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Relaxing Order Basis Computation

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Context

Let $K$ be a field, $F = \sum_{i=0}^{d} F_i x^i \in K[x]^{m \times n}$ a matrix of power series, $\sigma$ a positive integer and $(F, \sigma)$ be the $K[x]$-module defined by the set of $v \in K[x]^{m \times n}$ such that $v F \equiv 0 \mod x^\sigma$.

Definition of Order basis: $P \in K[x]^{m \times m}$ is a (left) $(\sigma, \delta)$-order basis of $F$ if the rows of $P$ form a $\delta$-row reduced basis of $(F, \sigma)$ (see [1]).

Order basis are used in: column reduction [2]; minimal nullspace basis [3]; block Wiedemann algorithm [4];...

Two existing algorithms

Input: $F \in K[x]^{m \times m}$, $\sigma \in N^n$ and $\delta \in Z^n$
Output: $P \in K[x]^{m \times m}$ a $(\sigma, \delta)$-order basis of $F$ and $\vec{u} \in Z^n$ the shifted $\delta$-row degree of $P$.

To simplify the presentation, let us assume w.l.o.g. that:
1. the procedure Basis$(F, \delta)$ handles the $(1, \delta)$-order basis case
2. $\nu = O(m)$ and the shift $\delta$ is balanced, as in [2]

M-Basis

Naive algorithm, iterative on the order $\sigma$, which costs $O(m^2 \sigma^2)$ op. in $K$.

- Quadratic complexity in the precision $\sigma$
- $\not\exists$ Easy to stop at any intermediate step
- $\not\exists$ Minimal knowledge on $F$, only coefficients $F_0, \ldots, F_k$ at step $k$

Algorithm 1: M-Basis($F, \sigma, \delta$)

1: $P_0, \vec{u} = \text{Basis}(F \mod x^\delta, \delta)$
2: for $k = 1$ to $\nu - 1$
3: $P^\prime = \frac{x^\delta}{x^\delta - P} \cdot F \mod x^{\delta+1}$
4: $P_{k, \vec{u}} = \text{Basis}(P^\prime, \vec{u})$
5: $P_k = P_{k, \vec{u}}$
6: return $P_k, \vec{u}$

PM-Basis

Recursive variant using a divide and conquer strategy on the order $\sigma$ which costs $O(m^2 \nu^2 \log(\nu)) = O(m^2 \sigma^2)$ operations in $K$.

- Quasi-linear complexity in the precision $\sigma$
- $\not\exists$ Not convenient for early termination
- $\not\exists$ Often requires to know coefficients of $F$ in advance

Algorithm 2: PM-Basis($F, \sigma, \delta$)

1: if $\sigma = 1$ then
2: return Basis($F \mod x^\delta, \delta$)
else
3: $P_0, \vec{u} = \text{PM-Basis}(F, \sigma/2, \delta)$
4: $P^\prime = (x^\delta/\nu P_0 \cdot F) \mod x^{\delta+1}$
5: $P_{k, \vec{u}} = \text{PM-Basis}(P^\prime, \sigma/2, \vec{u})$
6: return $P_k, \vec{u}$

Our contribution

1. Give an algorithm for order basis with the following properties:
   - Quasi-optimality: it takes a quasi-linear time in the precision $\sigma$.
   - Early termination: easy to stop at any intermediate step.
   - Relaxed algorithm: minimal knowledge on the input $F$ at each step.
2. Use 1 to improve the complexity of block Wiedemann approach.

Fast iterative algorithm

Iterative-PM-Basis

Iterative version of PM-Basis that regroups computations step by step

- Quasi-linear complexity in the precision $\sigma$
- Convenient for early termination
- Often requires to know coefficients of $F$ in advance

Algorithm 3: Iterative-PM-Basis($F, \sigma, \delta$)

1: $P_0, \vec{u} = \text{Basis}(F \mod x^\delta, \delta)$
2: $P = [P_0]$ and $S = [0, \ldots, 0, F]$ with $[\log_2(\sigma)]$ zeros
3: for $k = 1$ to $\nu - 1$
4: $t_k = \nu / k$ and $t_k' = \begin{cases} \log_2(\sigma) & \text{if } k = 2^i \\ \nu / (k - 2^i) & \text{otherwise} \end{cases}$
5: Merge first $t_k + 1$ elements of $P$ by multiplication product tree step $k$
6: $S[i + 1] = (x^\delta P[i] - S[i + 1]) \mod x^{2^i}$ middle product step $k$
7: $P_0, \vec{u} = \text{Iterative-PM-Basis}(S[i], \vec{u})$
8: return $P_k$ at the beginning of $P$

Relaxing the order basis algorithm

Problem:
At step $k = 2^i$, Iterative-PM-Basis requires $S[i][\log_2(\sigma)] + 1 \mod x^{2^i}$, that is $F \mod x^{2^i}$, to perform the middle product of step 6. However, we only need the middle product modulo $x$ at step $k$, and therefore $F \mod x^{2^i}$. The other coefficients of the middle product will be used in the next steps.

Solution:
Compute the middle products gradually with the additional constraint of not using any coefficient of the input before necessary, i.e. using a relaxed algorithm.

Definition of relaxed (or on-line) algorithm:
When computing the coefficient in $x^\delta$ of the output, a relaxed algorithm can read at most the coefficients in $1, \ldots, x^\delta$ of the input.

Relaxed middle product

Two methods for a relaxed middle product algorithm:

1. Compute a full $2n \times n$ product using a relaxed multiplication algorithm on polynomial of matrices ([5])
2. Compute just the middle product as in Figure 1 to gain asymptotically a factor 2 compared to method 1.

Algorithm 1: M-Basis

Let $A \in GL_n(K)$ with $O(N)$ non-zero elements and $S = \sum_{i=0}^{n} U A V x^i$ for random $U, V \in K^{n \times N}$. The block Wiedemann approach uses a $(\sigma, \delta)$-order basis of $F = [S[i]]$ in $K[x]^{m \times m}$ to solve sparse linear systems $Ay = b$.

Current approach:
Computing $S$ at precision $\sigma$ costs $O(n^2 \cdot N \cdot \sigma)$ operations in $K$, which is dominant since $n \ll N$. An $a$ priori bound $\delta$ on the order $\sigma$ is hard to find or may be loose. To circumvent this the paper [6] proposes a stopping criteria which has to be integrated into an iterative algorithm.

Benefits of our approach:

- Iterative-PM-Basis provides the first iterative algorithm with quasi-linear time complexity that can use stopping criteria from [6].
- Relaxed-PM-Basis improves the complexity of 1 on average by a constant factor because less coefficients of $S$ need to be computed.

References