Relaxing order basis computation
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To cite this version:

HAL Id: lirmm-01372532
https://hal-lirmm.ccsd.cnrs.fr/lirmm-01372532
Submitted on 27 Sep 2016

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Context

Let \( K \) be a field, \( F = \sum_{i=0}^{n} F_i x^i \in K[x]^{m \times n} \) a matrix of power series, \( \sigma \) a positive integer and \((F, \sigma)\) be the \( K[x]\)-module defined by the set of \( v \in K[x]^{m \times n} \) such that \( v F \equiv 0 \mod x^\sigma \).

Definition of Order basis: \( P \in K[x]^{m \times m} \) is a \((\sigma, \delta)\)-order basis of \( F \) if the rows of \( P \) form a \( \delta\)-row reduced basis of \( (F, \sigma) \) (see [1]).

Order basis are used in: column reduction [2]; minimal nullspace basis [3]; block Wiedemann algorithm [4]; ...

Two existing algorithms

- Naive algorithm, iterative on the order \( F \), which costs \( O(n^2\sigma^2) \) op. in \( K \).
  - Quadratic complexity in the precision \( \sigma \)
  - Easy to stop at any intermediate step
  - Minimal knowledge on \( F \), only coefficients \( F_0, \ldots, F_k \) at step \( k \)

Algorithm 1: M-Basis(F, \( \sigma, \sigma \))

1. \( P, \bar{u} = \text{Basis}(F \mod x, \bar{s}) \)
2. For \( k = 0 \) to \( \sigma - 1 \) do
3. \( P' = x^\sigma P \mod x^{k+1} \)
4. \( P_k, \bar{u} = \text{Basis}(P', \bar{u}) \)
5. \( P = P \times P_k \)
6. \( \text{return } P, \bar{u} \)

PM-Basis

Recursive variant using a divide and conquer strategy on the order \( \sigma \) which costs \( O(n^2 M(\sigma) \log \sigma) = O(n^2 \sigma^2) \) operations in \( K \).

- Quasi-linear complexity in the precision \( \sigma \)
- Not convenient for early termination
- Often requires to know coefficients of \( F \) in advance

Algorithm 2: PM-Basis(F, \( \sigma, \sigma \))

1. if \( \sigma = 1 \) then
2. \( \text{return Basis}(F \mod x, \bar{s}) \)
3. else
4. \( P_k, \bar{u} = \text{PM-Basis}(F, \sigma, 2, \bar{s}) \)
5. \( P' = (x^{-\sigma}/P_k) F \mod x^{k+1} \)
6. \( P_k, \bar{u} = \text{PM-Basis}(P', \sigma/2, \bar{u}) \)
7. \( \text{return } P_k, P_k, \bar{u} \)

Our contribution

1. Give an algorithm for order basis with the following properties:
   - Quasi-optimality: it takes a quasi-linear time in the precision \( \sigma \)
   - Early termination: easy to stop at any intermediate step;
   - Relaxed algorithm: minimal knowledge on the input \( F \) at each step.
2. Use 1 to improve the complexity of block Wiedemann approach.

Fast iterative algorithm

Iterative version of PM-Basis that regroups computations step by step

- Quasi-linear complexity in the precision \( \sigma \)
- Convenient for early termination
- Often requires to know coefficients of \( F \) in advance

Algorithm 3: Iterative-PM-Basis(F, \( \sigma, \sigma \))

1. \( P_0, \bar{u} = \text{Basis}(F \mod x, \bar{s}) \)
2. \( P = [P_0] \) and \( S = \{0, \ldots, n, F \} \) with \([\log_2(\sigma)]\) zeros
3. for \( k = 1 \) to \( \sigma - 1 \) do
4. \( \ell = \lfloor \log_2(k) \rfloor \) and \( \ell' = \begin{cases} \lfloor \log_2(\sigma) \rfloor, & \text{if } k = 2^\ell \\ \lfloor \log_2(k - 2^\ell) \rfloor, & \text{otherwise} \end{cases} \)
5. \( P_k, \bar{u} = \text{Basis}(F[k], S) \mod x^{\ell'} \)
6. \( P_k, \bar{u} = \text{Basis}(S + 1, S) \mod x^{\ell'} \)
7. Insert \( P_k, \bar{u} \) at the beginning of \( P \)
8. \( \text{return } [P] \)

Relaxing the order basis algorithm

Problem: At step \( k = 2^\ell \), Iterative-PM-Basis requires \( S/\lfloor \log_2(\sigma) \rfloor + 1 \) mod \( x^{\ell'} \), that is \( F \mod x^{\ell'} \), to perform the middle product of step 6. However, we only need the middle product modulo \( x \) at step \( k \), and therefore \( F \mod x^{\ell'+2} \). The other coefficients of the middle product will be used in the next steps.

Solution:

Compute the middle products gradually with the additional constraint of not using any coefficient of the input before necessary, i.e. using a relaxed algorithm.

Definition of relaxed (or on-line) algorithm:

When computing the coefficient in \( x^\ell \) of the output, a relaxed algorithm can read at most the coefficients in \( 1, \ldots, x^\ell \) of the input.

M-Basis

- Naive algorithm, iterative on the order \( \sigma \), which costs \( O(n^2\sigma^2) \) op. in \( K \).
  - Quadratic complexity in the precision \( \sigma \)
  - Easy to stop at any intermediate step
  - Minimal knowledge on \( F \), only coefficients \( F_0, \ldots, F_k \) at step \( k \)

Iterative-PM-Basis

Iterative version of PM-Basis that regroups computations step by step

- Quasi-linear complexity in the precision \( \sigma \)
- Convenient for early termination
- Often requires to know coefficients of \( F \) in advance

Algorithm 3: Iterative-PM-Basis(F, \( \sigma, \sigma \))

1. \( P_0, \bar{u} = \text{Basis}(F \mod x, \bar{s}) \)
2. \( P = [P_0] \) and \( S = \{0, \ldots, n, F \} \) with \([\log_2(\sigma)]\) zeros
3. for \( k = 1 \) to \( \sigma - 1 \) do
4. \( \ell = \lfloor \log_2(k) \rfloor \) and \( \ell' = \begin{cases} \lfloor \log_2(\sigma) \rfloor, & \text{if } k = 2^\ell \\ \lfloor \log_2(k - 2^\ell) \rfloor, & \text{otherwise} \end{cases} \)
5. \( P_k, \bar{u} = \text{Basis}(F[k], S) \mod x^{\ell'} \)
6. \( P_k, \bar{u} = \text{Basis}(S + 1, S) \mod x^{\ell'} \)
7. Insert \( P_k, \bar{u} \) at the beginning of \( P \)
8. \( \text{return } [P] \)

Relaxed middle product

Two methods for a relaxed middle product algorithm:

- Compute a full \( 2n \times n \) product using a relaxed multiplication algorithm on polynomial of matrices \((15)\)
- Compute just the middle product as in Figure 1 to gain asymptotically a factor 2 compared to method 1.

Application to block Wiedemann algorithm

Let \( A \in GL_k(K) \) with \( O(N) \) non-zero elements and \( S = \sum_{i=0}^{N-1} U V^i x^i \) for random \( U, V \in K^{\times N} \). The block Wiedemann approach uses a \( (\sigma, \delta)\)-order basis of \( F = [S] \times [U] \) in \( K[x]^{m \times n} \) to solve sparse linear systems \( A y = b \).

Current approach:

Computing \( S \) at precision \( \sigma \) costs \( O(n^{\sigma-1} N \sigma) \) operations in \( K \), which is dominant since \( n \ll N \). An \( a \) priori bound \( \delta \) on the order \( \sigma \) is hard to find or may be loose. To circumvent this the paper [6] proposes a stopping criterion which has to be integrated into an iterative algorithm.

Benefits of our approach:

- Iterative-PM-Basis provides the first iterative algorithm with quasi-linear time complexity that can use stopping criteria from [6].
- Relaxed-PM-Basis improves the complexity of 1 on average by a constant factor because less coefficients of \( S \) need to be computed.

References