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# Induced minors and well-quasi-ordering

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## Abstract

A graph  $H$  is an induced minor of a graph  $G$  if it can be obtained from an induced subgraph of  $G$  by contracting edges. Otherwise,  $G$  is said to be  $H$ -induced minor-free. Robin Thomas showed in [*Graphs without  $K_4$  and well-quasi-ordering*, Journal of Combinatorial Theory, Series B, 38(3):240 – 247, 1985] that  $K_4$ -induced minor-free graphs are well-quasi ordered by induced minors.

We provide a dichotomy theorem for  $H$ -induced minor-free graphs and show that the class of  $H$ -induced minor-free graphs is well-quasi-ordered by the induced minor relation if and only if  $H$  is an induced minor of the gem (the path on 4 vertices plus a dominating vertex) or of the graph obtained by adding a vertex of degree 2 to the complete graph on 4 vertices.

Similar dichotomy results were previously given by Guoli Ding in [*Subgraphs and well-quasi-ordering*, Journal of Graph Theory, 16(5):489–502, 1992] for subgraphs and Peter Damaschke in [*Induced subgraphs and well-quasi-ordering*, Journal of Graph Theory, 14(4):427–435, 1990] for induced subgraphs.

*Keywords:* Well-quasi-ordering, induced minors, combinatorial dichotomies.

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## 1 Introduction

A *well-quasi-order* (*wqo* for short) is a quasi-order which contains no infinite decreasing sequence, nor an infinite collection of pairwise incomparable elements (called an *antichain*). One of the most important results in this field is arguably the theorem by Robertson and Seymour which states that graphs are well-quasi-ordered by the minor relation [14]. Other natural containment relations are not so generous; they usually do not wqo all graphs. In the last decades, much attention has been brought to the following question: given a partial order  $(S, \preceq)$ , what subclasses of  $S$  are well-quasi-ordered by  $\preceq$ ? For instance, Fellows et al. proved in [7] that graphs with bounded feedback-vertex-set are well-quasi-ordered by topological minors. Other papers considering this question include [1, 3–6, 8, 9, 13, 15].

One way to approach this problem is to consider graph classes defined by excluded substructures. In this direction, Damaschke proved in [4] that a class of graphs defined by one forbidden induced subgraph  $H$  is wqo by the induced subgraph relation iff  $H$  is the path on four vertices. Similarly, a bit later Ding proved in [5] an analogous result for the subgraph relation. Other authors also considered this problem (see for instance [2, 10, 11]). In this paper, we provide the answer to the same question for the induced minor relation, which we denote  $\leq_{\text{im}}$ . Before stating our main result, let us introduce two graphs which play a major role in this paper:  $\hat{K}_4$  is obtained by adding a vertex of degree two to  $K_4$  and the gem by adding a dominating vertex to  $P_4$ . (cf. Figure 1).

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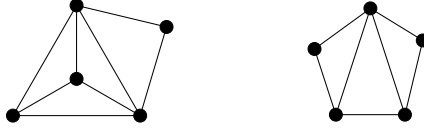


Fig. 1. The graph  $\hat{K}_4$  (on the left) and the gem (on the right).

## 2 Induced minors and well-quasi-ordering

Our main result is the following.

**Theorem 2.1 (Dichotomy Theorem)** *Let  $H$  be a graph. The class of  $H$ -induced minor-free graphs is wqo by  $\leq_{\text{im}}$  iff  $H \leq_{\text{im}} \hat{K}_4$  or  $H \leq_{\text{im}} \text{gem}$ .*

Our proof naturally has two parts: for different values of  $H$ , we need to show wqo of  $H$ -induced minor-free graphs or exhibit an  $H$ -induced minor-free infinite antichain. Due to space limitations, we only present the main ideas of the proof of the dichotomy theorem.

### 2.1 Classes that are wqo

The following two theorems describe the structure of graphs with  $H$  forbidden as an induced minor, when  $H$  is  $\hat{K}_4$  and the gem, respectively.

**Theorem 2.2 (Decomposition of  $\hat{K}_4$ -induced minor-free graphs)** *Let  $G$  be a 2-connected graph of  $\text{Excl}_{\text{im}}(\hat{K}_4)$ . Then:*

- either  $G \not\leq_{\text{im}} K_4$ ;
- or  $G$  is a subdivision of a graph on at most 9 vertices;
- or  $V(G)$  has a partition  $(C, M)$  such that  $G[C]$  is an induced cycle,  $G[M]$  is a complete multipartite graph and every vertex of  $C$  is either adjacent in  $G$  to all vertices of  $M$ , or to none of them.

**Theorem 2.3 (Decomposition of gem-induced minor-free graph)** *Let  $G$  be a 2-connected gem-induced minor-free graph. Then  $G$  has a subset  $X \subseteq V(G)$  of at most six vertices such that every connected component of  $G \setminus X$  is either a cograph, or a path whose internal vertices are of degree two in  $G$ .*

Using these structural results, we are able to show the wqo of the two classes with respect to induced minors.

## 2.2 Classes that are not wqo

For classes not covered by previous subsection, that is for any graph  $H$  which is not an induced minor of one of  $\hat{K}_4$  and  $\text{gem}$ , we need to show that the  $H$ -induced minor-free graphs are not wqo by  $\leq_{\text{im}}$ . The idea is to consider an infinite antichain for induced minors, and to show that infinitely many of its elements are  $H$ -induced minor-free. Let  $\overline{G}$  denote the complement of any graph  $G$ . Using the infinite antichain  $\{\overline{C_n}\}_{n \geq 6}$ , we are able to prove the following lemma.

**Lemma 2.4** *If the class of  $H$ -induced minor-free graphs is wqo by  $\leq_{\text{im}}$ , then  $\overline{H}$  is disjoint union of paths.*

Using Lemma 2.4 and two antichains introduced in [6] and in [12], we can deduce the following properties of  $\overline{H}$ . (Let  $\text{cc}(G)$  denote the number of connected components of a graph  $G$ .)

**Lemma 2.5** *If  $H$ -induced minor-free graphs are wqo by  $\leq_{\text{im}}$ , then (a)  $\overline{H}$  has at most 4 connected components; (b) the largest connected component of  $\overline{H}$  has at most 4 vertices; (c) if  $\text{cc}(\overline{H}) = 3$  then  $|\text{V}(H)| \leq 5$ ; and (d) if  $\text{cc}(\overline{H}) = 4$  then  $|\text{V}(H)| \leq 4$ .*

Table 1 enumerates all the possible cases for  $\overline{H}$ , where each line corresponds to a fixed number of vertices and each column to a fixed value of  $\text{cc}(\overline{H})$ . A gray cell means that either that no such graph exists, or that the cell corresponds to cases where  $H$ -induced minor-free graphs are not wqo by  $\leq_{\text{im}}$  (according to Lemma 2.5). The complement of any of the twelve remaining graphs can easily be shown to be induced minor of  $\hat{K}_4$  or  $\text{gem}$ .

$ \text{V}(H)  \setminus \text{cc}(\overline{H})$	1	2	3	4	$\geq 5$
1	$K_1$				(a)
2	$K_2$	$2 \cdot K_1$			(a)
3	$P_3$	$K_2 + K_1$	$3 \cdot K_1$		(a)
4	$P_4$	$P_3 + K_1$	$K_2 + 2 \cdot K_1$	$4 \cdot K_1$	(a)
5	(b)	$P_4 + K_1$	$P_3 + 2 \cdot K_1$	(d)	(a)
$\geq 6$	(b)	(b)	(c)	(d)	(a)

Table 1

If  $H$ -induced minors-free graphs are wqo by  $\leq_{\text{im}}$ , then  $\overline{H}$  belongs to this table.

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