Relaxing order basis computation
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To cite this version:

HAL Id: lirmm-01372532
https://hal-lirmm.ccsd.cnrs.fr/lirmm-01372532
Submitted on 27 Sep 2016

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Let $K$ be a field, $F = \sum_{i=0}^{\infty} F_i x^i \in K[x]^{m 	imes n}$ a matrix of power series, $\sigma$ a positive integer and $(F, \sigma)$ be the $K[x]$-module defined by the set of $v \in K[x]^{m \times n}$ such that $v F \equiv 0 \mod x^{\sigma}$.

**Definition of Order basis:** $P \in K[x]^{m \times m}$ is a (left) $(\sigma, \delta)$-order basis of $F$ if the rows of $P$ form a $\delta$-row reduced basis of $(F, \sigma)$ (see [1]).

**Order basis are used in:** column reduction [2]; minimal nullspace basis [3]; block Wiedemann algorithm [4]; ... 

**Two existing algorithms**

**Input:** $F \in K[x]^{m \times n}$, $\sigma \in \mathbb{N}$ and $\delta \in \mathbb{Z}^n$

**Output:** $P \in K[x]^{n \times n}$ a $(\sigma, \delta)$-order basis of $F$ and $\bar{u} \in \mathbb{Z}^n$ the shifted $\delta$-row degree of $P$.

To simplify the presentation, let us assume w.l.o.g. that:

- the procedure Basis$(F, \delta)$ handles the $(1, \delta)$-order basis case
- $n = O(m)$ and the shift $\delta$ is balanced, as in [2]

**Fast iterative algorithm**

**Iterative-PM-Basis**

Iterative version of PM-Basis that regroups computations step by step

- Quasi-linear complexity in the precision $\sigma$
- Convenient for easy termination
- Often requires to know coefficients of $F$ in advance

**Algorithm 3: Iterative-PM-Basis**

1. $P_0, \bar{u} := \text{Basis}(F \mod x, \bar{u})$
2. for $k = 1$ to $\sigma - 1$
3. $P' = x^k P \mod x^{\delta+1}$
4. $P_{k+1}, \bar{u} := \text{Basis}(F', \bar{u})$
5. $P = P_{k+1}$
6. return $P$, $\bar{u}$

**Relaxed middle product**

Two methods for a relaxed middle product algorithm:

1. Compute a full $2n \times n$ product using a relaxed multiplication algorithm on polynomial of matrices ([5])
2. Compute just the middle product as in Figure 1 to gain asymptotically a factor 2 compared to method 1.

**Application to block Wiedemann algorithm**

Let $A \in GL_n(K)$ with $O(N)$ non-zero elements and $S = \sum_{i=0}^{\infty} U^i V^T$ for random $U, V \in K^{n \times N}$. The block Wiedemann approach uses a $(\sigma, \delta)$-order basis of $F = [S^T | I]^T \in K[x]^{m \times n}$ to solve sparse linear systems $Ay = b$.

**Current approach:** Computing $S$ at precision $\sigma$ costs $O(n^2 \log \sigma)$ operations in $K$, which is dominant since $\sigma \ll N$. An $a$ priori bound $\delta$ on the order $\sigma$ is hard to find or may be loose. To circumvent this the paper [6] proposes a stopping criteria which has to be integrated into an iterative algorithm.

**Benefits of our approach:**

- **Iterative-PM-Basis** provides the first iterative algorithm with quasi-linear time complexity that can use stopping criteria from [6].
- **Relaxed-PM-Basis** improves the complexity of $O$ on average by a constant factor because less coefficients of $S$ need to be computed.

**References**