Relaxing order basis computation
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Let $K$ be a field, $F = \sum_{i=0}^{n} F_i x^i \in K[x]^{m \times n}$ a matrix of power series, $\sigma$ a positive integer and $(F, \sigma)$ be the $K[x]$-module defined by the set of $\nu \in K[x]^{m \times n}$ such that $\nu^F \equiv 0 \mod x^\sigma$.

**Definition of Order basis:** $P \in K[x]^{m \times n}$ is a (left) $(\sigma, \delta)$-order basis of $F$ if the rows of $P$ form a $\delta$-row reduced basis of $(F, \sigma)$ (see [1]).

**Order basis are used in:** column reduction [2], minimal nullspace basis [3]; block Wiedemann algorithm [4]; ...

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**Two existing algorithms**

**Input:** $F \in K[x]^{m \times n}, \sigma \in \mathbb{N}$ and $\delta \in \mathbb{Z}^n$

**Output:** $P \in K[x]^{m \times n}$ a $(\sigma, \delta)$-order basis of $F$ and $\tilde{u} \in \mathbb{Z}^n$ the shifted $\delta$-row degree of $P$.

To simplify the presentation, let us assume w.l.o.g. that:

1. the procedure Basis $(F, \tilde{\sigma})$ handles the $(1, \tilde{\sigma})$-order basis case
2. $n = O(m)$ and the shift $\tilde{\sigma}$ is balanced, as in [2]

**M-Basis**

Naive algorithm, iterative on the order $\sigma$, which costs $O(m^2 \sigma^2)$ op. in $K$.

- Quadratic complexity in the precision $\sigma$
- Easy to stop at any intermediate step
- Minimal knowledge on $F$, only coefficients $F_0, \ldots, F_k$ at step $k$

**Algorithm 1: M-Basis ($F, \sigma$)**

1. $P, \tilde{u} := $ Basis$(F \mod x, \tilde{\sigma})$
2. for $k = 1 \text{ to } \sigma - 1$
3. $P' := x^2 \cdot P \cdot F \mod x^{2k+1}$
4. $P_k, \tilde{u} := $ Basis$(P', \tilde{u})$
5. $P := P_k \cdot P$
6. return $P, \tilde{u}$

**PM-Basis**

Recursive variant using a divide and conquer strategy on the order $\sigma$ which costs $O(m^2M(\log(\sigma)) = O(m^3 \sigma)$ operations in $K$.

- Quasi-linear complexity in the precision $\sigma$
- Not convenient for early termination
- Often requires to know coefficients of $P$ in advance

**Algorithm 2: PM-Basis ($F, \sigma$)**

1. if $\sigma = 1$
2. return Basis$(F \mod x, \tilde{\sigma})$
3. else
4. $P_k, \tilde{u} := $ PM-Basis$(F, \sigma/2, \tilde{u})$
5. $P' := (x^{\sigma/2} P_k \cdot F) \mod x^{\sigma/2}$
6. $P_k, \tilde{u} := $ PM-Basis$(F', \sigma/2, \tilde{u})$
7. return $P_k P_k$

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**Our contribution**

1. Give an algorithm for order basis with the following properties:
   - Quasi-optimality: it takes a quasi-linear time in the precision $\sigma$
   - Early termination: easy to stop at any intermediate step
   - Relaxed algorithm: minimal knowledge on the input $F$ at each step.
   - Use 1 to improve the complexity of block Wiedemann approach.

**Fast iterative algorithm**

**Iterative-PM-Basis**

Iterative version of PM-Basis that regroups computations step by step

- Quadratic complexity in the precision $\sigma$
- Convenient for early termination
- Often requires to know coefficients of $F$ in advance

**Algorithm 3: Iterative-PM-Basis ($F, \sigma, \tilde{\sigma}$)**

1. $P_0, \tilde{u} := $ Basis$(F \mod x, \tilde{\sigma})$
2. $P := \left\{ \frac{\log_2(\sigma)}{2} \right\}$
3. $t := \nu_\sigma(k) + \nu_\sigma(k - 2^{t-1})$
4. $P_k, \tilde{u} := $ Basis$(S(t+1) \mod x, \tilde{u})$
5. $S(t+1) := (x^{\sigma} P(k) \mod x^{\sigma/2})$
6. Insert $P_k$ at the beginning of $P$
7. return $[P, P_0]$

**Relaxed middle product**

Two methods for a relaxed middle product algorithm:

1. Compute a full $2n \times v$ product using a relaxed multiplication algorithm on polynomial of matrices (15)
2. Compute just the middle product as in Figure 1 to gain asymptotically a factor 2 compared to method 1.

**Relaxed middle product**

Using this relaxed middle product within Iterative-PM-Basis, we obtain a new order basis algorithm relaxed w.r.t. $F$, which costs $O(k^2 M(\sigma) \log_2(\sigma))$.

- Quasi-linear complexity in the precision $\sigma$ (with an extra $\log_2(\sigma)$)
- Convenient for early termination
- Requires minimal knowledge on $F$

**Application to block Wiedemann algorithm**

Let $A \in GL_N(K)$ with $(\nu(N)$ non-zero elements and $S = \sum_{i=0}^{n-1} U A V x^i$ for random $U, V \in K^{n \times n}$. The block Wiedemann approach uses a $(\sigma, \delta)$-order basis of $F = [S^T | I]^T \in K[x]^n$ to solve sparse linear systems $A y = b$.

**Current approach:**
Computing $S$ at precision $\sigma$ costs $O(n^3 \cdot N \sigma)$ operations in $K$, which is dominant since $\nu \ll N$. An a priori bound $\delta$ on the order $\sigma$ is hard to find or may be loose. To circumvent this the paper [6] proposes a stopping criterion which has to be integrated into an iterative algorithm.

**Benefits of our approach:**

1. Iterative-PM-Basis provides the first iterative algorithm with quasi-linear time complexity that can use stopping criteria from [6].
2. Relaxed-PM-Basis improves the complexity of $1$ on average by a constant factor because less coefficients of $S$ need to be computed.

**References**