Relaxing order basis computation
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Let $K$ be a field, $F = \sum_{i=0}^{n-1} F_i x^i \in K[x]^{n \times n}$ a matrix of power series, $\sigma$ a positive integer and $(F, \sigma)$ be the $K[x]$-module defined by the set of $v \in K[x]^{n \times n}$ such that $v F \equiv 0 \mod x^\sigma$.

**Definition of order basis:** $P \in K[x]^{m \times m}$ is a (left) $(\sigma, \delta)$-order basis of $F$ if the rows of $P$ form a $\delta$-row reduced basis of $(F, \sigma)$ (see [1]).

**Order basis are used in:** column reduction [2]; minimal nullspace basis [3]; block Wiedemann algorithm [4]; ...

## Two existing algorithms

**Input:** $F \in K[x]^{m \times m}, \sigma \in \mathbb{N}^*$ and $\delta \in \mathbb{Z}^n$

**Output:** $P \in K[x]^{n \times n}$ a $(\sigma, \delta)$-order basis of $F$ and $\bar{u} \in \mathbb{Z}^m$ the shifted $\delta$-row degree of $P$.

To simplify the presentation, let us assume w.l.o.g. that:

1. the procedure basis $(F, \delta)$ handles the $(1, \delta)$-order basis case
2. $n = O(m)$ and the shift $\delta$ is balanced, as in [2]

## M-Basis

**Naive algorithm, iterative on the order $\sigma$, which costs $O(m^\sigma \sigma^2)$ op. in $K$.**

- Quadratic complexity in the precision $\sigma$
- Easy to stop at any intermediate step
- Minimal knowledge on $F$, only coefficients $F_0, \ldots, F_k$ at step $k$

**Algorithm 1:** M-Basis $(F, \sigma, \delta)$

1. $P, \bar{u} :=$ Basis $(F \text{ mod } x^{\delta}, \delta)$
2. for $k = 1$ to $\sigma - 1$
3. $F' := x^{\sigma - k} P \text{ mod } x^{\delta + 1}$
4. $P = F, P = P, P$
5. return $(P,\bar{u})$

## PM-Basis

**Recursive variant using a divide and conquer strategy on the order $\sigma$ which costs $O(m^\sigma \sigma^2 \log \sigma)$ = $O(m^\sigma \sigma^2)$ operations in $K$.**

- Quadratic complexity in the precision $\sigma$
- Not convenient for early termination
- Often requires to know coefficients of $F$ in advance

**Algorithm 2:** PM-Basis $(F, \sigma, \delta)$

1. if $\sigma = 1$
2. return Basis $(F \text{ mod } x^{\delta}, \delta)$
3. else
4. $P = P, \bar{u} :=$ PM-Basis $(F, \sigma$, $\sigma/2, \delta)$
5. $F' := (x^{\sigma/2} P, F \text{ mod } x^{\delta/2})$
6. $P, \bar{u} :=$ PM-Basis $(F', \sigma/2, \delta)$
7. return $(P, \bar{u})$

## Relaxing the order basis algorithm

**Problem:** At step $k = 2^i$, iterative-PM-Basis requires $S\left[\left[\log_2(\sigma)\right]+1\right] \mod 2^{i+1}$, that is $F \mod x^{2^{i+1}}$ to perform the middle product of step 6. However, we only need the middle product modulo $x$ at step $k$, and therefore $F \mod x^{2^{i-1}}$. The other coefficients of the middle product will be used in the next steps.

**Solution:** Compute the middle products gradually with the additional constraint of not using any coefficient of the input before necessary, i.e., using a relaxed algorithm.

**Definition of relaxed (or on-line) algorithm:** When computing the coefficient in $x^\ell$ of the output, a relaxed algorithm can read at most the coefficients in $1, \ldots, \ell$ of the input.

## Our contribution

- Give an algorithm for order basis with the following properties:
  - Quasi-optimality: it takes a quasi-linear time in the precision $\sigma$.
  - Early termination: easy to stop at any intermediate step;
  - Relaxed algorithm: minimal knowledge on the input $F$ at each step.
- Use 1 to improve the complexity of block Wiedemann approach.

## Fast iterative algorithm

**Iterative-PM-Basis**

**Iterative version of PM-Basis that regroups computations step by step**

- Quasi-linear complexity in the precision $\sigma$
- Convenient for early termination
- Often requires to know coefficients of $F$ in advance

**Algorithm 3:** Iterative-PM-Basis $(F, \sigma, \delta)$

1. $P_0, \bar{u} :=$ Basis $(F \mod x^{\delta}, \delta)$
2. $P = [P_0]$ and $S = [0, \ldots, 0, F]$ with $\log_2(\sigma)$ zeros
3. for $k = 1$ to $\sigma - 1$
4. $\ell := 2^k(k)$ and $\ell' := \left\lfloor \log_2(\sigma) \right\rfloor$ if $k = 2^k$
5. $\ell := (\ell - 2^k)$ otherwise
6. Merge first $\ell + 1$ elements of $P$ by multiplication product tree step 7
7. $S(\ell + 1) = (x^{-\ell'} P[1] \cdot S[\ell]) \mod x^{2^\ell}$ middle product step 5
8. $P_0, \bar{u} :=$ Basis $(S(\ell + 1) \mod x^\delta, \bar{u})$
9. Insert $P_1$ at the beginning of $P$
10. return $[1; P]$