1, z_2, \ldots z_q\}$ and the edges of the two $(u, v)$-paths $uy_1y_2 \ldots y_pv, uz_1z_2 \ldots z_qv$.
Let $\mathcal{F}$ be an instance of 3-SAT with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$. The ordering of the clauses $C_1, C_2, \ldots, C_m$ induces an ordering of the occurrences of a variable $x$ and its negation $\overline{x}$ in these. With each variable $x_i$ we associate a copy of $W[u_i, v_i, p_i + 1, q_i + 1]$ where $x_i$ occurs $p_i$ times and $\overline{x_i}$ occurs $q_i$ times in the clauses of $\mathcal{F}$. Identify end vertices of these graphs by setting $v_i = u_{i+1}$ for $i = 1, 2, \ldots, n - 1$. Let $s = u_1$ and $t = v_n$ and denote by $G'$ the resulting graph. In $G'$ we respectively denote by $y_{i,j}$ and $z_{i,j}$ the vertices $y_j$ and $z_j$ in the copy of $W$ associated with the variable $x_i$.
Next, for each clause $C_i$ we associate this with 3 edges from $G'$ as follows: assume $C_i$ contains variables $x_j, x_k, x_l$ (negated or not). If $x_j$ is not negated in $C_i$ and this is the $r$th copy of $x_j$ (in the order of the

clauses that use $x_j$), then we associate $C_i$ with the edge $y_{j,r}y_{j,r+1}$ and if $C_i$ contains $\overline{x_j}$ and this is the $k$th occurrence of $\overline{x_j}$, then we associate $C_i$ with the edge $z_{j,k}z_{j,k+1}$. We make similar associations for the other two literals of $C_i$. Thus for each clause $C_i$ we now have a set $E_i$ of three distinct edges $e_{i,1}, e_{i,2}, e_{i,3}$ from $G'$ and $E_i \cap E_j = \emptyset$ for $i \neq j$.
Now it is easy to check that $G'$ has an $(s,t)$-path which avoids at least one edge from each of the sets $E_1, E_2, \ldots, E_m$ if and only if $\mathcal{F}$ is satisfiable. Indeed, the $(s,t)$-path goes through the '$z$-vertices' of the copy of $W$ associated with $x_i$ if and only if $x_i$ is set to TRUE to satisfy $\mathcal{F}$.

Let us go back to the original problem. Let $H$ be the multigraph consisting of vertices $c_0, c_1, \ldots, c_m$ and three edges (denoted $f_{i,1}, f_{i,2}, f_{i,3}$) from $c_{i-1}$ to $c_i$ for $i \in \{1, \ldots, m\}$. Let $G$ denote the multigraph we obtain from $G'$ and $H$ by identifying $t$ and $c_0$. Let $x = s$ and $y = c_m$. Finally, form three disjoint pairs of arcs $(e_{i,1}, f_{i,1}), (e_{i,2}, f_{i,2}), (e_{i,3}, f_{i,3})$ between $E_i$ and $\{f_{i,1}, f_{i,2}, f_{i,3}\}$ for every $i \in \{1 \ldots m\}$.
By the observations above it is easy to check that $G$ has an $(x,y)$-path which avoids at least one arc from each of the forbidden pairs if and only if $\mathcal{F}$ is satisfiable. $\diamond$

**Theorem 2.2** *It is NP-complete to decide whether a given digraph contains an antidirected path between given vertices $x, y$.*

**Proof:** The following proof is due to Anders Yeo (private communication, April 2014). Let $G = (V, E)$ be a graph with two specified vertices $x, y \in V$ and pairs of distinct edges $\mathcal{P} = \{(e_1, f_1), (e_2, f_2), \ldots, (e_p, f_p)\}$, all from $E$. We will show how to construct a digraph $D_G$ with specified vertices $s, t$ such that $D_G$ contains an antidirected $(s,t)$-path if and only if $G$ has an $(x,y)$-path which avoids at least one edge from each pair in $\mathcal{P}$. Since the construction can be done in polynomial time this and Theorem 2.1 will imply the result.

Let $k$ be the maximum number of pairs in $\mathcal{P}$ involving the same edge from $E$. Let $P$ be an antidirected path of length $2k+2$ which starts with a forward arc (and hence ends with a backward arc). Now construct $D_G$ as follows: start from $G$ and first replace every edge $uv$ with a private copy $P_{uv}$ of $P$ (no internal vertices are common to two such paths). Then for each pair $(e_i, f_i) \in \mathcal{P}$ we identify one sink of $P_{e_i}$ with one source of $P_{f_i}$ so that the resulting vertex has in- and out-degree 2. By the choice of the length of $P$ we can identify in pairs, i. e. no three vertices will be identified. Note that all the original vertices of $G$ will be sources in $D_G$. The remaining (new vertices) will be called *internal* vertices.

Finally let $s = x$ and $t = y$. We claim that $D_G$ has an antidirected $(s,t)$-path if and only if $G$ has an $(x,y)$-path which uses at most one edge from each of the pairs in $\mathcal{P}$. Suppose first that $xx_1x_2 \ldots x_{r-1}x_ry$ is a path in $G$ which uses at most one edge from each of the pairs in $\mathcal{P}$. Then $P_{xx_1}P_{x_1x_2} \ldots P_{x_{r-1}x_r}P_{x_ry}$ is an antidirected $(s,t)$-path in $D_G$ (no vertex is repeated since the identifications above where only done for paths corresponding to pairs in $\mathcal{P}$). Conversely, suppose $D_G$ contains an antidirected $(s,t)$-path $Q$. By the way we identified vertex pairs according to $\mathcal{P}$, the internal vertices have in- and out-degree at most 2, and if an internal vertex is on two paths $P_{e_i}, P_{f_i}$ then it has both its in-neighbours on $P_{e_i}$ and both its out-neighbours on $P_{f_i}$. This implies that $Q$ will either completely traverse a path $P_{e_i}$ or not touch any internal vertex of that path. Hence it cannot traverse both $P_{e_i}$ and $P_{f_i}$ if $(e_i, f_i) \in \mathcal{P}$, and it follows that if we delete all internal vertices of $Q$ and add back the edges of $G$ corresponding to each of the traversed paths, we obtain an $(x,y)$-path in $G$ that uses at most one edge from each pair in $\mathcal{P}$. $\diamond$

# 3 Properties of antistrong digraphs

For every digraph $D$ we can associate an undirected bipartite graph which contains all the information we need to study antistrong connectivity. The **bipartite representation** [1, Page 19] of a digraph $D = (V, A)$ is the bipartite graph $B(D) = (V' \cup V'', E)$, where $V' = \{v' | v \in V\}$, $V'' = \{v'' | v \in V\}$ and $E = \{v'w'' | vw \in A\}$.

**Proposition 3.1** *Let $D = (V, A)$ be a digraph with $|V| \geq 3$. The following are equivalent.*

(a) *$D$ is antistrong*

(b) *$B(D)$ is connected.*

(c) *For every choice of distinct vertices $x, y$, the digraph $D$ contains both an antidirected $(x, y)$-trail $T_{x,y}$ of even length starting on a forward arc and an antidirected $(x, y)$-trail $\bar{T}_{x,y}$ of even length starting on a backward arc.*

**Proof:** Suppose (a) holds. Then, following the edges corresponding to the arcs of a forward antidirected $(x, y)$-trail, $B(D)$ contains an $(x', y'')$-path for every pair of distinct vertices $x, y \in V$. Now, if $x$ and $y$ are distinct vertices of $D$, we choose a third vertex $z$ in $D$ ($z \neq x$ and $z \neq y$), and the union of an $(x', z'')$-path and a $(z'', y')$-path contains an $(x', y')$-path in $B(D)$. Similarly we obtain an $(x'', y'')$-path in $B(D)$ for every pair of distinct vertices $x, y \in V$. Finally, for any $x \in V$, an $(x', x'')$-path in $B(D)$ can be found in the union of an $(x', y')$-path and a $(y', x'')$-path, where $y$ is a vertex of $D$ distinct from $x$. Hence $(a) \Rightarrow (b)$ holds. Conversely, $(b) \Rightarrow (a)$ holds, since any $(x', y'')$-path in $B(D)$ corresponds to a forward antidirected $(x, y)$-path in $D$ which starts and ends with a forward arc.
Now to prove $(b) \Rightarrow (c)$, it suffices to remark that $T_{x,y}$ and $\bar{T}_{x,y}$ correspond to an $(x', y')$-path and an $(x'', y'')$-path in $B(D)$, respectively. Finally, to see that $(c) \Rightarrow (b)$ holds, it suffices to show that if (c) holds, then $B(D)$ contains an $(x', y'')$-path for all $x, y \in V$ (possibly equal). This follows by considering a neighbour $z''$ of $x'$ and a $(z'', y'')$-path in $B(D)$. $\diamond$

Proposition 3.1 implies the next two results.

**Corollary 3.2** *One can check in linear time whether a digraph is antistrong.*

**Corollary 3.3** *No bipartite digraph is antistrong.*

Recall that a digraph is **k-strong** if it has at least $k+1$ vertices and it remains strong after deletion of any set of at most $k-1$ vertices. The digraph obtained from three disjoint independent sets $X, Y, Z$ each of size $k$ by adding all arcs from $X$ to $Y$, from $Y$ to $Z$, and from $Z$ to $X$ is $k$-strong. However, $B(D)$ has three connected components. This shows that no condition on the strong connectivity will guarantee that a digraph is antistrong.

Recall that $D$ is $k$-arc-antistrong if it contains $k$ arc-disjoint forward antidirected $(x, y)$-trails for every ordered pair of distinct vertices $x, y$. We can check in time $O(mk)$ whether a digraph has $k$ arc-disjoint forward antidirected $(x, y)$-trails for given vertices $x, y$, because they correspond to edge-disjoint $(x', y'')$-paths in $B(D)$ whose existence can be checked by using flows, see e.g. [1, Section 5.5]. So we can check in polynomial time if a digraph is $k$-arc-antistrong.

**Theorem 3.4** *If $D$ is $2k$-arc-antistrong, then it contains $k$ arc-disjoint antistrong spanning subdigraphs.*

**Proof:** Since $D$ is $2k$-arc-antistrong, $B(D)$ is $2k$-edge-connected. We can now use Nash-Williams' theorem (see [1, Theorem 9.4.2] for instance) to deduce that $B(D)$ has $k$ edge-disjoint spanning trees. Proposition 3.1 now gives the required set of $k$ arc-disjoint antistrong spanning subdigraphs of $D$. $\diamond$

**Theorem 3.5** *There exists a polynomial time algorithm which for a given digraph $D$ and a natural number $k$ either returns $k$ arc-disjoint spanning antistrong subdigraphs of $D$ or correctly answers that no such set exists.*

**Proof:** This follows from the fact that such subdigraphs exist if and only if $B(D)$ has $k$ edge-disjoint spanning trees, and the existence of such trees can be checked via Edmonds' algorithm for matroid partition [5]. $\diamond$

The corresponding problem for containing two arc-disjoint *strong* spanning subdigraphs is NP-complete (see e.g. [1, Theorem 13.10.1]).

**Theorem 3.6** *It is NP-complete to decide whether a digraph $D$ contains two spanning strong subdigraphs $D_1, D_2$ which are arc-disjoint.*

# 4 Antistrong connectivity augmentation

Note that every complete digraph on at least 3 vertices is antistrong. Hence it is natural to ask for the minimum number of new arcs one has to add to a digraph in order to make it antistrong.

**Theorem 4.1** *There exists a polynomial time algorithm for finding, for a given digraph $D = (V, A)$ on at least 3 vertices, a minimum cardinality set of new arcs $F$ such that the digraph $D' = (V, A\dot{\cup}F)$ is antistrong.*

**Proof:** Let $D$ be a digraph on $n \geq 3$ vertices which is not antistrong. By Proposition 3.1, its bipartite representation $B(D)$ is not connected. First observe that in the bipartite representation each new arc added to $D$ will correspond to an arc from a vertex $u'$ of $V'$ to a vertex $v'' \in V''$ such that $u \neq v$ back in $V$. So we are looking for the minimum number of new edges of type $u'v''$ with $u \neq v$ whose addition to $B(D)$ makes it connected while preserving the bipartition $V', V''$. Note that, as long as $n \geq 3$, in which case $B(D)$ has at least 6 vertices, we can always obtain a connected graph by adding edges that are legal according to the definition above. So the number of edges we need is exactly the number of connected components of $B(D)$ minus one.[1]

To find an optimal augmentation we add all missing edges between $V'$ and $V''$ to $B(D)$, except for those of the form $v'v''$ and give the new edges cost 1, while all original edges get cost 0. Now find a minimum weight spanning tree in the resulting weighted complete bipartite graph. The edges of cost 1 correspond to an optimal augmenting set back in $D$. $\diamond$

The complexity of the analogous question for $k$-arc-antistrong connectivity is open.

**Problem 4.2** *Given a digraph $D$ and a natural number $k$, can we find in polynomial time a minimum cardinality set of new arcs whose addition to $D$ results in a digraph $D'$ which is $k$-arc-antistrong?*

Problem 4.2 is easily seen to be equivalent to the following problem on edge-connectivity augmentation of bipartite graphs.

**Problem 4.3** *Given a natural number $k$ and a bipartite graph $B = (X, Y, E)$ with $|X| = |Y| = p$ which admits a perfect matching $M$ in its bipartite complement, find a minimum cardinality set of new edges $F$ such that $F \cap M = \emptyset$ and $B + F$ is $k$-edge-connected and bipartite with the same bipartition as $B$.*

Theorem 4.1 can be extended to find the minimum number of new arcs whose addition to $D$ gives a digraph with $k$ arc-disjoint antistrong spanning subdigraphs $D_1, \ldots, D_k$, provided that $V(D)$ is large enough to allow the existence of $k$ such subdigraphs. Note that since each $D_i$ needs at least $2n - 1$ arcs and we do not allow parallel arcs, we need $n$ to be large enough, in particular we must have $n \geq 2k + 1$.

**Theorem 4.4** *There exists a polynomial time algorithm for determining, for a given digraph $D$ on at least 3 vertices, whether one can add some edges to $D$ such that the resulting digraph is simple (no parallel arcs) and has $k$ arc-disjoint antistrong spanning subdigraphs. In the case when such a set exists, the algorithm will return a minimum cardinality set of arcs $A'$ such that $D' = (V, A \cup A')$ contains $k$ arc-disjoint antistrong spanning subdigraphs.*

**Proof:** This follows from the fact that the minimum set of new arcs is exactly the minimum number of new edges, not of the form $v'v''$ that we have to add to $B(D)$ such that the resulting bipartite graph is simple and has $k$ edge-disjoint spanning trees. This number can be found using matroid

[1] This number is also equal to $(2n - 1) - r(A)$ where $r$ is the rank function of the matroid $M(D)$ which we define in Section 5.

techniques as follows. Add all missing edges from $V'$ to $V''$ and give those of the form $v'v''$ very large cost (larger than $2nk$) and the other new edges cost 1. Now, if the resulting complete bipartite digraph $K_{n,n}$ has $k$-edge-disjoint spanning trees of total cost less than $2nk$, then the set of new edges added will form a minimum augmenting set and otherwise no solution exists. Recall from matroid theory that $k$ edge-disjoint spanning trees in $K_{n,n}$ correspond to $k$ edge-disjoint bases in the cycle matroid $M(K_{n,n})$ of $K_{n,n}$ which again corresponds to an independent set of size $k(2n-1)$ in the union $M = \bigvee_{i=1}^{k} M(K_{n,n})$. This means that we can solve the problem by finding a minimum cost base $B$ of $M$ and then either return the arcs which correspond to edges of cost 1 in $B$ or decide that no solution exists when the cost of $B$ is more than $2kn$. We leave the details to the reader. $\diamond$

# 5 A matroid for antistrong connectivity

Having seen the equivalence between antistrong connectivity of digraph $D$ on $n$ vertices and connectivity of its bipartite representation $B(D)$ (see Proposition 3.1), and recalling from matroid theory that $B(D)$ is connected if and only if the cycle matroid $M(B(D))$ has rank $|V(B(D))|-1$, it is natural to ask how antistrong connectivity can be expressed as a matroid property on $D$ itself.

For $F \subseteq A$, we denote by $h(F)$ and $t(F)$ the numbers of vertices that are heads, respectively tails, of one or more arcs in $F$.

Recall that the independent sets of the cycle matroid $M(G)$ of a graph $G = (V, E)$ are those subsets $I \subseteq E$ for which we have $|I'| \leq \nu(I') - 1$ for all $\emptyset \neq I' \subseteq I$, where $\nu(I')$ is the number of end vertices of the edges in $I'$. Inspired by this we define set $I$ of arcs in a digraph $D = (V, A)$ to be **independent** if

$$|I'| \leq h(I') + t(I') - 1 \quad \text{for all } \emptyset \neq I' \subseteq I, \tag{1}$$

A set $S \subseteq A$ is **dependent** if it is not independent.

**Proposition 5.1** *Let $D = (V, A)$ be a digraph. A subset $I \subseteq A$ is independent if and only if the corresponding edge set $I$ in $B(D)$ forms a forest. Every inclusion-minimal dependent set $S \subseteq A$ corresponds to a cycle in $B(D)$ and conversely.*

**Proof:** Suppose $I \subseteq A$ is independent and consider the corresponding edge set $\tilde{I}$ in $B(D)$. If $\tilde{I}$ is not a forest, then some subset $\tilde{I}' \subseteq \tilde{I}$ will be a cycle $C$ in $B(D)$ with $p$ vertices in each of $V', V''$ for some $p \geq 2$. The set $\tilde{I}'$ corresponds to a set $I' \subseteq I$ with $h(I') + t(I') - 1 = p + p - 1 < 2p = |I'|$, contradicting that $I$ is independent. The other direction follows from the fact that every forest $F$ in $B(D)$ spans at least $|E(F)| + 1$ vertices in $B(D)$ and every subset of a forest is again a forest. The last claim follows from the fact that every minimal set of edges which does not form a forest in $B(D)$ forms a cycle in $B(D)$. $\diamond$

The previous proposition implies that a set of arcs of a digraph is dependent if and only if it contains a closed trail of even length consisting of alternating forward and backward arcs. We will refer to such a trail as a **closed antidirected trail**, or **CAT** for short.

**Theorem 5.2** *Let $D = (V, A)$ be a digraph and $\mathcal{I}$ be the family of all independent sets of arcs in $D$. Then $M(D) = (A, \mathcal{I})$ is a graphic matroid with rank equal to the size of a largest collection of arcs containing no closed alternating trail.*

**Proof:** It follows immediately from Proposition 5.1 that a set $I$ belongs to $\mathcal{I}$ if and only if the corresponding edge set $\tilde{I}$ is independent in the cycle matroid on $B(D)$. $\diamond$

**Theorem 5.3** *A digraph $D$ is antistrong if and only if $M(B(D))$ has rank $2|V| - 1$.*

**Proof:** The rank of $M(B(D))$ equals the size of a largest acyclic set of edges in $B(D)$. This has size $2|V| - 1$ precisely when $B(D)$ has a spanning tree $H$. Back in $D$, the arcs corresponding to $E(H)$ contain antidirected forward trails between any pair of distinct vertices. $\diamond$

# 6 Antistrong orientations of graphs

Recall that, by Robbins' theorem (see e.g. [1, Theorem 1.6.1]) a graph $G$ has a strongly connected orientation if and only if $G$ is 2-edge-connected. For antistrong orientations we have the following consequence of Proposition 3.1 which implies that there is no lower bound to the (edge-) connectivity which guarantees an antistrong orientation of a graph.

**Proposition 6.1** *No bipartite graph can be oriented as an antistrong digraph.*

The purpose of this section is to characterize graphs which can be oriented as antistrong digraphs.

**Theorem 6.2** *Suppose $G = (V, E)$ and $|E| = 2|V| - 1$. Then $G$ has an antistrong orientation if and only if*

$$|E(H)| \leq 2|V(H)| - 1 \text{ for all nonempty subgraphs } H \text{ of } G, \text{ and} \tag{2}$$
$$|E(H)| \leq 2|V(H)| - 2 \text{ for all nonempty bipartite subgraphs } H \text{ of } G. \tag{3}$$

We derive Theorem 6.2 from the following characterization of graphs which can be oriented as digraphs with no closed antidirected trail (CAT).

**Theorem 6.3** *A graph $G = (V, E)$ has an orientation with no CAT if and only if $G$ satisfies (2) and (3). In particular no $n$ vertex graph with at least $2n$ edges and no $n$ vertex bipartite graph with at least $2n - 1$ edges admits a CAT-free orientation.*

It is not hard to see that Theorem 6.3 implies Theorem 6.2. Assume that Theorem 6.3 holds and consider a graph $G = (V, E)$ with $|E| = 2|V| - 1$. Suppose that $G$ has an antistrong orientation $D$. Then $B(D)$ is connected by Proposition 3.1. As $B(D)$ has $2|V| - 1 = |V(B(D))| - 1$ edges it is a tree. So $D$ is a CAT-free orientation of $G$ and, by Theorem 6.3, conditions (2) and (3) hold for $G$. Conversely, if (2) and (3) hold for $G$, then $G$ has a CAT-free orientation by Theorem 6.3, and we can deduce as above that this orientation is also an antistrong orientation of $G$.

We next show that (2) and (3) are necessary conditions for a CAT-free orientation. For the necessity of (2), suppose that some nonempty subgraph $H$ has $|E(H)| \geq 2|V(H)|$ and that $D$ is any orientation of $G$. Then $B(D)$ has at least $2|V(H)|$ edges between $V(H)'$ and $V(H)''$, implying that it contains a cycle. Hence $D$ is not CAT-free. The necessity of (3) can be seen as follows. Suppose $H$ is a bipartite subgraph on $2|V(H)| - 1$ edges and let $\vec{H}$ be an arbitrary orientation of $H$. Since no bipartite graph has an antistrong orientation it follows that $B(\vec{H})$ is not connected, and, as it has $2|V(H)| - 1 = |V(B(\vec{H}))| - 1$ edges, it contains a cycle. This corresponds to a CAT in $\vec{H}$.

Most of the remainder of this section is devoted to a proof of sufficiency in Theorem 6.3. We first show that, for an arbitrary graph $G' = (V', E')$, the edge sets of all subgraphs $G$ of $G'$ which satisfy (2) and (3) are the independent sets of a matroid on $E'$. We then show that this matroid is the matroid union of the cycle matroid and the 'even bicircular matroid' of $G'$ (defined below). This allows us to partition the edge-set of a graph $G$ which satisfies (2) and (3) into a forest and an 'odd pseudoforest'. We then use this partition to define a CAT-free orientation of $G$. We first recall some results from matroid theory. We refer a reader unfamiliar with submodular functions and matroids to [7].

Suppose $E$ is a set and $f : 2^E \to \mathbb{Z}$ is a submodular, nondecreasing set function which is nonnegative on $2^E \setminus \{\emptyset\}$. Edmonds [6], see [7, Theorem 13.4.2], showed that $f$ induces a matroid $M_f$ on $E$ in which $S \subseteq E$ is independent if $|S'| \leq f(S')$ for all $\emptyset \neq S' \subseteq S$. The rank of a subset $S \subseteq E$ in $M_f$ is given by the min-max formula

$$r_f(S) = \min_{\mathcal{P}} \left\{ \left| S \setminus \bigcup_{T \in \mathcal{P}} T \right| + \sum_{T \in \mathcal{P}} f(T) \right\}, \tag{4}$$

where the minimum is taken over all subpartitions $\mathcal{P}$ of $S$ (where a **subpartition** of $S$ is a collection of pairwise disjoint nonempty subsets of $S$). Note that the matroid $M(D)$ defined in the previous section is induced on the arc-set of the digraph $D$ by the set function $h + t - 1$.

Given a graph $G = (V, E)$ and $S \subseteq E$ let $G[S]$ be the **subgraph induced** by $S$ i.e. the subgraph of $G$ with edge-set $S$ and vertex-set all vertices incident to $S$. Let $\nu, \beta : 2^E \to \mathbb{Z}$ by putting $\nu(S)$ equal to the number of vertices incident to $S$, and $\beta(S)$ equal to the number of bipartite components of $G[S]$. It is well known that $\nu$ is submodular, nondecreasing, and nonnegative on $2^E$ and that $M_{\nu-1}(G)$ is the cycle matroid of $G$. The function $\nu - \beta$ is also known to be submodular, nondecreasing, and nonnegative on $2^E$ and hence induces a matroid $M_{\nu-\beta}(G)$ on $E$ which we call the **even bicircular matroid** of $G$, see for example [11]. The independent sets of $M_{\nu-\beta}(G)$ are the edge sets of the **odd pseudoforests** of $G$, i. e. subgraphs in which each connected component contains at most one cycle, and if such a cycle exists then it is odd.

The above mentioned properties of $\nu$ and $\nu-\beta$ imply that $2\nu-1-\beta$ is submodular, nondecreasing, and nonnegative on $2^E \setminus \{\emptyset\}$. We will show that the independent sets in $M_{2\nu-1-\beta}(G)$ are the edge sets of the subgraphs which satisfy (2) and (3).

**Lemma 6.4** *Let $G = (V, E)$ be a graph and $\mathcal{I} = \{I \subseteq E : G[I]$ satisfies (2) and (3)$\}$. Then $\mathcal{I}$ is the family of independent sets of the matroid $M_{2\nu-1-\beta}(G)$. In addition, the rank of a subset $S \subseteq E$ in this matroid is $r_{2\nu-1-\beta}(S) = \min_{\mathcal{P}} \left\{|S \setminus \bigcup_{T\in\mathcal{P}} T| + \sum_{T\in\mathcal{P}} (2\nu(T) - 1 - \beta(T))\right\}$ where the minimum is taken over all subpartitions $\mathcal{P}$ of $S$.*

**Proof.** We first suppose that some $S \subseteq E$ is not independent in $M_{2\nu-1-\beta}(G)$. Then we may choose a nonempty $S' \subseteq S$ with $|S'| > 2\nu(S') - 1 - \beta(S')$, and subject to this condition, such that $|S'|$ is as small as possible. The minimality of $S'$ implies that $H = G[S']$ is connected. So $\beta(S') = 1$ if and only if $H$ is bipartite (and 0 otherwise) and we may now deduce that that $H \subseteq G[S]$ fails to satisfy (2) or (3).

We next suppose that $G[S]$ fails to satisfy (2) or (3) for some $S \subseteq E$. Then there exists a nonempty subgraph $H$ of $G[S]$ such that either $|E(H)| > 2|V(H)|-1$, or $H$ is bipartite and $|E(H)| > 2|V(H)|-2$. Then $S' = E(H)$ satisfies $|S'| > 2\nu(S') - 1 - \beta(S')$ so $S$ is not independent in $M_{2\nu-1-\beta}(G)$.

The expression for the rank function of $M_{2\nu-1-\beta}(G)$ follows immediately from (4). $\diamond$

The **matroid union** of two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ on the same ground set $E$ is the matroid $M_1 \vee M_2 = (E, \mathcal{I})$ where $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}_1$ and $I_1 \in \mathcal{I}_1\}$. Suppose $f_1, f_2 : E \to \mathbb{Z}$ are submodular, nondecreasing, and nonnegative on $2^E \setminus \{\emptyset\}$. Then $f_1 + f_2$ will also be submodular, nondecreasing, and nonnegative on $2^E \setminus \{\emptyset\}$ and hence will induce the matroid $M_{f_1+f_2}$. Every independent set in $M_{f_1} \vee M_{f_2}$ is independent in $M_{f_1+f_2}$, but the converse does not hold in general. Katoh and Tanigawa [9, Lemma 2.2] have shown that the equality $M_{f_1+f_2} = M_{f_1} \vee M_{f_2}$ does hold whenever the minimum in formula (4) for the ranks $r_{f_1}(S)$ and $r_{f_2}(S)$ is attained for the same subpartition of $S$, for all $S \subset E$. This allows us to deduce

**Lemma 6.5** *For any graph $G = (V, E)$, we have $M_{2\nu-1-\beta}(G) = M_{\nu-1}(G) \vee M_{\nu-\beta}(G)$.*

**Proof.** This follows from the above mentioned result of Katoh and Tanigawa, and the facts that $r_{\nu-1}(S) = \sum_{T\in\mathcal{P}}(\nu(T) - 1)$ and $r_{\nu-\beta}(S) = \sum_{T\in\mathcal{P}}(\nu(T) - \beta(T))$ where $\mathcal{P}$ is the partition of $S$ given by the connected components of $G[S]$ (since $r_{\nu-1}(S)$ and $r_{\nu-\beta}(S)$ are equal to the number of edges in a maximum forest and a maximum odd pseudoforest, respectively, in $G[S]$). $\diamond$

Lemma 6.4 and Lemma 6.5 immediately give the following.

**Lemma 6.6** *Let $G = (V, E)$ be a graph. Then $G$ satisfies (2) and (3) if and only if $E$ can be partitioned into a forest and an odd pseudoforest.*

We provide an alternative graph theoretic proof of this lemma in the Appendix.

We next show that every graph whose edge set can can be partitioned into a spanning tree and an odd pseudoforest has a CAT-free orientation.

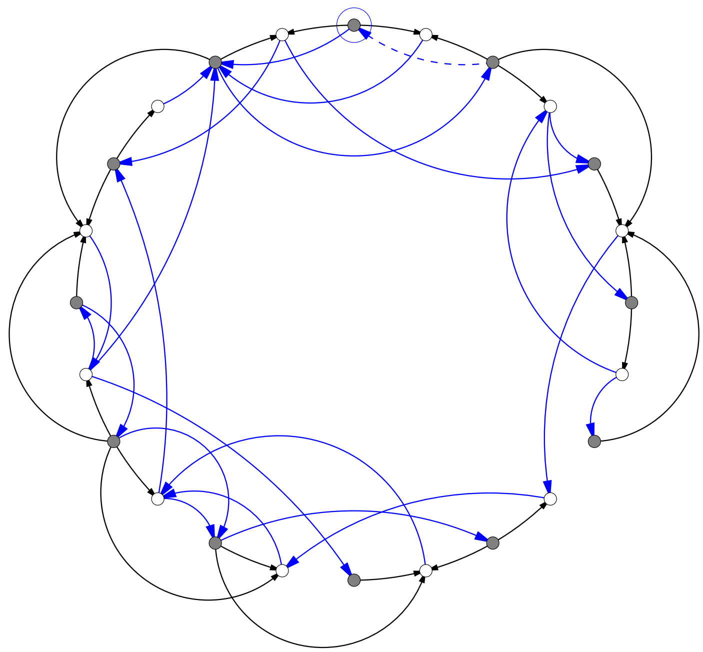

Figure 1: A CAT-free orientation of the union of a spanning tree $T$ and a spanning pseudoforest $P$; $T$ governs the bipartition $X, Y$ (white/grey), its edges are drawn outside (or on) the disk spanned by the vertices. The edges of $P$ are embedded in the interior of that disk, the root vertex is the encircled topmost one, the precious edge is the dashed one.

**Theorem 6.7** *Let $G$ be the edge-disjoint union of a spanning tree $T$ and an odd pseudoforest $P$.*[2] *Then $G$ has a CAT-free orientation. In addition, such an orientation can be constructed in linear time given $T$ and $P$.*

**Proof:** Let $X, Y$ be the unique (up to renaming the two sets) bipartition of $T$ and orient all edges of $T$ from $Y$ to $X$. If $P$ has no edges we are done since there are no cycles in $G$. Let $P_1, \ldots, P_k$ be the connected components of $P$. We shall show that we can orient the edges of $P_1, \ldots, P_k$ in such a way that none of the resulting arcs of these (now oriented) pseudoforests $\vec{P_1}, \ldots, \vec{P_k}$ can belong to a closed antidirected trail. Clearly this will imply the lemma. For each $P_i$ we choose a root vertex $r_i$ of $P_i$ as follows. If $P_i$ is a tree then we choose $r_i$ to be an arbitrary vertex of $P_i$. If $P_i$ contains an odd cycle $C_i$ then we choose $r_i$ to be a vertex of $C_i$ such that $r_i$ has at least one neighbour $s_i \in C_i$ which belongs to the same set in the bipartition $(X, Y)$ as $r_i$ (this is possible since $C_i$ is odd). We will refer to the edge $r_i s_i$ as a **precious edge** of $P_i$. Put $T_i = P_i - r_i s_i$ if $P_i$ contains a cycle and otherwise put $T_i = P_i$.

We orient the edges of $T_i$ as follows. Every edge of $P_i$ with one end in $X$ and the other in $Y$ is oriented from $X$ to $Y$. Every edge $uv$ of $T_i$ with $u, v \in X$ is oriented towards $r_i$ in $T_i$ (so if $v$ is closer to $r_i$ than $u$ in $T_i$ we orient the edge from $u$ to $v$ and otherwise we orient it from $v$ to $u$, see Figure 1). Every edge $pq$ of $T_i$ with $p, q \in Y$ is oriented away from $r_i$ in $T_i$. Finally, if $P_i$ contains a precious edge $r_i s_i$, then we orient $r_i s_i$ from $r_i$ to $s_i$ if $r_i, s_i \in X$, and from $s_i$ to $r_i$ if $r_i, s_i \in Y$. Let $D$ denote the resulting orientation of $G$. The digraph $D$ can be constructed in linear time if we traverse each tree $T_i$ by a breath first search rooted at $r_i$.

We use induction on $|E(P)|$ to show that $D$ is CAT-free. As noted above, this is true for the base case when $E(P) = \emptyset$. Suppose that $E(P) \neq \emptyset$ and choose an edge $uv$ in some $P_i$ according to the following criteria. If $P_i$ is not a cycle then choose $v$ to be a vertex of degree one in $P_i$ distinct from $r_i$ and $u$ to be the neighbour of $v$ in $P_i$. If $P_i$ is an odd cycle then choose $v = r_i$ and $u = s_i$. We will show that $uv$ belongs to no CAT in $D$. By symmetry, we may suppose that $v \in X$.

[2] Note that $G$ may have parallel edges, but no more than two copies of any edge, in which case one copy is in $T$ and the other in $P$

We first consider the case when $v$ is a vertex of degree one in $P_i$. Below $d^+(v), d^-(v)$ denote the out-degree, respectively, the in-degree of the vertex $v$. We have two possible subcases:

- $u \in Y$. Since $v \in X$, we oriented $uv$ from $v$ to $u$. All the other edges incident to $v$ belong to $T$ and were oriented towards $v$. Then $d^+(v) = 1$ and the arc $vu$ cannot be part of a CAT.
- $u \in X$. Since $v \in X$, we oriented $uv$ from $v$ to $u$ (as $u$ is closer to $r_i$ than $v$ in $T_i$). As previously we have $d^+(v) = 1$ and the arc $vu$ cannot be part of a CAT.

Since $D - uv$ is CAT-free by induction, $D$ is also CAT-free.

We next consider the case when $P_i$ is an odd circuit, $v = r_i$ and $u = s_i$. Let $t_i$ be the neighbour of $r_i$ in $P_i$ distinct from $s_i$. We again have two possible subcases:

- $t_i \in X$. Since $r_i \in X$, we oriented the edge $t_i r_i$ from $t_i$ to $r_i$. Then $d^+(r_i) = 1$, and the arc $r_i s_i$ cannot be part of a CAT.
- $t_i \in Y$. Let $q_i$ be the neighbour of $s_i$ in $P_i$ which is distinct from $r_i$. The choice of $r_i$ implies that $q_i \in Y$, and hence that $s_i q_i$ is oriented from $s_i$ to $q_i$. Then $d^+(s_i) = 1$, and the arc $r_i s_i$ cannot be part of a CAT.

Since $D - r_i s_i$ is CAT-free by induction, $D$ is also CAT-free. $\diamond$

**Proof of Theorem 6.3 (sufficiency):** Let $G = (V, E)$ be a graph satisfying (2) and (3). By Lemma 6.6, $E$ can be partitioned into a forest $F$ and an odd pseudoforest $P$. By adding a suitable set of edges to $G$, we may assume that $|E| = 2|V| - 1$. (This follows by considering the matroid $M_{2\nu-1-\beta}(2K_n)$ on the edge set of the graph $2K_n$ with vertex set $V$ in which all pairs of vertices are joined by two parallel edges. It is easy to check that $2K_n$ has an edge-disjoint forest and odd pseudoforest with a total of $2|V| - 1$ edges. Thus the rank of $M_{2\nu-1-\beta}(2K_n)$ is $2|V| - 1$. Since $E$ is an independent set in $M_{2\nu-1-\beta}(2K_n)$, it can be extended to an independent set with $2|V| - 1$ edges.) The fact that $|E| = 2|V| - 1$ implies that $F$ is a spanning tree of $G$. We can now apply Theorem 6.7 to deduce that $G$ has a CAT-free orientation. $\diamond$

We have seen that Theorem 6.3 implies Theorem 6.2, and hence that a graph $G = (V, E)$ has an antistrong orientation if and only if the rank of $M_{2\nu-1-\beta}(G)$ is equal to $2|V| - 1$. We can now apply the rank formula (4) to characterize graphs which admit an antistrong orientation.

**Theorem 6.8** *A graph $G = (V, E)$ has an antistrong orientation if and only if*

$$e(\mathcal{Q}) \geq |\mathcal{Q}| - 1 + b(\mathcal{Q}) \tag{5}$$

*for all partitions $\mathcal{Q}$ of $V$, where $e(\mathcal{Q})$ denotes the number of edges of $G$ between the different parts of $\mathcal{Q}$ and $b(\mathcal{Q})$ the number of parts of $\mathcal{Q}$ which induce bipartite subgraphs of $G$.*

**Proof:** Suppose that $G$ has no antistrong orientation. Then the rank of $M_{2\nu-1-\beta}(G)$ is less than $2|V| - 1$ so there exists a subpartition $\mathcal{P}$ of $E$ such that

$$\alpha(\mathcal{P}) := \left| E \setminus \bigcup_{T \in \mathcal{P}} T \right| + \sum_{T \in \mathcal{P}} (2\nu(T) - 1 - \beta(T)) < 2|V| - 1. \tag{6}$$

We may assume that $\mathcal{P}$ has been chosen such that:

(i) $\alpha(\mathcal{P})$ is as small as possible;

(ii) subject to (i), $|\mathcal{P}|$ is as small as possible;

(iii) subject to (i) and (ii), $|\bigcup_{T \in \mathcal{P}} T|$ is as large as possible.

Let $\mathcal{P} = \{E_1, E_2, \ldots, E_t\}$ and let $H_i = (V_i, E_i)$ be the subgraph of $G$ induced by $E_i$ for all $1 \le i \le t$. We will show that $H_i$ is a (vertex-)induced connected subgraph of $G$ and that $V_i \cap V_j = \emptyset$ for all $i \ne j$.

First, suppose that $H_i$ is disconnected for some $1 \le i \le t$. Then we have $H_i = H_i' \cup H_i''$ for some subgraphs $H_i' = (V_i', E_i')$ and $H_i'' = (V_i'', E_i'')$ with $V_i' \cap V_i'' = \emptyset$. Let $\mathcal{P}' = (\mathcal{P} \setminus \{E_i\}) \cup \{E_i', E_i''\}$. We have

$$2\nu(E_i) - 1 - \beta(E_i) > 2\nu(E_i') - 1 - \beta(E_i') + 2\nu(E_i'') - 1 - \beta(E_i'')$$

since, $\nu(E_i) = \nu(E_i') + \nu(E_i'')$ and $\beta(E_i) = \beta(E_i') + \beta(E_i'')$. This implies that $\alpha(\mathcal{P}') < \alpha(\mathcal{P})$ and contradicts (i). Hence $H_i$ is connected and $\beta(E_i) \in \{0, 1\}$ for all $1 \le i \le t$.

Next, suppose that $V_i \cap V_j \ne \emptyset$ for some $1 \le i < j \le t$. Let $\mathcal{P}' = (\mathcal{P} \setminus \{E_i, E_j\}) \cup \{E_i \cup E_j\}$. We have

$$2\nu(E_i) - 1 - \beta(E_i) + 2\nu(E_j) - 1 - \beta(E_j) \ge 2\nu(E_i \cup E_j) - 1 - \beta(E_i \cup E_j)$$

since, if $|V_i \cap V_j| = 1$, then $\nu(E_i) + \nu(E_j) = \nu(E_i \cup E_j) + 1$ and $\beta(E_i) + \beta(E_j) \le \beta(E_i \cup E_j) + 1$, and, if $|V_i \cap V_j| \ge 2$, then $\nu(E_i) + \nu(E_j) \ge \nu(E_i \cup E_j) + 2$ and $\beta(E_1) + \beta(E_j) \le 2$. This implies that $\alpha(\mathcal{P}') \le \alpha(\mathcal{P})$. Since $|\mathcal{P}'| < |\mathcal{P}|$ this contradicts (i) or (ii). Hence $V_i \cap V_j = \emptyset$ for all $i \ne j$.

Finally, suppose that $H_i \ne G[V_i]$. Then some $e \in E \setminus \bigcup_{T \in \mathcal{P}} T$ has both end vertices in $E_i$. Let $E_i' = E_i + e$ and $\mathcal{P}' = \mathcal{P} - E_i + E_i'$. This implies that $\alpha(\mathcal{P}') \le \alpha(\mathcal{P})$. Since $|\mathcal{P}'| = |\mathcal{P}|$ and $|\bigcup_{T \in \mathcal{P}'} T| > |\bigcup_{T \in \mathcal{P}} T|$, this contradicts (i) or (iii). Hence $H_i = G[V_i]$.

Let $\mathcal{Q}$ be the partition of $V$ obtained from $\{V_1, V_2, \ldots, V_t\}$ by adding the remaining vertices of $G$ as singletons. Then $\left|E \setminus \bigcup_{T \in \mathcal{P}} T\right| = e(\mathcal{Q})$ and $\sum_{T \in \mathcal{P}} (2\nu(T) - 1 - \beta(T)) = 2|V| - |\mathcal{Q}| - b(\mathcal{Q})$. We can now use (6) to deduce that $e(\mathcal{Q}) < |\mathcal{Q}| - 1 + b(\mathcal{Q})$.

Suppose, on the other hand, that $e(\mathcal{Q}) < |\mathcal{Q}| - 1 + b(\mathcal{Q})$ for some partition $\mathcal{Q} = \{V_1, V_2, \ldots, V_s\}$ of $V$. Let $G[V_i] = (V_i, E_i)$ for $1 \le i \le s$ and $\mathcal{P} = \{E_i : E_i \ne \emptyset, 1 \le i \le s\}$. Then $\left|E \setminus \bigcup_{T \in \mathcal{P}} T\right| = e(\mathcal{Q})$ and

$$\sum_{T \in \mathcal{P}} (2\nu(T) - 1 - \beta(T)) = 2|V| - |\mathcal{Q}| - b(\mathcal{Q}) - \sum_{T \in \mathcal{P}} (\beta(T) - \beta^*(T)) \le 2|V| - |\mathcal{Q}| - b(\mathcal{Q}),$$

where $\beta^*(T) = \min\{\beta(T), 1\}$. A straightforward calculation now gives

$$\left|E \setminus \bigcup_{T \in \mathcal{P}} T\right| + \sum_{T \in \mathcal{P}} (2\nu(T) - 1 - \beta(T)) < 2|V| - 1$$

and hence $G$ has no antistrong orientation. $\diamond$

**Corollary 6.9** *Every 4-edge-connected nonbipartite graph has an antistrong orientation.*

**Proof:** Suppose $G = (V, E)$ is 4-edge-connected and not bipartite and let $\mathcal{Q}$ be a partition of $V$. If $\mathcal{Q} = \{V\}$ then $e(\mathcal{Q}) = 0 = |\mathcal{Q}| - 1 + b(\mathcal{Q})$ since $G$ is not bipartite, and if $\mathcal{Q} \ne \{V\}$ then $e(\mathcal{Q}) \ge 2|\mathcal{Q}| \ge |\mathcal{Q}| - 1 + b(\mathcal{Q})$ since $G$ is 4-edge-connected. Hence $G$ has an antistrong orientation by Theorem 6.8. $\diamond$

**Corollary 6.10** *Every nonbipartite graph with three edge-disjoint spanning trees has an antistrong orientation.*

**Proof:** We give two proofs of this corollary.

Suppose $G = (V, E)$ is a nonbipartite graph with three edge-disjoint spanning trees and let $\mathcal{Q}$ be a partition of $V$. If $\mathcal{Q} = \{V\}$ then $e(\mathcal{Q}) = 0 = |\mathcal{Q}| - 1 + b(\mathcal{Q})$ since $G$ is not bipartite, and if $\mathcal{Q} \ne \{V\}$ then $e(\mathcal{Q}) \ge 3(|\mathcal{Q}| - 1)$ since $G$ has three edge-disjoint spanning trees. Since $|\mathcal{Q}| \ge 2$, $2(|\mathcal{Q}| - 1) \ge |\mathcal{Q}| \ge b(\mathcal{Q})$ and $e(\mathcal{Q}) \ge |\mathcal{Q}| - 1 + b(\mathcal{Q})$. Hence $G$ has an antistrong orientation by Theorem 6.8.

We could also remark that if $T_1$, $T_2$ and $T_3$ denote three edge-disjoint spanning trees of $G$, then there exists $e \in G$ such that $T_1 + e$ is not bipartite. Then depending if $e \in T_2$ or not, $\{T_1 + e, T_3\}$ or

$\{T_1+e, T_2\}$ is an edge-disjoint pair of a spanning odd pseudo-tree and a spanning tree of $G$. Let $H$ denote this subgraph of $G$. Then using Theorem 6.7, $H$ has a CAT-free orientation which is also an antistrong orientation of $H$ since $|E(H)| = 2|V(H)| - 1$. So $G$ has also an antistrong orientation. $\diamond$

Corollary 6.10 is tight in the sense that there exist graphs with many edge-disjoint trees, two spanning and the others missing just three vertices, which have no antistrong orientation. Consider the graph $G$ obtained by identifying one vertex of a complete bipartite graph $K_{k,k}$ and a complete graph $K_4$. Then $G$ has no antistrong orientation. Indeed, consider the partition $\mathcal{Q}$ of $V(G)$ into four parts: the copy of $K_{k,k}$, and one part for each remaining vertex of $K_4$. We have $e(\mathcal{Q}) = 6 < |\mathcal{Q}| - 1 + b(\mathcal{Q}) = 4 - 1 + 4$ and then $G$ has no antistrong orientation by Theorem 6.8.

Since $M_{2\nu-1-\beta}(G) = M_{\nu-1}(G) \vee M_{\nu-\beta}(G)$, we can use Edmonds' matroid partition algorithm [5] to determine the rank of $M_{2\nu-1-\beta}(G)$ in polynomial time, and hence determine whether $G$ has an antistrong orientation. Moreover, when such an orientation exists, we can use Edmonds' algorithm to construct an edge-disjoint spanning tree and pseudoforest with a total of $2|V|-1$ edges, and then use the construction from the proof of Theorem 6.7 to obtain the desired antistrong orientation in polynomial time. This gives

**Corollary 6.11** *There exists a polynomial time algorithm which finds, for a given input graph $G$, either an antistrong orientation $D$ of $G$, or a certificate, in terms of a subpartition $\mathcal{P}$ which violates (5), that $G$ has no such orientation.*

# 7 Connected bipartite 2-detachments of graphs

We now show a connection between antistrong orientations of a graph $G$ and so-called detachments of $G$. We need only the special case of 2-detachments (see e.g. [10] for results on detachments).

A **2-detachment** of a graph $G = (V, E)$ is any graph $H = (V' \cup V'', E')$ which can be obtain from $G$ by replacing every vertex $v \in V$ with two new vertices $v', v''$ and then for each original edge $uv$ adding precisely one of the four edges $u'v', u'v'', u''v', u''v''$ to $E'$.

**Lemma 7.1** *A graph $G = (V, E)$ has an antistrong orientation if and only if $G$ has a 2-detachment $H = (V' \cup V'', E')$ which is connected and bipartite with bipartition $V', V''$ (we call such a 2-detachment* **good***).*

**Proof:** Suppose $G$ has a good 2-detachment $H = (V' \cup V'', E')$. Then there are no edges of the form $u'v'$ and no edges of the form $u''v''$. Hence the orientation $D$ that we get by orienting the edges of the form $u'v''$ from $u$ to $v$ will be an antistrong orientation of $G$ by Proposition 3.1. Conversely, if $D$ is an antistrong orientation of $G$, then $B(D)$ is a good 2-detachment of $G$. $\diamond$

We can now use Theorem 6.8 and the subsequent remark to deduce the following.

**Theorem 7.2** *A graph $G = (V, E)$ has a good 2-detachment if and only if*

$$e(\mathcal{Q}) \geq |\mathcal{Q}| - 1 + b(\mathcal{Q}) \tag{7}$$

*for all partitions $\mathcal{Q}$ of $V$. Furthermore, there exists a polynomial time algorithm which returns such a 2-detachment when it exists and otherwise returns a certificate, in terms of a partition violating (7), that no such detachment exists.*

# 8 Non-separating antistrong spanning subdigraphs

While there are polynomial time algorithms for checking the existence of two edge-disjoint spanning trees [5], or two arc-disjoint branchings (spanning out-trees) in a digraph (see e.g. [1, Corollary 9.3.2]), checking whether we can delete a strong spanning subdigraph and still have a connected digraph is difficult. Let $UG(D)$ denote the underlying undirected graph of a digraph $D$.

**Theorem 8.1** *[3] It is NP-complete to decide whether a given digraph $D$ contains a spanning strong subdigraph $H$ such that $UG(D - A(H))$ is connected.*

If we replace "strong" by "antistrong" above, the problem becomes solvable in polynomial time.

**Theorem 8.2** *We can decide in polynomial time for a given digraph $D = (V, A)$ on $n$ vertices whether $D$ contains a spanning antistrong subdigraph $H = (V, A')$ such that $UG(D - A')$ is connected.*

**Proof:** We may assume that $D$ is antistrong, since this can be checked in linear time by verifying that $B(D)$ is connected. Let $M_1 = (A, \mathcal{I})$ be the cycle matroid of of the underlying graph $UG(D)$ of $D$ and let $M_2 = M(D) = (A, \mathcal{I}(D))$ be the matroid from Section 5 whose bases are the antistrong sets consisting of $2n - 1$ arcs. Let $M = M_1 \vee M_2$ be the union of the matroids $M_1, M_2$, that is, a set $X$ of arcs is independent in $M$ if and only we can partition $X$ into $X_1, X_2$ such that $X_i$ is independent in $M_i$. For each of the matroids $M_1, M_2$ we can check in polynomial time whether a given subset $X$ of arcs is independent in $M_1$ and $M_2$ (for $M_1$ we need to check that there is no cycle in $UG(D)[X]$ and for $M_2$ we need to check that there is no cycle in the subgraph of $B(D)[E_X]$ induced by the edges $E_X$ corresponding to $X$ in $B(D)$). Thus it follows from Edmonds' algorithm for matroid partitioning [5] that we can find a base of $M$ in polynomial time using the independence oracles of $M_1, M_2$. The desired digraph $H$ exists if and only if the size of a base in $M$ is $(2n - 1) + (n - 1) = 3n - 2$. $\diamond$

A similar proof gives the following.

**Theorem 8.3** *We can decide in polynomial time whether a digraph $D$ contains $k + \ell$ arc-disjoint spanning subdigraphs $D_1, \ldots, D_{k+\ell}$ such that $D_1, \ldots, D_k$ are antistrong and $UG(D_{k+1}), \ldots, UG(D_{k+\ell})$ are connected.*

# 9 Remarks and open problems

We saw in Theorem 3.6 that it is NP-complete to decide whether a given digraph contains two arc-disjoint spanning strong subdigraphs. We would be interested to know what happens if we modify the problem as follows.

**Question 9.1** *Can we decide in polynomial time whether $D$ contains arc-disjoint spanning subdigraphs $D_1, D_2$ such that $D_1$ is antistrong and $D_2$ is strongly connected?*

Inspired by Theorem 8.2 it is natural to ask the following intermediate question.

**Question 9.2** *Can we decide in polynomial time whether $D$ contains arc-disjoint spanning subdigraphs $D_1, D_2$ such that $D_1$ is antistrong and $UG(D_2)$ is 2-edge-connected?*

The following conjecture was raised in [2].

**Conjecture 9.3** *[2] There exists a natural number $k$ such that every $k$-arc-strong digraph has arc-disjoint strong spanning subdigraphs $D_1, D_2$.*

Perhaps the following special case may be easier to study.

**Conjecture 9.4** *There exists a natural number $k$ such that every digraph $D$ which is both $k$-arc-strong and $k$-arc-antistrong has arc-disjoint strong spanning subdigraphs $D_1, D_2$.*

*

**Acknowledgement.** Bang-Jensen and Jackson wish to thank Jan van den Heuvel for stimulating discussions about antistrong connectivity. They also thank the Mittag-Leffler Institute for providing an excellent working environment.

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# A Appendix: a graph theoretical proof of Lemma 6.6

In this appendix, we give a graph theoretical proof of Lemma 6.6, recalled below.

**Lemma 6.6** *Let $G = (V, E)$ be a graph. Then $G$ satisfies*

$$|E(H)| \leq 2|V(H)| - 1 \text{ for all nonempty subgraphs } H \text{ of } G, \text{ and} \tag{2}$$
$$|E(H)| \leq 2|V(H)| - 2 \text{ for all nonempty bipartite subgraphs } H \text{ of } G \tag{3}$$

*if and only if $E$ can be partitioned into a forest and an odd pseudoforest.*

**Proof:** Recall that a **pseudoforest** is a graph in which each connected component contains at most one cycle, and it is called **odd** if it does not contain even cycles. A theorem due to Whiteley [12] (see also [7] p.367 for a short proof based on Edmonds' branching theorem) asserts that a graph satisfies condition (2) if and only if its edge set can be partitioned into a forest and a pseudoforest. So let us denote by a **2-decomposition** of a graph $G = (V, E)$ a pair $(G_b, G_r)$ of spanning subgraphs $G_b = (V, E_b)$ and $G_r = (V, E_r)$ such that $\{E_b, E_r\}$ is a partition of $E$ and $G_b$ is a forest of $G$ and $G_r$ is a pseudoforest of $G$. We will call any sub-structure — edge, component, subgraph etc. — of $G_r$ or of $G_b$ **red** or **black**, respectively, and for a subgraph $H$ of $G$ we denote by $H_r$ and $H_b$ the subgraph of $H$ induced by its red or black edges, respectively.

Without loss of generality we may assume that $G$ is connected, and that $G_b$ is a spanning tree of $G$ (otherwise we could move edges from $G_r$ to $G_b$ to make $G_b$ connected). The **canonical bipartition** of a 2-decomposition $(G_b, G_r)$ of a connected graph $G$ is the unique bipartition given by any 2-colouring of $G_b$. Moreover, an edge of $G_r$ which does not cross this bipartition is called (as previously) a **precious edge** in $(G_b, G_r)$. A 2-decomposition of a 2-decomposable graph is **nice** if every red cycle contains at least one precious edge.

First we establish the next claim.

**Claim 1** *A connected graph which has a 2-decomposition admits a nice 2-decomposition if and only if (3) holds.*

**Proof:** First observe that for any subgraph $H$ of $G$ with at least one black edge, we have $|E(H)| = |E(H_b)| + |E(H_r)| \leq (|V(H_b)| - 1) + |V(H_r)| = 2|V(H)| - 1$. For any red subgraph $H$ with at least one edge, we get $|E(H)| = |E(H_r)| \leq |V(H_r)| \leq 2|V(H)| - 2$. In particular (2) holds for every 2-decomposable graph.

The necessity is quite clear. Indeed, consider a nice 2-decomposition $(G_b, G_r)$ of $G$ and assume that (3) does not hold. Thus there exists $H$ a bipartite subgraph of $G$ with $|E(H)| = 2|V(H)| - 1$. So equality holds in the previous computation and we have $|E(H_b)| = |V(H_b)| - 1$ and $|E(H_r)| = |V(H_r)|$. In particular $H_b$ is a spanning tree of $H$ and $H_r$ contains at least one cycle $C$. As $(G_b, G_r)$ is nice, $C$ contains a precious edge $xy$. As $H_b$ is connected, there exists a black path $P$ from $x$ to $y$ and $P$ has even length because $x$ and $y$ belong to the same part of the canonical bipartition of $(G_b, G_r)$. So $P \cup xy$ forms an odd cycle of $H$, a contradiction.

Now let us prove the sufficiency. Let $(G_b, G_r)$ be a 2-decomposition of $G$. A red component $R$ of the decomposition is **bad** if it is not a tree and its (hence unique) cycle does not contain any precious edges. If we remove from a bad component $R$ all its precious edges, we obtain several connected components, one of which contains the cycle of $R$. We call this component the **core** of $R$ and denote it by $c(R)$. For convenience we use $c(R)$ below to denote both a vertex set and the red subgraph induced by these vertices. Note that $G[c(R)]$ is bipartite as $c(R)$ contains no precious edge. A **sequence** of the decomposition $(G_b, G_r)$ is a list $\mathcal{R} = (c(R_1), \ldots, c(R_i))$ of the cores of its bad red components in decreasing order of cardinality.

Among all the 2-decompositions of $G$, we choose one whose sequences $\mathcal{R} = (c(R_1), \ldots, c(R_i))$ satisfy

$(a)$ $i$ is minimal, and

$(b)$ subject to $(a)$, $|c(R_i)|$ is minimal.

We will prove that this 2-decomposition $(G_b, G_r)$ is nice, that is, $\mathcal{R} = \emptyset$. Assume that it is not the case and consider $\{X_1, \ldots, X_p\}$ the black components of $G[c(R_i)]$ (that is, the connected components of $G_b[c(R_i)]$). If $p = 1$, then $G[c(R_i)]$ is connected in black, and as $G_r[c(R_i)]$ is unicyclic, the bipartite graph $G[c(R_i)]$ violates (3), a contradiction. So we must have $p \geq 2$. Now denote by $W_1, \dots, W_q$ the connected components of $G_b \setminus c(R_i)$ and construct a graph $T'$ on $\{X_1, \dots, X_p, W_1, \dots, W_q\}$ by connecting two vertices of $T'$ if there exists an edge in $G_b$ between the corresponding components. In other words, we contract the (connected) vertex sets $X_1, \dots, X_p, W_1, \dots, W_q$ in $G_b$ to single vertices. So $T'$ is a tree. Finally we consider $T$ the minimal subtree of $T'$ containing $\{X_1, \dots, X_p\}$. By definition the leaves of $T$ are in $\{X_1, \dots, X_p\}$ and as $p \geq 2$, $T$ has at least two such leaves. So we consider a leaf $X_k$ of $T$ which does not contain entirely the red cycle of $R_i$ (this could occur even without violating (3) if $X_k$ is not connected in red for instance). We denote by $W_{k'}$ the only neighbour of $X_k$ in $T$. Now, we specify two edges, one black and one red in order to 'change their color' and obtain a contradiction. First denote by $uv$ the unique black edge between $X_k$ and $W_{k'}$. We suppose that $u \in X_k$ and $v \in W_{k'}$ (so $v \notin c(R_i)$). Now we look at a 1-orientation of $c(R_i)$ (this is an orientation of $c(R_i)$ in which every vertex has out-degree at most 1) and consider a maximal oriented red path leaving $u$ with all its vertices in $X_k$. As $X_k$ does not contain entirely the red cycle of $R_i$, this path ends at a vertex $x \in X_k$ which has a unique red out-neighbour $y \in X_{k''}$ for some $k'' \neq k$. We select this red edge $xy$.

Notice that the unique black path from $x$ to $y$ contains the edge $uv$. Indeed the unique path from $X_k$ to $X_{k''}$ in $T$ corresponds to the unique black path $P$ from $X_k$ to $X_{k''}$ in $G$. As $X_k$ is a leaf of $T$, the first edge of $P$ is $uv$ and as $X_k$ and $X_{k''}$ are respectively connected in black, the unique black path from $x$ to $y$ contains $P$ and so it contains the edge $uv$. This implies that $(G_b + xy) - uv$ is also a spanning tree of $G$. Moreover, its bipartition is the same as the bipartition of $G_b$. Indeed as $xy$ is an edge lying inside the core of $R_i$, it is not precious and $G_b + xy$ is still bipartite and has the same bipartition as $G_b$. Removing $uv$ does not affect the bipartition (because $(G_b + xy) - uv$ is connected).

To conclude, we focus on the red part of the new 2-decomposition $((G_b + xy) - uv, (G_r + uv) - xy)$. By construction, the component $X$ of $G_r - xy$ containing $u$ (and also $x$) is a red tree. Remark that $X = R_i$ if and only if $xy$ is an edge of the cycle of $R_i$. If $v$ does not belong to $R_i$, then by adding $uv$ we attached $X$ to another red component in $(G_r + uv) - xy$). As $X$ contains at least one vertex, namely $u$, $|c(R_i)|$ has decreased, a contradiction to $(b)$ in the choice of $(G_b, G_r)$ (or to $(a)$ if $X = R_i$). If $v$ belongs to $R_i$ but $v$ does not belong to $X$ (in this case we have $X \neq R_i$), then $v \in R_i \setminus c(R_i)$ and by adding $uv$ we attached $X$ to a vertex of $R_i \setminus c(R_i)$. Once again, $|c(R_i)|$ has decreased, a contradiction to $(b)$ in the choice of $(G_b, G_r)$. Finally if $v$ belongs to $X$ then adding $uv$ produces a new red unicyclic component. However as the red path in $G_r$ from $v$ to $u$ starts in $R_i \setminus c(R_i)$ and ends in $c(R_i)$, it contains a precious edge. So that newly created red unicyclic component is not bad, and $|c(R_i)|$ has decreased. Hence, again, we either contradict $(b)$, or $(a)$ if $X = R_i$. contradicting $(a)$. $\diamond$

Now to finish the proof, we will show how to go from a nice 2-decomposition of a connected graph to a decomposition into a spanning tree and an odd pseudoforest (i.e. a pseudoforest in which every cycle has odd length). Let $G = (V, E)$ be a connected graph which admits a nice 2-decomposition and consider a nice 2-decomposition $(G_b, G_r)$ of $G$ with a minimum number of even red cycles. We will show by contradiction that this decomposition has no even red cycle. Assume it is not the case and denote by $C_1, \dots, C_l$ the even red cycles of $G_r$. In each of these, select a precious edge $e_i = x_i y_i$ and let $X = \{x_1, y_1, x_2, y_2, \dots, x_l, y_l\}$. Exchanging two edges between $G_b$ and $G_r$ will modify the structure of $G_b$, and some previously selected precious edges could become not precious any more. To avoid this we will find a vertex $u$ with the following property

$\mathcal{P}$ : There exists a component $B$ of $G_b \setminus u$ such that one of the following hold:

- $B \cap X$ contains only one element and this is not in the same component of $G_r$ as $u$ (*Case A*).
- $(B \cup \{u\}) \cap X$ contains exactly two elements and they are the end vertices of some $e_i$ (*Case B*).

First assume we have found such a vertex $u$ and let us see how to conclude, depending of which the two cases A or B we are in.

*Case A*. Denote by $x_i$ the only element of $B \cap X$ and by $B_i$ the red component of $G_r$ containing $x_i$. As $u$ does not belong to $B_i$, we can find an edge $vw$ along the black path in $G_b$ from $u$ to $x_i$ such that $w$ and $x_i$ are in the same component of $G_b - vw$, $w$ belongs to $B_i$ and $v$ does not belong to $B_i$.

So we exchange the colors of $vw$ and $x_iy_i$. The graph $B_i - x_iy_i$ is a tree and when we add $vw$ to $G_r$ we connect this tree to another component of $G_r$. The component of $G_b - vw$ containing $v$ is a tree containing all the vertices of $X$ except $x_i$. So the precious edges $e_j$ with $j \neq i$ are still precious edges, and this is also the case in $(G_b - vw) + x_iy_i$ which is a spanning tree of $G$. So, we reduce the number of even red cycle of the nice 2-decomposition $(G_b, G_r)$, a contradiction.

*Case B.* Denote by $x_i$ and $y_i$ the two elements of $(B \cup \{u\}) \cap X$ and also by $B_i$ the red component of $G_r$ containing the precious edge $x_iy_i$. If the black path $P$ in $G_b$ between $x_i$ and $y_i$ is not totally contained in $B_i$ then we can select a vertex $u'$ not belonging to $B_i$ along this path and end up in the previous case with $u$ replaced by $u'$. So $P$ is totally contained in $B_i$. Then, as $P + x_iy_i$ is an odd cycle (because $x_iy_i$ is precious), we can find along $P$ two consecutive vertices $vw$ which are in the same part of the bipartition induced by the bipartite graph $G_r[B_i]$. So we exchange $x_iy_i$ and $vw$. As previously $G_b + x_iy_i - vw$ is a spanning tree of $G$ such that all the edges $e_j$ with $j \neq i$ are still precious and $vw$ is also precious. The graph $G_r - x_iy_i + vw$ is now a pseudoforest, and we have reduced the number of even red cycles of the nice 2-decomposition $(G_b, G_r)$, a contradiction.

Finally, let us see how to find a vertex $u$ in $G$ which has property $\mathcal{P}$. Consider $T'$ the minimal subtree of $G_b$ containing all the vertices of the set $X$. In particular all the leaves of $T'$ are elements of $X$. Then build the tree $T$ from $T'$ by replacing iteratively each vertex of degree 2 in $T'$ and not belonging to $X$ by an edge linking its two neighbours in $T'$. The vertices of $T$ are now vertices of $X$ or have degree at least three in $T$. Assume first that a leaf $f$ of $T$ has its neighbour $f'$ in $X$. Denote by $B$ the component of $G_b \setminus f'$ containing $f$. By construction $f$ is the unique element of $B \cap X$. We select $u = f'$. If $f$ and $f'$ are in different components of $G_r$ then we are in *Case A*, otherwise we are in *Case B*.

So we can assume that all the leaves of $T$ are neighbours of vertices of $T$ which are not in $X$ and have degree at least three in $T$. Consider now a leaf $f'$ of the tree obtained from $T$ by removing its leaves. Denote by $L$ the set of neighbours of $f'$ in $T$ which are leaves of $T$. If $|L| = 2$ and $L$ consists of the end vertices of some $e_i$ then we choose $u = f'$ and are in *Case B*. Otherwise, let $B_i$ be the component of $G_r$ containing $f'$ and consider a vertex $f$ of $L$ not belonging to $B_i$ (this exists as $|B_i \cap X| = 2$). Then we choose $u = f'$ and $B$ to be the component of $G_b \setminus f'$ containing $f$ and we are in *Case A*. $\diamond$