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# FPT Algorithms for Plane Completion Problems\*

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## Abstract

The PLANE SUBGRAPH (resp. TOPOLOGICAL MINOR) COMPLETION problem asks, given a (possibly disconnected) plane (multi)graph  $\Gamma$  and a connected plane (multi)graph  $\Delta$ , whether it is possible to add edges in  $\Gamma$  without violating the planarity of its embedding so that it contains some subgraph (resp. topological minor) that is topologically isomorphic to  $\Delta$ . We give FPT algorithms that solve both problems in  $f(|E(\Delta)|) \cdot |E(\Gamma)|^2$  steps. Moreover, for the PLANE SUBGRAPH COMPLETION problem we show that  $f(k) = 2^{\mathcal{O}(k \log k)}$ .

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## 1 Introduction

*Completion problems* on graphs are defined as follows: Consider a graph class  $\mathcal{P}$  and ask whether we may add edges to a given graph  $G$  in order to obtain a graph  $G^+$ , where  $G^+ \in \mathcal{P}$ . Numerous results have appeared for the case where the objective is to minimize the number of edges added in  $G$  [13, 9, 11, 8, 3].

In this paper, we consider the PLANE SUBGRAPH (resp. TOPOLOGICAL MINOR) COMPLETION (PSC) (resp. PTMC) problem which, given a (possibly disconnected) plane graph  $\Gamma$ , called the *host graph*, and a connected plane graph  $\Delta$ , called the *pattern graph*, asks whether it is possible to add edges in  $\Gamma$  such that the resulting graph remains plane and contains some subgraph (resp. topological minor) that is topologically isomorphic to  $\Delta$ . Both  $\Gamma$  and  $\Delta$  are allowed to have multiple edges but not loops. When the input graph  $\Gamma$  is planar triangulated, both PSC and PTMC are NP-complete. Indeed, let  $G$  be any planar triangulated graph. Note here, that as any planar triangulated graph is 3-connected,  $G$  is 3-connected and from Whitney's Theorem [12] admits a unique embedding on the plane (up to equivalence), say  $\Gamma$ . Let also  $\Delta$  be the cycle on  $n = |V(G)|$  vertices. Then  $\Delta$  also has unique embedding on the plane (up to equivalence). Since  $\Gamma$  is triangulated no edge can be added to it while preserving its planarity. Thus, both PSC and PTMC become equivalent to the HAMILTON CYCLE PROBLEM which is NP-complete on planar triangulated graphs [4] (see also [7]). This observation further implies that PSC and PTMC parameterized by the number of added edges  $k$ , and in particular even for  $k = 0$ , are NP-complete. Thus, PSC and PTMC are not FPT when parameterized by the number of added edges unless  $P = NP$ . Thus, in order to obtain a tractable algorithm, we need to find an alternative way to parameterize these problems. In particular, we will consider  $|E(\Delta)|$  as our parameter. Our two main results are the following.

► **Theorem.** PSC parameterized by the number of edges of the pattern graph  $\Delta$ , say  $k$ , can be solved in  $2^{\mathcal{O}(k \log k)} \cdot m^2$  time, where  $m = |E(\Gamma)|$ .

► **Theorem.** PTMC parameterized by the number of edges of the pattern graph  $\Delta$ , say  $k$ , can be solved in  $f(k) \cdot m^2$  time, where  $m = |E(\Gamma)|$  and  $f$  is a computable function.

For the PTMC algorithm our approach is the following. Let  $\Gamma$  and  $\Delta$  be an input of the problem as above. We first apply a series of transformations on our input graph  $\Gamma$  that turn it into a combinatorial structure  $\mathbf{G}$  whose treewidth is bounded by a function of  $|E(\Delta)|$ . Then, we apply a series of transformations on our input graph  $\Delta$  that also turn it into a combinatorial structure  $\mathbf{D}$ . Finally, we show that  $(\Delta, \Gamma)$  is a yes-instance of our problem if and only if an MSO-expressible relation holds for  $\mathbf{G}$  and  $\mathbf{D}$ , thus translating our problem into a purely combinatorial one. Then by employing Courcelle's Theorem we prove our algorithm. We remark here that a similar approach could also solve the PLANE SUBGRAPH COMPLETION problem. However, with a more careful analysis we are able to derive an algorithm with much better bounds on the dependence on the parameter.

Our approach towards solving PSC is the following. Let  $\Gamma$  and  $\Delta$  be an input of PSC, where  $|E(\Delta)| = k$  for some positive integer  $k$ . We construct a family  $\mathcal{G}$  consisting of  $\mathcal{O}(n)$  combinatorial structures depending only on  $\Gamma$  whose underlying graphs have treewidth  $\mathcal{O}(k)$ . We also construct a family  $\mathcal{H}$  consisting of  $2^{\mathcal{O}(k \log k)}$  combinatorial structures depending only on  $\Delta$ , again by applying series of appropriate transformations on them (different than the transformations for PTMC). For the graphs  $\Gamma$  and  $\Delta$  and the families  $\mathcal{G}$  and  $\mathcal{H}$ , it holds that  $(\Delta, \Gamma)$  is a yes-instance if and only if some structure  $\mathbf{D} \in \mathcal{H}$  is contained as a *contraction* in a structure  $\mathbf{G} \in \mathcal{G}$ , denoted  $\mathbf{D} \leq_c \mathbf{G}$ . Therefore, we again translate our problem into one of

combinatorial nature. Finally, for a fixed pair of structures  $(\mathbf{D}, \mathbf{G}) \in \mathcal{H} \times \mathcal{G}$  with the above properties, we can decide in  $2^{\mathcal{O}(k \log k)} \cdot m$  time whether  $\mathbf{D} \leq_c \mathbf{G}$ . Therefore, by testing for all pairs  $(\mathbf{D}, \mathbf{G}) \in \mathcal{H} \times \mathcal{G}$  whether  $\mathbf{D} \leq_c \mathbf{G}$ , we decide in  $2^{\mathcal{O}(k \log k)} \cdot m^2$  steps whether  $(\Delta, \Gamma)$  is a yes-instance.

The paper is organized as follows. In Section 2 we give the necessary definitions. In Section 3 we present the algorithm for the PSC problem and in Section 4 we present the algorithm for the PTMC problem. In the concluding Section 5 we discuss about other completion problems that can be solved by modifying our results, such as the PLANE INDUCED SUBGRAPH COMPLETION, the PLANE MINOR COMPLETION, the PLANAR ROOTED TOPOLOGICAL MINOR, and the PLANAR DISJOINT PATHS COMPLETION problems. The lemmas whose proofs have been omitted due to lack of space are marked with  $(\star)$ .

## 2 Definitions

For a positive integer  $n$ , we denote  $[n] = \{1, 2, \dots, n\}$ . Given a set  $S$ , a near-partition of  $S$  is a family of sets  $S_1, S_2, \dots, S_k$ , where  $S_i \cap S_j = \emptyset$ , for every  $i \neq j$ , and  $\cup_{i \in [k]} S_i = S$  (note that by the definition it is possible that  $S_i = \emptyset$  for some  $i \in [k]$ ). Unless stated otherwise, the graphs considered do not have loops but may have multiple edges. In a graph  $G$  we will denote by  $V(G)$  the set of its vertices and  $E(G)$  the set of its edges. We denote by  $\text{dist}_G(u, v)$  the distance of two vertices  $u$  and  $v$  in the graph  $G$ . Also, given a graph  $G$ , a vertex  $u \in V(G)$ , and  $V_0 \subseteq V(G)$ , we denote by  $N_G(u)$  the neighborhood of  $u$  in  $G$  and by  $N_G(V_0) := \bigcup_{v \in V_0} N_G(v) \setminus V_0$ . Given a vertex  $v$  with exactly two neighbors  $v_1$  and  $v_2$ , the dissolution of  $v$  is the operation where we delete  $v$  and add an edge  $\{v_1, v_2\}$  (even if one existed already).

Let  $G$  be a graph. A subset  $S$  of its vertices is a *separator* of  $G$  if the graph  $G - S := (V(G) \setminus S, E[V(G) \setminus S])$  is not connected. The size of a separator  $S$  is equal to  $|S|$ . The vertex contained in a separator of size 1 will be called a *cut-vertex*, while the vertices of a separator of size 2 will be called a *cut-pair*. For every integer  $k > 1$ , a graph  $G$  with at least  $k + 1$  vertices is *k-connected* if  $G$  has no separators of size less than  $k$ . For definitions not explicitly stated on the paper as well as more details on general graphs, see [6]

We say that a graph is *plane* when it is embedded without crossings between its edges on the sphere  $\Sigma = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ . We treat a plane graph as its embedding in  $\Sigma$  and we simply refer to it as *plane graph*. That is, we do not distinguish between a vertex of the graph and the point of the sphere used in the drawing to represent the vertex or between an edge and the open line segment representing it. We often use the term “general graph” in order to stress that a graph is treated as a combinatorial structure and not as a topological (i.e., embedded) one. Also, given a plane graph  $\Gamma$  we use the term *general graph* of  $\Gamma$  to refer to  $\Gamma$  as a combinatorial structure. We use capital greek letters for plane graphs and capital latin letters for general graphs.

We denote by  $\subseteq$ ,  $\subseteq_{sp}$ ,  $\subseteq_{in}$ ,  $\leq_m$ , and  $\simeq$  the usual subgraph, spanning subgraph, induced subgraph, minor, and isomorphism relation between two graphs, respectively. Given a graph  $G$  and  $V_0 \subseteq V(G)$ , we denote by  $G[V_0]$  the subgraph of  $G$  induced by  $V_0$ . We call  $V_0$  *connected* if  $G[V_0]$  is connected.

Let  $\Gamma$  be a plane graph and  $u \in V(\Gamma)$ . Then a tuple  $(u_1, \dots, u_k)$ , with possible repetitions, will be called a *cyclic neighborhood* of  $u$ , and denoted by  $\mathcal{N}_\Gamma(u)$ , if  $\{u, u_1\}, \dots, \{u, u_k\}$  are exactly the edges incident to  $u$ , as we meet them starting from  $(u, u_1)$  and proceeding clockwise.

Let  $A$  be a subset of  $\mathbb{R}^n$ . We define  $\text{int}(A)$  to be the interior of  $A$ ,  $\text{cl}(A)$  its closure and  $\text{bd}(A) = \text{cl}(A) \setminus \text{int}(A)$  its border. Given a plane graph  $\Gamma$  we denote its *faces* by  $F(\Gamma)$ , i.e.,

$F(\Gamma)$  is the set of the connected components of  $\Sigma \setminus \Gamma$  (in the operation  $\Sigma \setminus \Gamma$  we treat  $\Gamma$  as the set of points of  $\Sigma$  corresponding to its vertices and its edges). Given a graph  $G$  we denote by  $\mathcal{C}(G)$  the set of the connected components of  $G$ . For every  $f \in F(\Gamma)$  we denote by  $B_\Gamma(f)$  the graph induced by the vertices and edges of  $\Gamma$  whose embeddings are subsets of  $\mathbf{bd}(f)$  and we call it the *boundary* of  $f$ . We also denote by  $V(B_\Gamma(f))$  and  $E(B_\Gamma(f))$  the vertices and the edges of  $B_\Gamma(f)$ , respectively.

We define a *closed walk* of a graph  $G$  to be a cyclic ordering  $w = (v_1, \dots, v_l, v_1)$  of vertices of  $V(G)$  such that for any two consecutive vertices, say  $v_i, v_{i+1}$ , there is an edge between them in  $G$ , i.e.,  $\{v_i, v_{i+1}\} \in E(G)$ . Note here that there may exist two distinct indices  $i, j$  such that  $v_i, v_j \in w$  and  $v_i = v_j$  (the walk can revisit a vertex). We will denote by  $\ell_w = l$  the length of the respective closed walk  $w$ . We say that a walk  $w$  of a plane graph  $\Gamma$  is *facial* if there exists  $f_i \in F(\Gamma)$  and  $\Theta_j \in \mathcal{C}(B_\Gamma(f_i))$  such that the vertices of  $w$  are the vertices of  $V(\Theta_j)$  and the cyclic ordering of  $w$  indicates the way these vertices are met when making a closed walk along  $\Theta_j$  while always keeping  $f_i$  on the same side of the walk.

Given that  $\Gamma$  is a plane graph and  $\mathbf{w} = \{w_1, \dots, w_p\}$  is a non-empty set of closed walks of  $\Gamma$ , we say that  $\mathbf{w}$  is a *facial mapping* if there exists some face  $f$  of  $\Gamma$  such that  $\mathcal{C}(B_\Gamma(f)) = \{\Theta_1, \Theta_2, \dots, \Theta_p\}$  and  $w_j$  is a facial walk of  $\Theta_j$ ,  $j \in [p]$ . We define the length of the facial mapping  $\mathbf{w}$  to be  $\ell_{\mathbf{w}} = \sum_{i=1}^p \ell_{w_i}$ . Given a plane graph  $\Gamma$  and  $f \in F(\Gamma)$ , we define  $\mathbf{w}(f)$  as the facial mapping of  $\Gamma$  corresponding to  $f$  and define its length  $\ell_f$  to be the length  $\ell_{\mathbf{w}(f)}$  of its corresponding facial mapping. Observe that for every face  $f \in F(\Gamma)$ , its facial mapping  $\mathbf{w}(f)$  is unique (up to permutations). Let  $C_1, C_2$  be two disjoint closed curves of  $\Sigma$ . Let also  $D_i$  be the open disk of  $\Sigma \setminus C_i$  that does not contain points of  $C_{3-i}$ ,  $i \in [2]$ . The *annulus between  $C_1$  and  $C_2$*  is the set  $\Sigma \setminus (D_1 \cup D_2)$  and we denote it by  $A[C_1, C_2]$ . Notice that  $A[C_1, C_2]$  is a closed set.

Let  $\Gamma$  and  $\Delta$  be two plane graphs. We say that  $\Gamma$  and  $\Delta$  are *topologically isomorphic* if they are isomorphic via a bijection  $g : V(\Gamma) \rightarrow V(\Delta)$  and there exists a function  $h : F(\Gamma) \rightarrow F(\Delta)$ , such that for every  $f \in F(\Gamma)$ ,  $g(\mathbf{w}(f)) = \mathbf{w}(h(f))$  (where  $g(\mathbf{w}(f))$  is the result of applying  $g$  to every vertex of every closed walk in  $\mathbf{w}$ ). We call the function  $\alpha : V(\Gamma) \cup F(\Gamma) \rightarrow V(\Delta) \cup F(\Delta)$  such that  $\alpha = g \cup h$ , a *topological isomorphism* between  $\Gamma$  and  $\Delta$ .

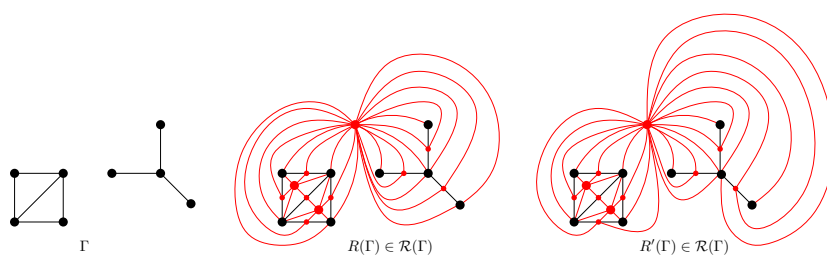
We say that a general graph  $G$  is *uniquely embeddable* if any two plane graphs  $\Gamma$  and  $\Gamma'$  that are embeddings of  $G$  in the sphere, are topologically isomorphic. We say that a plane graph  $\Gamma$  is *uniquely embedded* if its general graph  $G$  is uniquely embeddable, i.e.,  $\Gamma$  is the unique embedding of  $G$ , up to topological isomorphism. Given two plane graphs  $\Gamma_1$  and  $\Gamma_2$  we say that they are the *same* graph if they are topologically isomorphic (and not just isomorphic).

Let  $\Gamma$  and  $\Delta$  be two plane graphs and let  $Z \subseteq V(\Gamma)$ . We say that  $\Delta$  is a  *$Z$ -embedded subgraph* of  $\Gamma$ , and write  $\Delta \leq_{es}^Z \Gamma$ , if  $\Delta$  is topologically isomorphic to some subgraph of  $\Gamma \setminus Z$ . When  $Z = \emptyset$ , we say that  $\Delta$  is an *embedded subgraph* of  $\Gamma$  and write  $\Delta \leq_{es} \Gamma$ .

Let  $\Gamma$  and  $\Delta$  be two plane graphs and let  $Z \subseteq V(\Gamma)$ . We say that  $\Delta$  is a  *$Z$ -embedded topological minor* of  $\Gamma$ , and write  $\Delta \leq_{etm}^Z \Gamma$  if there exist a function  $\rho_1 : V(\Delta) \rightarrow V(\Gamma)$  and a function  $\rho_2 : E(\Delta) \rightarrow \mathcal{P}(\Gamma)$ , where  $\mathcal{P}(\Gamma)$  denotes the set of all paths of  $\Gamma$  such that

1. For every  $v \in V(\Delta)$ ,  $\rho_1(v) \notin Z$ .
2. For every  $e = \{u, v\} \in E(\Delta)$ , the path  $\rho_2(e)$  of  $\Gamma$  has  $\rho_1(u)$  and  $\rho_1(v)$  as its endpoints and if  $e_1 \neq e_2$ , then  $\rho_2(e_1)$  and  $\rho_2(e_2)$  are internally vertex-disjoint.
3. If  $\Gamma \langle \rho_2 \rangle$  is the graph obtained by the union of all paths in  $\rho_2(E(\Delta))$  after we dissolve all vertices that are not vertices in  $\rho_1(V(\Delta))$ , then there is a topological isomorphism  $\alpha : V(\Delta) \cup F(\Delta) \rightarrow V(\Gamma \langle \rho_2 \rangle) \cup F(\Gamma \langle \rho_2 \rangle)$  between  $\Delta$  and  $\Gamma \langle \rho_2 \rangle$  where  $\alpha|_{V(\Delta)} = \rho_1$ .

When  $Z = \emptyset$ , we just write  $\Delta \leq_{etm} \Gamma$ .



■ **Figure 1** A disconnected plane graph  $\Gamma$  and two members of  $\mathcal{R}_\Gamma$ .

If in the 3rd condition of the above definition we replace topological isomorphism by isomorphism and consider general graphs, say  $H$  and  $G$ , we define the relation of  $H$  being a  $Z$ -topological minor of  $(G, Z)$ .

For definitions not explicitly stated on the paper as well as more details on plane graphs, see [10].

## 2.1 Radial Enhancements

Let  $\Gamma$  be a plane graph. A *subdivided radial enhancement* of  $\Gamma$  is defined as a plane graph that can be constructed as follows: consider  $\Gamma$ , subdivide every edge once, add a vertex  $v_f$  inside each face  $f$  of  $\Gamma$ . Consider a permutation  $(H_1, H_2, \dots, H_s)$  of the connected components of  $B_\Gamma(f)$  and a facial walk of each connected component. Then add edges connecting  $v_f$  with the vertices incident to  $B_\Gamma(f)$  in such a way that the first vertices in the cyclic neighborhood of  $v_f$  are the vertices of  $H_1$  and appear in the order of the fixed facial walk. Then the vertices of  $H_2$  follow, etc. Observe that in the resulting embedding, every face that is incident to an edge of  $E(\Gamma)$  is (planar) triangulated. This triangulation may have multiple edges unless the boundary of each face of  $\Gamma$  is a cycle, as can be seen in the two distinct examples of a subdivided radial enhancement of a disconnected plane graph  $\Gamma$  in Figure 1.

Notice that the vertices of the resulting plane graph can be partitioned into three independent sets: the *original vertices* of  $\Gamma$  denoted by  $V_o(\Gamma)$ , the *subdivision vertices* denoted by  $V_s(\Gamma)$ , which are the ones that were introduced after subdividing the edges, and the *radial vertices* denoted by  $V_r(\Gamma)$ , which are the ones that were added inside each face. Notice also that the edges of the resulting plane graph can be partitioned into two independent sets: the *subdivision edges* denoted by  $E_s(\Gamma)$  and the *radial edges*, denoted by  $E_r(\Gamma)$ , that were introduced after adding the radial vertices.

We denote by  $\mathcal{R}_\Gamma$  the set of all different (in terms of topological isomorphism) subdivided radial enhancements of  $\Gamma$ . Observe that if  $\Gamma$  is connected, then the boundary of each face of  $\Gamma$  is connected and we obtain the following.

► **Observation 1.** *For every connected plane graph  $\Gamma$ ,  $R(\Gamma)$  is uniquely defined and thus  $\mathcal{R}_\Gamma$  contains only one member.*

From the subdivided radial enhancement's construction we obtain the following.

► **Observation 2.** *For every plane graph  $\Gamma$  and every  $R(\Gamma) \in \mathcal{R}_\Gamma$  it holds that  $|E(R(\Gamma))| = \mathcal{O}(|E(\Gamma)|)$ .*

Given a plane graph  $\Gamma$  and a graph  $R(\Gamma) \in \mathcal{R}_\Gamma$ , for every integer  $i > 1$ , we denote by  $R^i(\Gamma)$  the graph  $R(R^{i-1}(\Gamma))$ , where  $R^1(\Gamma) = R(\Gamma)$ . We define then  $V_o^i(\Gamma) = V(R^{i-1}(\Gamma))$ ,  $V_s^i = V_s(R^{i-1}(\Gamma))$ , and  $V_r^i = V_r(R^{i-1}(\Gamma))$ . For notation consistency, we will denote by

$V_s^1(\Gamma) = V_s(\Gamma)$ ,  $V_r^1(\Gamma) = V_r(\Gamma)$  and  $V_o^1(\Gamma) = V_o(\Gamma)$ . We also define the sets of edges  $E_s^i(\Gamma)$  which are the edges obtained in  $R^i(\Gamma)$  after subdividing the edges  $E_s^{i-1}(\Gamma)$  and  $E_r^i(\Gamma) = E(R^i(\Gamma)) \setminus E_s^i(\Gamma)$ .

**► Lemma 2.1** ( $\star$ ). *For every plane graph  $\Gamma$  (with possibly multiple edges) every member of  $\mathcal{R}_\Gamma$  is connected. Moreover, if  $\Gamma$  is  $i$ -connected, then  $R(\Gamma)$  is  $(i+1)$ -connected, for  $i \in [2]$ .*

**► Remark.** If  $\Gamma$  is 2-connected then  $R(\Gamma)$  can also be shown to be 4-connected. However, 3-connectivity is sufficient for our purposes.

## 2.2 Graph Structures

A key-concept in our algorithms is the notion of the vertex and the edge structure which is formally defined as follows. Let  $G$  be a simple planar graph,  $k, l \in \mathbb{N}$ ,  $(S_1, S_2, \dots, S_k)$  be a near-partition of  $V(G)$  and  $E_1, E_2, \dots, E_l$  be a near-partition of  $E(G)$ . A *vertex structure*  $\mathbf{G}$  is a tuple  $(G, S_1, S_2, \dots, S_k)$  and an *edge structure*  $\mathbf{G}'$  is a tuple  $(G, E_1, E_2, \dots, E_l)$ .

Let  $\mathbf{G} = (G, A, X_1, \dots, X_l)$  and  $\mathbf{D} = (D, B, Y_1, \dots, Y_l)$  be vertex structures, where  $l \in \mathbb{N}$ . We say that  $\mathbf{D}$  is a *contraction* of  $\mathbf{G}$ , denoted by  $\mathbf{D} \leq_c \mathbf{G}$ , if and only if there exists a function  $\sigma : V(G) \rightarrow V(D)$  satisfying the following *contraction properties*:

1. if  $u, v \in V(D)$ ,  $u \neq v \Leftrightarrow \sigma^{-1}(u) \cap \sigma^{-1}(v) = \emptyset$ ,
2. for every  $u \in V(D)$ ,  $G[\sigma^{-1}(u)]$  is connected,
3.  $\{u, v\} \in E(D) \Leftrightarrow G[\sigma^{-1}(u) \cup \sigma^{-1}(v)]$  is connected,
4.  $\sigma(A) \subseteq B$ , and
5. for every  $i \in [l]$  and every  $x \in Y_i$  it holds that  $|\sigma^{-1}(x)| = 1$  and  $\sigma^{-1}(x) \in X_i$ .

In particular, a graph  $D$  is a *contraction* of a graph  $G$  if  $(D, V(D)) \leq_c (G, V(G))$  and we write  $D \leq_c G$ . Notice that  $\leq_c$  defined for graphs is the usual contraction relation where only conditions 1, 2, and 3 apply. Observe that for any two vertex structures  $\mathbf{G}$  and  $\mathbf{D}$ , where  $G$  and  $D$  respectively are their associated planar graphs,  $\mathbf{D} \leq_c \mathbf{G}$  implies that  $D \leq_c G$ .

We will also need the following proposition, which follows from the results in [1].

**► Proposition 1.** *There exists an algorithm that receives as input a vertex structure  $\mathbf{G}$ , whose graph has  $m$  edges and treewidth at most  $h$ , and a vertex structure  $\mathbf{D}$ , whose graph is connected and has  $k$  edges, and outputs whether  $\mathbf{D} \leq_c \mathbf{G}$  in  $2^{\mathcal{O}(k+h+k \log h)}$  steps.*

Let  $\mathbf{G} = (G, S_1, \dots, S_l)$  be a vertex structure on a planar graph  $G$ , where  $l \in \mathbb{N}$ . Given a possibly empty  $Q \subseteq V(G)$ , notice that the tuple  $(Q, S_1 \setminus Q, \dots, S_l \setminus Q)$  also forms a near-partition of  $V(G)$ . Then, we can define the following operator on vertex structures:

$$\mathbf{d}(\mathbf{G}, Q) := (G, Q, S_1 \setminus Q, \dots, S_l \setminus Q).$$

Obviously,  $\mathbf{d}(\mathbf{G}, Q)$  is also a vertex structure on  $G$ .

Let  $\Gamma$  be a plane graph and consider an  $R(\Gamma) \in \mathcal{R}_\Gamma$ . By Lemma 2.1 and Observation 1, the graph  $R^3(\Gamma)$  is uniquely defined according to  $R(\Gamma)$ . The following operators on  $(\Gamma, R(\Gamma))$  uniquely define a vertex and an edge structure:

$$\begin{aligned} \mathbf{p}(\Gamma, R(\Gamma)) &:= (R^3(\Gamma), V(\Gamma), V_s^1(\Gamma), V_r^1(\Gamma), V_s^2(\Gamma), V_r^2(\Gamma), V_s^3(\Gamma), V_r^3(\Gamma)) \\ \mathbf{e}(\Gamma, R(\Gamma)) &:= (R^3(\Gamma), E_s^3(\Gamma), E_r^3(\Gamma)). \end{aligned}$$

The underlying graph of the above structure is the general graph of  $R^3(\Gamma)$  and the vertex sets that form the partition of  $V(R^3(\Gamma))$  are the original vertices  $V(\Gamma)$ , followed by the sets of the subdivision and the radial vertices of each of the three subdivided radial enhancements.



Moreover, the edges are separated to those that have been obtain in  $R^3(\Gamma)$  only by subdividing original edges of the graph and those that where obtained after adding radial vertices and edges and subdividing those edges.

Throughout the rest of the paper we will only use structures defined by those three operators. The main purpose is to associate three subdivided radial enhancements to a given plane graph so that (i) the resulting graph is 3-connected and therefore uniquely embeddable, so we can disregard the embedding and treat it as a combinatorial object, and (ii) the vertices and edges of the original graph and each subdivided radial enhancement are distinguishable. In addition, both in PSC and the PTMC problems we try to match the faces of the pattern graph to faces, or parts of faces, of the host graph, the radial enhancements and the corresponding structures seem to be the appropriate tool to use, since we actually only need to match the radial vertices that are added inside each face.

Given a graph  $G$  and a non-negative integer  $k$ , we define the *ball* around a vertex  $v$  of  $G$  as the subgraph  $B_G^k(v)$  of  $G$  induced by the set of vertices at distance at most  $k$  from  $v$ . Consider now the subgraph  $\tilde{G}$  of  $G$  induced by the set of vertices that lay outside a given ball  $B_G^k(v)$ , i.e.,  $\tilde{G} = G \setminus B_G^k(v)$ , and consider the set  $\mathcal{C}(\tilde{G})$  of all its connected components. Then by contracting all the edges of every  $C \in \mathcal{C}(\tilde{G})$  to a single vertex in  $G$ , denoted  $v_C$ , we obtain the *k-contracted graph* around  $v$ , that will be denoted by  $G_v$ . Given a vertex structure  $\mathbf{G} = (G, \emptyset, S_1, \dots, S_l)$  and a non-negative integer  $k$ , we define the *k-contracted vertex structure* around a vertex  $v$  of the graph  $G$  as  $\mathbf{G}_v^{(k)} := (G_v, \{v_C \mid C \in \mathcal{C}(\tilde{G})\}, S'_1, \dots, S'_l)$ , where  $S'_i = S_i \cap B_G^k(v)$  for every  $i \in [l]$ .

### 3 An FPT algorithm for the PSC problem

Given a plane graph  $\Gamma$  we define *the set of non-edges of  $\Gamma$* :  $\overline{E(\Gamma)} = \binom{V(\Gamma)}{2} \setminus E(\Gamma)$ . A set of non-edges  $S \subseteq \overline{E(\Gamma)}$  will be called *insertable* if there is a way to add the edges to  $\Gamma$  such that no two edges of  $E(\Gamma) \cup S$  intersect (apart from any common endpoints). Finally, we define the following relation between two plane graphs  $\Gamma$  and  $\Delta$ . We say that  $\Delta \preceq \Gamma$  if there exists a set  $S \subseteq \overline{E(\Gamma)}$  of insertable edges of  $\Gamma$  such that  $\Delta \leq_{es} \Gamma'$ , where  $\Gamma'$  is obtained from  $\Gamma$  after adding  $S$ . Then PSC asks, given two plane graphs  $\Gamma$  and  $\Delta$ , whether  $\Delta \preceq \Gamma$ .

The main idea of our algorithm is to create two families of vertex structures, one from the host graph  $\Gamma$  and the other from the pattern graph  $\Delta$ , such that  $\Delta \preceq \Gamma$  if and only if there are two structures  $\mathbf{D}$  and  $\mathbf{G}$  from each of the above families such that  $\mathbf{D} \leq_c \mathbf{G}$ . Then, we bound the size of these families and use the algorithm from Proposition 1 to check all pairs of their members for the required property. From now on, in this section, whenever we refer to a structure we will assume that it is a vertex structure.

We define the first family of structures based on the host graph. Given a plane graph  $\Gamma$ , a subdivided radial enhancement of it,  $R(\Gamma)$ , and a positive integer  $k$ , we define the following family of structures:

$$\mathcal{G}_{\Gamma, R(\Gamma), k} := \{\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{(k)} \mid v \in V(\Gamma)\}.$$

Obviously,  $|\mathcal{G}_{\Gamma, R(\Gamma), k}| = |V(\Gamma)|$ , regardless of the choice of  $R(\Gamma)$  and  $k$ .

► **Lemma 3.1** ( $\star$ ). *Let  $\Gamma$  be a plane graph,  $R(\Gamma)$  a subdivided radial enhancement of  $\Gamma$ ,  $k \in \mathbb{N}$ ,  $v \in V(\Gamma)$ , and  $\mathbf{G}_v := \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{(k)} \in \mathcal{G}_{\Gamma, R(\Gamma), k}$ . Then the underlying graph  $G_v$  of the structure  $\mathbf{G}_v$  has treewidth at most  $3(k+1)$  and size  $\mathcal{O}(|E(\Gamma)|)$ .*

In order to define the second family of structures based on the pattern graph we need the following two definitions.



A *facial extension* of a connected plane graph  $\Delta$  is a connected plane graph  $\Delta^+$  satisfying the following properties:

1.  $\Delta \subseteq \Delta^+$ ,
2.  $V(\Delta^+) \setminus V(\Delta)$  is an independent set in  $\Delta^+$ , and
3. for every distinct  $x, y \in V(\Delta^+) \setminus V(\Delta)$ ,  $N_{\Delta^+}(x) \not\subseteq N_{\Delta^+}(y)$ .

We will denote by  $\mathcal{F}_\Delta$  the family of all facial extensions of the graph  $\Delta$ .

Given a connected plane graph  $\Delta$  and a subset  $L \subseteq E(\Delta)$  of its edges, we denote by  $\text{span}(\Delta, L)$  the set of all spanning subgraphs of  $\Delta$  that contain all the edges in  $E(\Delta) \setminus L$ . Note that such subgraphs could also contain some edges in  $L$ . A *pattern-guess* of a connected plane graph  $\Delta$  is an element  $\Delta^*$  of  $\text{span}(\Delta^+, E(\Delta))$ , for  $\Delta^+ \in \mathcal{F}_\Delta$ . That is, a spanning subgraph of a facial extension  $\Delta^+$  of  $\Delta$  containing at least all the edges in  $E(\Delta^+) \setminus E(\Delta)$ . The family of all possible pattern-guesses  $\Delta^*$  of  $\Delta$  will be denoted by  $\mathcal{PG}_\Delta$ .

Now, given a connected plane graph  $\Delta$  we define the following family of structures:  $\mathcal{H}_\Delta := \{\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), V(\Delta^*) \setminus V(\Delta)) \mid \Delta^* \in \mathcal{PG}_\Delta, R(\Delta^*) \in \mathcal{R}_{\Delta^*}\}$ .

► **Lemma 3.2** ( $\star$ ). *If  $\Delta$  is a connected plane graph then  $|\mathcal{H}_\Delta| = 2^{\mathcal{O}(|E(\Delta)| \cdot \log |E(\Delta)|)}$  and, for any structure  $\mathbf{D} \in \mathcal{H}_\Delta$ , the underlying graph  $D$  of  $\mathbf{D}$  has size and diameter bounded by  $\mathcal{O}(|E(\Delta)|)$ .*

► **Lemma 3.3** ( $\star$ ). *Let  $\mathbf{G} = (G, \emptyset, S_1, \dots, S_l)$  and  $\mathbf{D} = (D, B, Z_1, \dots, Z_l)$  be two structures, where  $B$  is an independent set and  $l \in \mathbb{N}$ . Then  $\mathbf{D} \leq_c \mathbf{G}$  if and only if there exists some  $v \in V(G)$  such that  $\mathbf{D} \leq_c \mathbf{G}_v^{(k)}$ , where  $k := \text{diam}(D)$ .*

The next theorem ensures the correctness of our algorithm.

► **Theorem 3.4**. *Let  $\Gamma$  be a plane graph and  $\Delta$  be a connected plane graph. It holds that  $\Delta \preceq \Gamma$  if and only if for every  $R(\Gamma) \in \mathcal{R}_\Gamma$  there exist two structures  $\mathbf{G} \in \mathcal{G}_{\Gamma, R(\Gamma), c}$  and  $\mathbf{D} \in \mathcal{H}_\Delta$ , such that  $\mathbf{D} \leq_c \mathbf{G}$ , where  $c$  is a constant such that  $\max_{\Delta^*} \{\text{diam}(R^3(\Delta^*))\} \leq c$ .*

**Proof.** First of all, we know that such a constant  $c$  exists from Lemma 3.2 and that in fact  $c = \mathcal{O}(|E(\Delta)|)$ . Let us first assume that  $\Delta \preceq \Gamma$ . Then there exists an insertable set of non-edges  $S \subseteq \overline{E(\Gamma)}$  and two plane graphs  $\Gamma' = (V(\Gamma), E(\Gamma) \cup S)$  and  $\Gamma_0$ , such that  $\Gamma_0 \subseteq \Gamma'$  and  $\Delta \simeq_{tp} \Gamma_0$ . Without loss of generality we may assume that all edges of  $S$  are also edges of  $\Gamma_0$ . Let then  $\alpha : V(\Gamma_0) \cup F(\Gamma_0) \rightarrow V(\Delta) \cup F(\Delta)$  be a topological isomorphism between  $\Gamma_0$  and  $\Delta$ . For every edge  $e = \{u, v\}$  of  $S$  let  $e_\alpha = \{\alpha(u), \alpha(v)\}$ . We define the sets  $S_\alpha = \{e_\alpha \mid e \in S\}$ ,  $S_\alpha^\Delta = S_\alpha \cap E(\Delta)$ , and  $S_\alpha^\Gamma = S_\alpha \setminus S_\alpha^\Delta$ .

We first construct a graph  $\Delta^+ \in \mathcal{F}_\Delta$ . For this, we add a set of vertices and edges embedded inside some of the faces of  $\Delta$  in such a way that edges intersect only at their common endpoints. In particular, for each face  $f \in F(\Delta)$  with facial mapping  $\mathbf{w}(f)$  do the following:

- For each edge  $e = \{u, v\}$  that lies inside the region enclosed by  $\alpha^{-1}(\mathbf{w}(f))$  in  $\Gamma$  and whose endpoints belong to  $\Gamma'$ , add the edge  $\{\alpha(u), \alpha(v)\}$  in the interior of  $f$  in  $\Delta$  in such a way that (i) edges intersect only at their common endpoints and (ii) after we extend the mapping  $\alpha$  so that it takes into account those edges of  $\Gamma$  that were added in  $\Delta$ , the following must hold: for any connected component that was inside  $f$  and, after the addition of the edges, is in a face  $f'$ , the preimages of the vertices of that connected component in  $\Gamma_0$  are inside the region enclosed by  $\alpha^{-1}(\mathbf{w}(f'))$ .
- Consider the faces  $f_1, f_2, \dots, f_j$  that form the partition of  $f$  after the addition of the new edges. For every such face  $f_i$  let  $p_i$  be the region enclosed by  $\alpha^{-1}(\mathbf{w}(f_i))$  in  $\Gamma'$ . Notice that since  $\Delta^+$  is connected, the boundary of  $\alpha^{-1}(\mathbf{w}(f_i))$  is connected. For every  $i \in [j]$

let  $\mathcal{C}_{p_i}$  be the set of all connected components that lie entirely in the region enclosed by  $\alpha^{-1}(\mathbf{w}(f_i))$  in  $\Gamma'$ . Let  $\mathcal{C}_{p_i}^\emptyset$  denote the set of all connected components in  $\mathcal{C}_{p_i}$  that do not have any neighbors in  $B_{\Gamma'}(f_i)$ . For every  $C_w \in \mathcal{C}_{p_i}$ , let  $S_w$  be its neighborhood in  $B_\Gamma(f_i)$ . Consider the Hasse diagram defined by the sets  $S_w$  and without loss of generality, let  $S_1, S_2, \dots, S_q$  be its maximal elements. Let then  $O_t = \{C_l \in \mathcal{C}_{p_i} \setminus \mathcal{C}_{p_i}^\emptyset \mid S_l \subseteq S_t\}$ ,  $t \in [q]$ . For every  $t \in [q]$ , add a vertex  $u_t$  in  $f_i$  and make it adjacent to the vertices in  $\alpha(S_t)$  (notice that since the boundary is again connected there is a unique way to construct the cyclic neighborhood of  $u_t$  up to cyclic permutations). We call  $O_t$  the *origin* of  $u_t$ .

The resulting graph  $\Delta^+$  is, by definition, a member of  $\mathcal{F}_\Delta$ .

To construct  $\Delta^*$  from  $\Delta^+$ , for every edge  $\{u, v\} \in S$ , we remove the edge  $\{\alpha(u), \alpha(v)\}$  from  $\Delta^+$ . Since  $\{\alpha(u), \alpha(v)\} \in E(\Delta)$ , it follows that

$$\Delta^* \in \text{span}(\Delta^+, E(\Delta)).$$

We now define a function  $g_0 : E(\Delta^*) \cup F(\Delta^*) \mapsto E(\Gamma) \cup F(\Gamma)$ . Let  $f \in F(\Delta^*)$  with facial mapping  $\mathbf{w}(f)$ . Observe that there is at least one face  $f' \in F(\Gamma)$  with facial mapping  $\mathbf{w}(f')$ , such that for every facial walk  $w = (u_1, \dots, u_k) \in \mathbf{w}(f)$  there is a facial walk  $w' \in \mathbf{w}(f')$  of length at least  $k$  and a subsequence  $(v_1, \dots, v_k)$  of  $w'$  (up to cyclic permutations) with the following properties:  $v_i = \alpha(u_i)$  if  $v_i \in \Delta$  and  $v_i \in V(C)$ , for some  $C$  in the origin of  $u_i$ , if  $u_i \in V(\Delta^*) \setminus V(\Delta)$ .

Notice that due to planarity the regions defined by those walks (unless the walks are trivial) are mutually nested. Of all such faces (if there are multiple), let  $f'$  be the one whose region contains all other regions. Then,  $g_0(f) = f'$ . We will call the connected component whose vertices belong to that walk the *outermost connected component*.

Recall that, by construction, the new vertices of  $V(\Delta^*) \setminus V(\Delta)$  form an independent set. Thus, for each edge  $e = \{u, v\} \in E(\Delta^*)$  at most one of its endpoints belongs in  $V(\Delta^*) \setminus V(\Delta)$ . If both endpoints  $u, v$  of  $e$  belong to  $V(\Delta)$ , then we define  $g_0(e) = \{\alpha^{-1}(u), \alpha^{-1}(v)\} \in E(\Gamma)$ . Otherwise exactly one of  $u$  and  $v$ , say  $v$ , belongs to  $V(\Delta^*) \setminus V(\Delta)$ . In this case, we define  $g_0(e) = \{\alpha^{-1}(u), v'\} \in E(\Gamma)$ , where  $v'$  is a neighbor of  $\alpha^{-1}(u)$  in the outermost connected component in the origin of  $v$ .

Let now  $R(\Gamma)$  be an arbitrary subdivided radial enhancement of  $\Gamma$ . In order to construct a subdivided radial enhancement  $R(\Delta^*)$  of  $\Delta$  recall that we first subdivide all edges of  $R(\Delta^*)$  and then add a radial vertex  $u_f$  inside each face  $f \in F(\Delta^*)$ . For every  $f$  let  $r_{g_0}(f)$  be the radial vertex of  $R(\Gamma)$  that was added in  $g_0(f)$ . Consider the cyclic neighborhood of  $r_{g_0}(f)$  in  $R(\Gamma)$ . Notice that it can be broken down in  $s_1, s_2, \dots, s_l$  segments where  $s_i$  is a facial walk  $w_i$  of  $\mathbf{w}(g_0(f))$ . Let  $w'_i$  be the subsequence of the walk that corresponds to a walk  $z_i$  in  $\mathbf{w}(f)$ . Add edges between the  $u_f$  and the vertices of the boundary of  $u_f$  in such a way that the cyclic neighborhood of  $u_f$  is  $(z_1, z_2, \dots, z_l)$ . Notice that for every subdivision vertex  $x$  of  $R(\Delta^*)$  that appears between  $u_i$  and  $u_{i+1}$  in the facial walk of  $w$ , there is a subdivision vertex  $v_x$  appearing between  $v_i$  and  $v_{i+1}$  in the walk  $w$  of  $\mathbf{w}(f)$ . We add an edge  $\{u_f, v_x\}$  so that  $v_x$  appears between  $u_i$  and  $u_{i+1}$  in the cyclic neighborhood of  $u_f$  (this can be done in a unique way). We extend the mapping  $g_0$  restricted to  $E(\Delta^*)$  to the mapping  $g_1$  by mapping every edge  $\{u_f, u_i\}$  to the edge  $\{r_{g_0}(f), v_i\}$ . We also map the edges  $\{u_f, x\}$  to the edges  $\{r_{g_0}(f), v_x\}$ . Notice that  $g_1$  can be extended to  $F(R(\Delta^*))$  similarly to  $g_0$ . In the same fashion we extend  $g_1$  to the function  $g_2$  on the graphs  $R^2(\Gamma)$  and  $R^2(\Delta^*)$  and then to  $g_3$  on the graphs  $R^3(\Gamma)$  and  $R^3(\Delta^*)$ . Recall that

$$\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset) = (R^3(\Gamma), \emptyset, V(\Gamma), V_s^1(\Gamma), V_r^1(\Gamma), \dots, V_s^3(\Gamma), V_r^3(\Gamma)),$$

and that

$$\begin{aligned} \mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) &= \\ &= (R^3(\Delta^*), V(\Delta^*) \setminus V(\Delta), V(\Delta), V_s^1(\Delta^*), V_r^1(\Delta^*), \dots, V_s^3(\Delta^*), V_r^3(\Delta^*)). \end{aligned}$$

Let now  $\sigma : V(R^3(\Gamma)) \rightarrow V(R^3(\Delta^*))$  such that:

$$\sigma(v) = \begin{cases} u & \text{if } v \in V(\Gamma), u \in V(\Delta), \text{ and } \alpha^{-1}(u) = v \in V(\Gamma) \\ z & \text{if } v \in V_s^i(\Gamma) \text{ and there exists } u \in V_s^i(\Delta^*) \text{ with } g^i(e) = e', i \in [3], \\ & \text{where } z \text{ (resp. } v) \text{ is the subdivision vertex of the edge } e \text{ (resp. } e') \\ w & \text{if } v \in V_r^i(\Gamma) \text{ and there exists } u \in V_r^i(\Delta^*) \text{ with } g^i(f) = f', i \in [3], \\ & \text{where } w \text{ (resp. } v) \text{ is the radial vertex added in face } f \text{ (resp. } f') \\ x & \text{where } x \in B \text{ such that the distance between } v \text{ and the vertices in} \\ & O_x \text{ in } R^3(\Gamma) \text{ is minimized} \end{cases}$$

It is quite straightforward to verify that  $\sigma$  satisfies the five required contraction properties and thus  $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) \leq_c \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)$ . Therefore, since these two structures satisfy the conditions of Lemma 3.3, we conclude that there exists some  $v \in V(\Gamma)$  such that  $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) \leq_c \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{\text{diam}(R^3(\Delta^*))}$ . Notice now that  $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) \in \mathcal{H}_\Delta$  and that  $\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^{\text{diam}(R^3(\Delta^*))}$  is a minor of  $\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)_v^c \in \mathcal{G}_{\Gamma, R(\Gamma), c}$  and we have proven the first direction.

Suppose now that for every  $R(\Gamma) \in \mathcal{R}_\Gamma$  there exist two structures  $\mathbf{G} \in \mathcal{G}_{\Gamma, R(\Gamma), c}$  and  $\mathbf{D} \in \mathcal{H}_\Delta$ , such that  $\mathbf{D} \leq_c \mathbf{G}$ . This is the same as saying that for every  $R(\Gamma) \in \mathcal{R}_\Gamma$  there exist a  $\Delta^+ \in \mathcal{F}_\Delta$ , a  $\Delta^* \in \text{span}(\Delta^+, E(\Delta))$ , and an  $R(\Delta^*) \in \mathcal{R}_{\Delta^*}$  such that  $\mathbf{d}(\mathbf{p}(\Delta^*, R(\Delta^*)), B) \leq_c \mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), \emptyset)$ . Let then  $\Gamma'$  be the plane graph that results from  $R^3(\Gamma)$  if we contract all connected components of  $R^3(\Gamma)[\sigma^{-1}(B)]$ . It follows immediately that  $\Gamma' \simeq_{tp} R^3(\Delta^*)$ . Let

$$\alpha : V(\Gamma') \cup F(\Gamma') \rightarrow V(R^3(\Delta^*)) \cup F(R^3(\Delta^*))$$

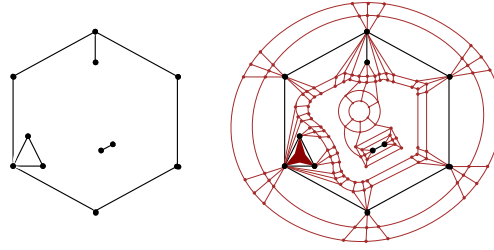
be a topological isomorphism between  $\Gamma'$  and  $R^3(\Delta^*)$ . Then, for each edge  $\{u, v\} \in E(\Delta) \setminus E(\Delta^*)$  there is a face  $f \in F(\Gamma)$  such that both  $\alpha^{-1}(u)$  and  $\alpha^{-1}(v)$  belong to a member of the facial mapping of  $f$ . Hence, the set  $S = \{\{\alpha^{-1}(u), \alpha^{-1}(v)\} \mid \{u, v\} \in E(\Delta) \setminus E(\Delta^*)\}$  is insertable in  $\Gamma$ . Hence,  $\Delta \preceq \Gamma$ .  $\blacktriangleleft$

► **Theorem 3.5** (\*). *There exists an algorithm that, given as input an  $n$ -edge plane graph  $\Gamma$  and a connected  $k$ -edge plane graph  $\Delta$ , decides whether  $\Delta \preceq \Gamma$  in  $2^{\mathcal{O}(k \log k)} \cdot n^2$  steps.*

## 4 An FPT algorithm for the PTMC problem

We need the following definitions and results before we are ready to prove the main result of this section.

Given a plane graph  $\Gamma$  and a non-negative integer  $k$ , we say that a graph  $\Gamma'$  is a  $k$ -face completion of  $\Gamma$  if it can be obtained from  $\Gamma$  in the following way; for every  $f \in F(\Gamma)$  we add a set  $E_f$  of at most  $k$  edges to  $\Gamma$  such that the endpoints of the edges in  $E_f$  are vertices of  $\Gamma$  that belong to the boundary of  $f$ , all the edges  $E_f$  lie inside  $f$ , they do not intersect  $\Gamma$  in any points other than their endpoints, and finally they do not intersect each other.



■ **Figure 2** This figure depicts the construction of  $\Gamma_{3,2}$  from  $\Gamma$  ( $\Gamma$  is the graph on the left).

Let  $r$  and  $q$  be integers such that  $r \in \mathbb{N}_{\geq 3}, q \in \mathbb{N}_{\geq 1}$ . A  $(r, q)$ -cylinder, denoted by  $C_{r,q}$ , is the Cartesian product of a cycle on  $r$  vertices and a path on  $q$  vertices. We will refer to  $r$  as the *length* and  $q$  as the *width* of  $C_{r,q}$ . Note here that  $C_{r,q}$  is a 3-connected graph and thus, by Whitney's Theorem, it is uniquely embeddable (up to homeomorphism) in the sphere. Furthermore,  $C_{r,q}$  has exactly two non-square faces  $f_1$  and  $f_2$  that are incident only with vertices of degree 3. We call one of the faces  $f_1$  and  $f_2$  the *interior* of  $C_{r,q}$  and the other the *exterior* of  $C_{r,q}$ . We call the vertices incident to the interior (exterior) of  $C_{r,q}$  *base* (*roof*) of  $C_{r,q}$ .

Let  $\Gamma$  be a plane graph. We give the definition of the graph  $\Gamma_{r,q}$  for  $r \in \mathbb{N}_{\geq 3}$  and  $q \in \mathbb{N}_{\geq 3}$ . Let  $f_i \in F(\Gamma)$  and let  $\Theta_1^i, \dots, \Theta_{\rho_i}^i$  be the connected components of  $B_\Gamma(f_i)$ . For each  $\Theta_j^i$ , we denote by  $\sigma_{j,i}$  the length of a facial walk of  $\Theta_j^i$ . We then add a copy  $C_j^i$  of  $(\sigma_{j,i} \cdot r, q)$ -cylinder in the embedding of  $\Gamma$  such that  $\Theta_j^i$  is contained in the interior of  $C_j^i$  and all  $\Theta_1^i, \dots, \Theta_{j-1}^i, \dots, \Theta_{j+1}^i, \dots, \Theta_{\rho_i}^i$  are contained in the exterior of  $C_j^i$ . Then we partition the base of  $C_j^i$  into  $\sigma_{j,i}$  parts  $Q_l, l \in \sigma_{j,i}$  each consisting of  $r$  consecutive base vertices. Let  $(u_{j,i}^1, u_{j,i}^2, \dots, u_{j,i}^{\sigma_{j,i}}, u_{j,i}^1)$  be a facial walk of  $\Theta_{j,i}$ . We join by  $r$  edges the vertex  $u_{j,i}^x$  to all the vertices of the set  $Q_l, l \in \sigma_{j,i}$ . We apply this enhancement for each connected component of the boundary of each face of  $\Gamma$  and we denote the resulting graph by  $\widehat{\Gamma}_{r,q}$ .

We call a face  $f_i$  of  $\widehat{\Gamma}_{r,q}$  non-trivial if  $B_{\widehat{\Gamma}_{r,q}}(f_i)$  has more than one connected components  $\Theta_1^i, \dots, \Theta_{\rho_i}^i$ . Notice that if  $f_i$  is non-trivial, each  $\Theta_j^i$  is the roof of some previously added cylinder. For each such cylinder, let  $J_j^i$  be a set of  $r$  consecutive vertices of its roof. We add inside  $f_i$  a copy  $C_{f_i}$  of  $C_{\rho_i \cdot r, q}$  such that its base is a subset of  $f_i$  and let  $\{I_1, \dots, I_{\rho_i}\}$  be a partition of its roof in  $\rho_i$  parts, each consisting of  $r$  consecutive base vertices. For each  $x \in \{1, \dots, \rho_i\}$  we add  $r$  edges each connecting a vertex of  $J_j^i$  with some vertex of  $I_x$  in a way that the resulting embedding remains plane (there is a unique way for this to be done). We apply this enhancement for each non-trivial face of  $\widehat{\Gamma}_{r,q}$  and the resulting graph is the graph  $\Gamma_{r,q}$ . Notice that  $\Gamma_{r,q}$  is not uniquely defined as its definition depends on the choice of the sets  $J_x$ . From now on, we always consider an arbitrary choice for  $\Gamma_{r,q}$  and we call  $\Gamma_{r,q}$  the  $(r, q)$ -cylindrical enhancement of  $\Gamma$ . Finally, given a plane graph  $\Gamma$  and  $r, q \in \mathbb{N}_{\geq 3}$ . Let  $V_{\Gamma_{r,q}}^0 = V(\Gamma)$  and  $V_{\Gamma_{r,q}}^n = V(\Gamma_{r,q}) \setminus V(\Gamma)$  and notice that  $\deg_{\Gamma_{r,q}}(v) \leq 4$ , for every  $v \in V_{\Gamma_{r,q}}^n$ . (For an example, see Figure 2.) Given a positive integer  $k$ , we denote by  $\tilde{\Gamma}_k$  the graph  $\Gamma_{2 \cdot k, 8 \cdot k}$ .

We are now ready to state one of the main results of this section.

► **Theorem 4.1** ( $\star$ ). *Let  $\Gamma$  and  $\Delta$  be plane graphs where  $\Delta$  is connected and  $k = |E(\Delta)|^{2|E(\Delta)|}$ . There exists a  $k$ -face completion  $\Gamma^+$  of  $\Gamma$  such that  $\Delta \leq_{etm} \Gamma^+$  if and only if  $\Delta \leq_{etm}^S \tilde{\Gamma}_k$  where  $S = V(\tilde{\Gamma}_k) \setminus V(\Gamma) = V_{\Gamma_{2 \cdot k, 8 \cdot k}}^n$ .*

Moreover, we have the following.

► **Theorem 4.2** ( $\star$ ). *There exists an algorithm that given two plane graphs  $\Gamma$  and  $\Delta$  and a set  $V \subseteq V(\Gamma)$  with  $\deg_\Gamma(z) \leq c$ , for every  $z \in V$  outputs a graph  $\Gamma'$ , with  $\Gamma' \subseteq_{sp} \Gamma$  and  $\mathbf{tw}(\Gamma') = \mathcal{O}(f(|E(\Delta)|))$ , for some computable function  $f$  such that  $\Delta \leq_{etm}^V \Gamma$  if and only if  $\Delta \leq_{etm}^V \Gamma'$ . This algorithm runs in  $\mathcal{O}_{|E(\Delta)|}(|E(\Gamma)|)$  steps.*

Let  $\Gamma$  be a connected plane graph and  $Z \subseteq V(\Gamma)$ , we define the following pair of vertex and edge structures:

$$\mathbf{G}_{\Gamma,Z} := (\mathbf{d}(\mathbf{p}(\Gamma, R(\Gamma)), Z), \mathbf{e}(\Gamma, R(\Gamma))).$$

Given two connected plane graphs  $\Delta$  and  $\Gamma$  and  $Z \subseteq V(\Gamma)$  we say that  $\mathbf{G}_{\Delta,\emptyset}$  is a *restricted topological minor* of  $\mathbf{G}_{\Gamma,Z}$ , denoted by  $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$ , if and only if there exist two functions  $f_1 : V(R^3(\Delta)) \rightarrow 2^{V(R^3(\Gamma))}$  and  $f_2 : E(R^3(\Delta)) \rightarrow 2^{E(R^3(\Gamma))}$  satisfying the following:

1. for every  $x \in V(\Delta)$ ,  $f_1(x) \in V(\Gamma) \setminus Z$  and  $|f_1(x)| = 1$ ,
2. for every  $x \in \cup_{i \in [3]} V(R_s^i(\Delta))$ ,  $f_1(x) \notin \cup_{i \in [3]} (V(R_r^i(\Gamma)))$  and  $|f_1(x)| = 1$ ,
3. for every  $x, y \in \cup_{i \in [3]} V(R_r^i(\Delta))$  is connected and  $f_1(x) \cap f_1(y) = \emptyset$ ,
4. for every  $xy \in E_s^3(\Delta)$ ,  $G[f_2(xy)]$  is a path between  $f_1(x)$  and  $f_1(y)$  and  $f_2(xy) \subseteq E_s^3(\Gamma)$ , and
5. for every  $xy \in E_r^3(\Delta)$ ,  $G[f_2(xy)]$  is a path between some vertex of  $f_1(x)$  and some vertex of  $f_1(y)$ .

► **Theorem 4.3.** *Let  $\Gamma, \Delta$  be two connected plane graphs and  $Z \subseteq V(\Gamma)$ . Then  $\Delta \leq_{etm}^Z \Gamma$  if and only if  $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,Z}$ .*

**Our algorithm for PTMC** . Let  $\Gamma$  and  $\Delta$  be two plane graphs, where  $\Delta$  is connected. From Theorem 4.1 we construct a cylindrical enhancement  $\tilde{\Gamma}_k$  of  $\Gamma$ , where the vertices of the set  $S = V_{\Gamma,2,k,8,k}^n$  have degree bounded by a constant such that  $\Delta, \Gamma$  are a yes instance if and only if  $\Delta \leq_{etm}^S \tilde{\Gamma}_k$ . Then, the algorithm of Theorem 4.1 with inputs  $\tilde{\Gamma}_k, \Delta, S$  outputs a graph  $\Gamma'$  with  $\Gamma' \subseteq_{sp} \Gamma$  and  $\mathbf{tw}(\Gamma') = \mathcal{O}(f(|E(\Delta)|))$ . Moreover, Theorem 4.3 translates  $\Gamma', \Delta$ , and  $S$  to two structures  $\mathbf{G}_{\Delta,\emptyset}$  and  $\mathbf{G}_{\Gamma,S}$ , for which  $\Delta \leq_{etm}^S \Gamma$  if and only if  $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,S}$ . Notice that the relation  $\mathbf{G}_{\Delta,\emptyset} \leq_{rtm} \mathbf{G}_{\Gamma,S}$  can be expressed in Monadic Second Order Logic. Finally, by observing that  $\mathbf{tw}(R^3(\Gamma)) = \mathcal{O}(f(|E(\Delta)|))$  we can employ Courcelle's Theorem [5] to obtain an  $f(|E(\Delta)|) \cdot m^2$  time algorithm, for some computable function  $f$ .

## 5 Extensions

Our approach for the PSC problem can also solve the PLANE INDUCED SUBGRAPH COMPLETION problem, with the same running time, where instead of an embedded subgraph we ask for an embedded *induced subgraph*. The only modification would be at the definition of a facial extension of  $\Delta$  where we would additionally require that every connected graph  $\Delta^+$  contains  $\Delta$  as an *induced* subgraph.

In the PTMC problem the connectivity of  $\Delta$  is only required in the proof of Theorem 4.1 (that has been omitted). We would like to remark here that if we disregard the embedding of  $\Delta$  then the Proposition holds for disconnected graphs as well. In this case by modifying the algorithm for PTMC we may obtain an FPT algorithm that given a plane graph  $\Gamma$  and a *planar* graph  $D$  decides whether there exists a face completion of  $\Gamma$ , say  $\Gamma^+$ , such that  $D$  is a rooted topological minor of  $\Gamma$ . That is, each vertex of  $D$  is mapped to a specified vertex of  $\Gamma$ . Notice that this approach also permits us to solve the PLANAR DISJOINT PATHS COMPLETION problem where we allow edge additions inside all faces of  $\Gamma$  (in contrast to [2] where edge additions are allowed only inside a specified face of  $\Gamma$ ).

Finally, with the same cylindrical enhancement that we apply for PTMC and the extra restriction that the sets of vertices of the enhanced graph that are contracted to a vertex of the pattern graph  $\Delta$  contain only vertices of the initial graph we can solve the PLANE MINOR COMPLETION problem. In these last two cases, however, only the existence of an FPT algorithm is verified (since both would be derived by Courcelle's Theorem).

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