Relaxing order basis computation
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To cite this version:

HAL Id: lirmm-01372532
https://hal-lirmm.ccsd.cnrs.fr/lirmm-01372532
Submitted on 27 Sep 2016

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Let $K$ be a field, $F = \sum_{i=0}^{d} F_i x^i \in K[x]^{m \times n}$ a matrix of power series, $σ$ a positive integer and $(F, σ)$ be the $K[x]$-module defined by the set of $v \in K[x]^{m \times n}$ such that $v F \equiv 0 \mod x^\sigma$.

**Definition of Order basis:** If $P \in K[x]^{m \times n}$ is a (left) $(σ, 0)$-order basis of $F$ if the rows of $P$ form a $σ$-row reduced basis of $(F, σ)$ (see [1]).

**Order basis are used in:** column reduction [2]; minimal nullspace basis [3]; block Wiedemann algorithm [4]; ...

### Two existing algorithms

**Input:** $F \in K[x]^{m \times n}$, $σ \in \mathbb{N}$ and $\ell \in \mathbb{Z}^n$.

**Output:** $P \in K[x]^{m \times n}$ a $(σ, \ell)$-order basis of $F$ and $\tilde{u} \in \mathbb{Z}^m$ the shifted $σ$-row degree of $P$.

To simplify the presentation, let us assume w.l.o.g. that:
1. the procedure $\text{Basis}(F, \ell)$ handles the $(1, \ell)$-order basis case
2. $n = O(m)$ and the shift $\ell$ is balanced, as in [2]

### M-Basis

**Naïve algorithm, iterative on the order $σ$, which costs $O(m^2 σ^2)$ op. in $K$.**

- Quadratic complexity in the precision $σ$
- Easy to stop at any intermediate step
- Minimal knowledge on $F$, only coefficients $F_{\ell \ldots \ell}$ at step $k$

**Algorithm 1: M-Basis($F, \sigma, k$)**

1. $P_0, \tilde{u} := \text{Basis}(F \mod x, \ell)$
2. For each $k = 1$ to $σ - 1$
3. $P_{k} := P_{k-1} \cdot F \mod x^{k+1}$
4. $P_{k}, \tilde{u} := \text{Basis}(P_{k}, \tilde{u})$
5. $P := P_{σ - 1}$
6. return $P, \tilde{u}$

### PM-Basis

**Recursive variant using a divide and conquer strategy on the order $σ$ which costs $O(m^2 \log(σ))^2 \sigma 2$ operations in $K$.**

- Quadratic complexity in the precision $σ$
- Not convenient for early termination
- Often requires to know coefficients of $F$ in advance

**Algorithm 2: PM-Basis($F, \sigma, k$)**

1. if $σ = 1$ then
2. return $\text{Basis}(F \mod x, \ell)$
3. else
4. $P_1, \tilde{u}_1 := \text{PM-Basis}(F, \sigma/2, \ell)$
5. $P_2 := (x^{σ/2} P_1 \cdot F \mod x^{σ/2})$
6. $P_k, \tilde{u}_k := \text{PM-Basis}(F_k, \sigma/2, \tilde{u}_k)$
7. return $P_k, \tilde{u}_k$

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### Our contribution

**Give an algorithm for order basis with the following properties:**
- **Quasi-optimality:** it takes a quasi-linear time in the precision $σ$.
- **Early termination:** easy to stop at any intermediate step.
- **Relaxed algorithm:** minimal knowledge on the input $F$ at each step.

**Use 1 to improve the complexity of block Wiedemann approach.**

### Fast iterative algorithm

**Fast iterative algorithm**

**Iterative version of PM-Basis that regroups computations step by step**

- Quasi-linear complexity in the precision $σ$
- Convenient for early termination
- Often requires to know coefficients of $F$ in advance

**Algorithm 3: Iterative-PM-Basis($F, x, σ$)**

1. $P_0, \tilde{u} := \text{Basis}(F \mod x, \ell)$
2. For each $k = 0$ to $σ - 2$
3. $P := P \cdot F \mod x^{k+1}$
4. $P, \tilde{u} := \text{Basis}(P, \tilde{u})$
5. $S[\ell + 1] := \left(\sum_{i=0}^{F} P[1] - S[\ell + 1] \mod x^{σ/2}\right)$ middle product step 5
6. $P_{k}, \tilde{u} := \text{Basis}(S[\ell + 1] \mod x, \ell)$ recursive leaf step 2
7. return $\prod_{P}$

### Iterative-PM-Basis algorithm

**Problem:** At step $k = 2^i$, Iterative-PM-Basis requires $S[\log_2(σ/2)] \equiv 1 \mod x^{σ/2}$, that is $F \mod x^{σ/2}$, to perform the middle product of step 6. However, we only need the middle product modulo $x$ at step $k$, and therefore $F \mod x^{σ/2}$. The other coefficients of the middle product will be used in the next steps.

**Solution:**

- Compute the middle products gradually with the additional constraint of not using any coefficient of the input before necessary, i.e. using a relaxed algorithm.

### Relaxed order basis algorithm

**Definition of relaxed (or on-line) algorithm:**

When computing the coefficient in $x^2$ of the output, a relaxed algorithm can read at most the coefficients in $1, \ldots, x^2$ of the input.

**Two methods for a relaxed product algorithm:**

1. Compute a full $2n \times n$ product using a relaxed multiplication algorithm on polynomial of matrices ([5])
2. Compute just the middle product as in Figure 1 to gain asymptotically a factor 2 compared to method 1.

**Relaxed middle product**

**Relaxed-PM-Basis algorithm**

Using this relaxed middle product within Iterative-PM-Basis, we obtain a new order basis algorithm relaxed w.r.t. $F$, which costs $O(k^2 M(σ)\log^2(σ))$.

- Quasi-linear complexity in the precision $σ$ (with an extra $\log_2(σ)$)
- Convenient for early termination
- Requires minimal knowledge on $F$.

### Application to block Wiedemann algorithm

Let $A \in \mathbb{GL}_N(K)$ with $O(N)$ non-zero elements and $S = \sum_{i=0}^{N-1} U AV^t x^i$ for random $U, V^t \in K^{n \times N}$. The block Wiedemann approach uses a $(\sigma, \ell)$-order basis of $F = [S \mid I] \in K[x]^{m \times n}$ to solve sparse linear systems $Ay = b$.

**Current approach:**

Computing $S$ at precision $σ$ costs $O(n^2 \cdot N \cdot σ)$ operations in $K$, which is dominant since $n \ll N$. An $a priori$ bound $σ$ on the order $\sigma$ is hard to find or may be loose. To circumvent this the paper [6] proposes a stopping criteria which has to be integrated into an iterative algorithm.

**Benefits of our approach:**

1. Iterative-PM-Basis provides the first iterative algorithm with quasi-linear-time complexity that can use stopping criteria from [6].

2. Relaxed-PM-Basis improves the complexity of $\Omega$ on average by a constant factor because less coefficients of $S$ need to be computed.

**References**


