A Fixed Parameter Algorithm for Plane Subgraph Completion

Dimitris Chatzidimitriou, Archontia C. Giannopoulou, Clément Requilé, Dimitrios M. Thilikos, Dimitris Zoros

A Fixed Parameter Algorithm for Plane Subgraph Completion[*]

Dimitris Chatzidimitriou[1], Archontia C. Giannopoulou[2], Clément Requilé[3], Dimitrios M. Thilikos[1,4], and Dimitris Zoros[1]

[1]Department of Mathematics, National and Kapodistrian University of Athens, Athens, Greece.
[2]Institute of Informatics, University of Warsaw, Warsaw, Poland.
[3]Freie Universität Berlin, Institut für Mathematik und Informatik, Berlin, Germany.
[4]AlGCo project team, CNRS, LIRMM, France.

The PLANE SUBGRAPH COMPLETION problem asks, given a (possibly disconnected) plane graph Γ and a connected plane graph Δ, whether it is possible to add edges in Γ such that the resulting graph remains planar and contains some subgraph that is topologically isomorphic to Δ. We give an algorithm that solves this problem in $2^{O(k \log k)} \cdot n^2$ steps where k and n are the number of vertices of Δ and Γ respectively.

1 Introduction

Completion problems are defined as follows: Consider a graph class $\mathcal{P}$ and ask whether we may add edges to a given graph G in order to obtain a graph G^+, where $G^+ \in \mathcal{P}$. Numerous results have appeared for the case where the objective is to minimize the number of added edges [12, 9, 10, 8, 4] in G. However, interestingly there is no known dichotomy asserting when the problem is solvable in polynomial time and when it is NP-complete.

In this paper, we consider the PLANE SUBGRAPH COMPLETION (PSC) problem which, given a (possibly disconnected) plane graph G and a connected plane graph H, asks whether it is possible to add edges in G such that the resulting graph remains planar and contains some subgraph that is topologically isomorphic to H. When the input graph G is planar triangulated, PSC is NP-complete. Indeed, let G be any planar triangulated graph. Note here, that as any planar triangulated graph is 3-connected, G is 3-connected and from Whitney's Theorem [11] admits a unique embedding on the plane (up to equivalence). Let also H be the cycle on $n = |V(G)|$ vertices. Then H also has unique embedding on the plane (up to equivalence). Since G is triangulated no edge can be added to it while preserving its planarity. Thus, PSC becomes equivalent to the HAMILTON CYCLE PROBLEM which is NP-complete on planar triangulated graphs [6] (see also [7]). This observation further implies that PSC parameterized by the number of added edges k, and in particular even for $k = 0$, is NP-complete. Thus, PSC is not FPT when parameterized by the number of added edges unless P = NP. Thus, in order to obtain a tractable algorithm, we need to find an alternative way to parameterize this problem. In particular, we will consider $|V(H)|$ as our parameter. Such alternative parameterization first appeared in [3].

To state our main result, and throughout the paper, we need the following definitions.

Plane graphs and completion subgraphs By plane graph Γ we mean a planar graph Γ with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$ drawn in the sphere. To simplify notations, we usually do not distinguish between a vertex of the graph and the point of the sphere used in the drawing to represent the vertex or between an edge and the open line segment representing it. We define the following relation on plane graphs. Let Γ and Δ be plane graphs. We say that $\Delta \preceq \Gamma$ if there exists a set $S \in \binom{V(\Gamma)}{2} \setminus E(\Gamma)$ such that $\Gamma' = (V(\Gamma), E(\Gamma) \cup S)$ is plane and Δ is topologically isomorphic to a subgraph of Γ'. Notice that $\preceq$ is not a transitive relation.

Our main result is the following.

Theorem 1. *There exists an algorithm that, given as input an n-vertex plane graph Γ and a connected k-vertex plane graph Δ decides whether $\Delta \preceq \Gamma$ in $2^{O(k \log k)} \cdot n^2$ steps.*

Let Γ and Δ be an input of PSC as above. In order to obtain our algorithm we consider a family $\mathcal{G}$ consisting of n *structures* depending only on Γ whose underlying graphs have treewidth at most $3k$ (Lemma 6). We also consider a family $\mathcal{D}$ consisting of $2^{O(k)}$ *structures* depending only on Δ (Lemma 4). For the graphs Γ and Δ and the families $\mathcal{G}$ and $\mathcal{D}$, it holds that $\Delta \preceq \Gamma$ if and only if some structure $\mathbf{D} \in \mathcal{D}$ is contained as a *special contraction* in a structure $\mathbf{G} \in \mathcal{G}$, denoted $\mathbf{D} \leq_c \mathbf{G}$ (Lemmata 5 and 7). Finally, for a fixed pair of structures $(\mathbf{D}, \mathbf{G}) \in \mathcal{D} \times \mathcal{G}$, we can decide in $2^{O(k \log k)} n$ time whether $\mathbf{D} \leq_c \mathbf{G}$ (Lemma 2). Therefore, by testing for all pairs $(\mathbf{D}, \mathbf{G}) \in \mathcal{D} \times \mathcal{G}$ whether $\mathbf{D} \leq_c \mathbf{G}$, we decide in $2^{O(k \log k)} \cdot n^2$ steps whether $\Delta \preceq \Gamma$.

2 Preliminaries

Contractions A *structure* $\mathbf{G}$ is a tuple whose first element is a graph G and the rest of its elements are subsets of $V(G)$ forming a partition of $V(G)$. Let $\mathbf{G} = (G, S_0, Y, S_1, S_2, S_3, S_4)$ and $\mathbf{D} = (D, Z_0, B, Z_1, Z_2, Z_3, Z_4)$ be structures. We say that $\mathbf{D}$ is a *contraction* of $\mathbf{G}$, denoted by $\mathbf{D} \leq_c \mathbf{G}$, if and only if there exists a function $\sigma : V(G) \to V(D)$ such that:

1. if $u, v \in V(D)$, $u \neq v \Leftrightarrow \sigma^{-1}(u) \cap \sigma^{-1}(v) = \emptyset$,

2. for every $u \in V(D)$, $G[\sigma^{-1}(u)]$ is connected,

3. $\{u, v\} \in E(D) \Leftrightarrow G[\sigma^{-1}(u) \cup \sigma^{-1}(v)]$ is connected,

4. for every $i \in \{0, \ldots, 4\}$ and every $x \in Z_i$ it holds that $|\sigma^{-1}(x)| = 1$ and $\sigma^{-1}(x) \in S_i$, and

5. $\sigma(Y) \subseteq B$.

We say that a graph G is a *contraction* of a graph D if $(D, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) \leq_c (G, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$ and we denote it by $D \leq_c G$. Notice that $\leq_c$ defined for graphs is the usual contraction relation where only conditions 1,2, and 3 apply.

The next Lemma follows using the results in [2].

Lemma 2. *There exists an algorithm that receives as input a structure $\mathbf{G}$, whose graph has n vertices and treewidth at most h, and a structure $\mathbf{D}$, whose graph is connected and has k vertices, and outputs whether $\mathbf{D} \leq_c \mathbf{G}$ in $2^{O(k+h+k \log h)} \cdot n$ steps.*

Radial enhancements Let Γ be a plane graph. A *radial enhancement* of Γ is defined as a plane graph that can be constructed as follows: take Γ, add a vertex v_f inside each face f of Γ and add edges connecting v_f with the vertices incident to f, so that in the resulting embedding, every face that is incident to an edge of $E(\Gamma)$ is triangulated (this triangulation

may have multiple edges). We denote by $\mathcal{R}_\Gamma$ the set of all different (in terms of topological isomorphism) radial enhancements of Γ. Observe that if Γ is connected, then $\mathcal{R}_\Gamma$ contains only one member that will be denoted as $R(\Gamma)$.

Given a connected plane graph Γ, we define $Q(\Gamma) = R(\mathsf{sub}(R(\Gamma)))$, where $\mathsf{sub}(\Gamma)$ is the graph obtained from Γ if we subdivide all its edges once.

If Γ and $\tilde{\Gamma}$ are plane graphs such that $\tilde{\Gamma} = Q(\Gamma)$ and $S \subseteq V(\Gamma)$, then we denote by $\mathbf{p}(\tilde{\Gamma}, \Gamma, S) = (S_0, S_1, S_2, S_3, S_4)$ where $S_0 = S$ and:

$$\begin{array}{rclcrcl}
S_1 & = & V(\Gamma) \setminus S_0 & \bigg| & S_3 & = & V(\mathsf{sub}(R(\Gamma))) \setminus (S_0 \cup S_1 \cup S_2) \\
S_2 & = & V(R(\Gamma)) \setminus (S_0 \cup S_1) & \bigg| & S_4 & = & V(Q(\Gamma)) \setminus (S_0 \cup S_1 \cup S_2 \cup S_3)
\end{array}$$

Also given a tuple $\mathbf{p}$ of sets of vertices of a plane graph Γ and a set $Q \subseteq V(\Gamma)$, we define $\mathbf{t}(\mathbf{p}, Q) = (Q, \mathbf{p} \setminus Q)$, where the operation $\mathbf{p} \setminus Q$ removes from each element of $\mathbf{p}$ the vertices in Q. And if $\mathbf{G} = (G, S_0, S_1, S_2, S_3, S_4)$, then we define $\mathbf{r}(\mathbf{G}) = (G, S_0, \emptyset, S_1, S_2, S_3, S_4)$.

3 The algorithm

Lemma 3. *Let Γ and Δ be plane graphs. If Δ is topologically isomorphic to a subgraph of Γ then for every $R_\Gamma \in \mathcal{R}_\Gamma$ there exists some $R_\Delta \in \mathcal{R}_\Delta$ such that $Q(R_\Gamma)$ is isomorphic to $Q(R_\Delta)$.*

A *face extension* of a connected plane graph Δ is a connected plane supergraph Δ^+ of Δ such that $V(\Delta^+) \setminus V(\Delta)$ is an independent set and $N_{\Delta^+}(x) \neq N_{\Delta^+}(y)$, for every $x, y \in V(\Delta^+) \setminus V(\Delta)$ with $x \neq y$, where $N_{\Delta^+}(x)$ is the neighbourhood of the vertex x in Δ^+. We then denote by $\mathcal{F}_\Delta$ the set of all face extension of the connected plane graph Δ.

Given a plane graph Δ and $L \subseteq E(\Delta)$, we denote by $\mathsf{span}(\Delta, L)$ the set of all spanning subgraphs of Δ that contain all edges in $E(\Delta) \setminus L$.

Lemma 4. *If Δ is a connected plane k-vertex graph, then $|\mathcal{F}_\Delta|, |\mathsf{span}(\Delta, E(\Delta))| \leq 2^{O(k)}$.*

Based on Lemma 3, we prove the following:

Lemma 5. *Let Γ be a plane graph and Δ a connected plane graph. It holds that $\Delta \preceq \Gamma$ if and only if there exists some $\Delta^+ \in \mathcal{F}_\Delta$ such that there exists some $\Delta^* \in \mathsf{span}(\Delta^+, E(\Delta))$ such that for every $R_\Gamma \in \mathcal{R}_\Gamma$ there exists a $R_{\Delta^*} \in \mathcal{R}_{\Delta^*}$ such that*

$$(Q(R_{\Delta^*}), \mathbf{t}(\mathbf{p}(Q(R_{\Delta^*}), R_{\Delta^*}, V(\Delta^*)), V(\Delta))) \leq_c (Q(R_\Gamma), \mathbf{r}(\mathbf{p}(Q(R_\Gamma), R_\Gamma, V(\Gamma)))).$$

Given a structure $\mathbf{G} = (G, S_0, \emptyset, S_1, S_2, S_3, S_4)$, a non-negative integer k and a vertex $v \in S_0$, we define $\mathbf{G}_v^{(k)} = (G_v, S_0', Y, S_1', S_2', S_3', S_4')$ such that G_v is the graph created from G if for every $C \in \mathcal{C}(\tilde{G})$, the set of all the connected components of $\tilde{G}$, we contract all edges of C to a single vertex v_C, where $\tilde{G} = G \setminus \mathbf{B}_G^k(v)$, and $\mathbf{B}_G^k(v)$ is the set of all the vertices at distance at most k in G from v. Also $Y = \{v_C \mid C \in \mathcal{C}(\tilde{G})\}$ and for $i \in \{0, \dots, 4\}$, $S_i' = S_i \cap \mathbf{B}_G^{(k)}(v)$.

The following is a direct consequence of [5].

Lemma 6. *If $\mathbf{G} = (G, S_0, \emptyset, S_1, S_2, S_3, S_4)$ is a structure, k is a non-negative integer and $v \in V(G)$, then the graph of $\mathbf{G}_v^{(k)}$ has treewidth at most $3k$.*

Lemma 7. *Let $\mathbf{G} = (G, S_0, \emptyset, S_1, S_2, S_3, S_4)$ and $\mathbf{D} = (D, Z_0, B, Z_1, Z_2, Z_3, Z_4)$ be two structures. Then $\mathbf{D} \leq_c \mathbf{G}$ if and only if there exists some $v \in S_0$ such that $\mathbf{D} \leq_c \mathbf{G}_v^{(|V(D)|)}$.*

Theorem 1 follows combining Lemmata 2, 4, 5, 6, and 7:

Input: An encoding of two plane graphs Γ and Δ.
Question: Is it true that $\Delta \preceq \Gamma$?

1. **for every** $\Delta^+ \in \mathcal{F}_\Delta$
2.   **for every** $\Delta^* \in \mathsf{span}(\Delta^+, E(\Delta))$
3.   **for every** $u \in V(\Gamma)$
4.   arbitrarily construct an $R_\Gamma \in \mathcal{R}_\Gamma$
5.   **for every** $R_{\Delta^*} \in \mathcal{R}_{\Delta^*}$
6.   construct $\mathbf{Q}(\Gamma)_u^{(|V(\Delta)|)}$ and $\mathbf{Q}(\Delta^*)$
7.   **if** $\mathbf{Q}(\Delta^*) \leq_c \mathbf{Q}(\Gamma)_u^{(|V(\Delta)|)}$, **return**
         true
8. **return** false
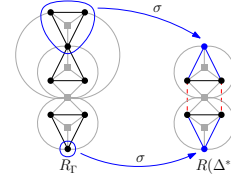
Γ and Δ are the graphs with the black and red edges, red edges represent a choice in $\mathsf{span}(\Delta^+, E(\Delta))$. Greyed vertices are added by the first radial enhancement. σ maps black vertices to black, grey to grey and blue regions to blue vertices.

4 Conclusion

We can drop the number of steps to $2^{O(k)} \cdot n^2$ by using the results of [1] and [2]. Our approach can also solve the PLANE INDUCED SUBGRAPH COMPLETION problem where we are asked, given a plane graph G and a connected plane graph H, whether it is possible to add edges in G such that the resulting graph remains planar and contains some *induced subgraph* that is topologically isomorphic to H. The only modification would be at the definition of a face extension of Δ where we would additionally require that every connected graph Δ^+ contains Δ as an *induced* subgraph.

References

[1] I. Adler, F. Dorn, F. V. Fomin, I. Sau, and D. M. Thilikos. Faster parameterized algorithms for minor containment. *Theor. Comput. Sci.*, 412(50):7018–7028, 2011.

[2] I. Adler, F. Dorn, F. V. Fomin, I. Sau, and D. M. Thilikos. Fast minor testing in planar graphs. *Algorithmica*, 64(1):69–84, 2012.

[3] I. Adler, S. G. Kolliopoulos, and D. M. Thilikos. Planar disjoint-paths completion. In *Parameterized and Exact Computation - 6th International Symposium, IPEC 2011, Saarbrücken, Germany, September 6-8, 2011. Revised Selected Papers*, pages 80–93, 2011.

[4] I. Bliznets, F. V. Fomin, M. Pilipczuk, and M. Pilipczuk. A subexponential parameterized algorithm for proper interval completion. In *Algorithms - ESA 2014 - 22th Annual European Symposium, Wroclaw, Poland, September 8-10, 2014. Proceedings*, pages 173–184, 2014.

[5] H. Bodlaender. Planar graphs with bounded treewidth. Technical Report RUU-CS-88-14, Department of Information and Computing Sciences, Utrecht University, 1988.

[6] V. Chvatal. Hamiltonian cycles. In E. Lawler, J. Lenstra, A. Kan, and D. S. (Eds.), editors, *The Traveling Salesman Problem*, pages 403–429. Wiley, New York, 1985.

[7] M. B. Dillencourt. Finding hamiltonian cycles in delaunay triangulations is np-complete. *Discrete Applied Mathematics*, 64(3):207 – 217, 1996.

[8] P. G. Drange, F. V. Fomin, M. Pilipczuk, and Y. Villanger. Exploring subexponential parameterized complexity of completion problems. In *31st International Symposium on Theoretical Aspects of Computer Science (STACS 2014), STACS 2014, March 5-8, 2014, Lyon, France*, pages 288–299, 2014.

[9] H. Kaplan, R. Shamir, and R. E. Tarjan. Tractability of parameterized completion problems on chordal, strongly chordal, and proper interval graphs. *SIAM J. Comput.*, 28(5):1906–1922, 1999.

[10] Y. Villanger, P. Heggernes, C. Paul, and J. A. Telle. Interval completion is fixed parameter tractable. *SIAM J. Comput.*, 38(5):2007–2020, 2009.

[11] H. Whitney. Congruent graphs and the connectivity of graphs. *American Journal of Mathematics*, 54(1):pp. 150–168, 1932.

[12] M. Yannakakis. Computing the minimum fill-in is np-complete. *SIAM Journal on Algebraic Discrete Methods*, 2(1):77–79, 1981.