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Doubled patterns are 3-avoidable

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Abstract

In combinatorics on words, a word $w$ over an alphabet $\Sigma$ is said to avoid a pattern $p$ over an alphabet $\Delta$ if there is no factor $f$ of $w$ such that $f = h(p)$ where $h : \Delta^* \rightarrow \Sigma^*$ is a non-erasing morphism. A pattern $p$ is said to be $k$-avoidable if there exists an infinite word over a $k$-letter alphabet that avoids $p$. A pattern is said to be doubled if no variable occurs only once. Doubled patterns with at most 3 variables and patterns with at least 6 variables are 3-avoidable. We show that doubled patterns with 4 and 5 variables are also 3-avoidable.

Keywords: Word; Pattern avoidance.

1 Introduction

A pattern $p$ is a non-empty word over an alphabet $\Delta = \{A, B, C, \ldots\}$ of capital letters called variables. An occurrence of $p$ in a word $w$ is a non-erasing morphism $h : \Delta^* \rightarrow \Sigma^*$ such that $h(p)$ is a factor of $w$. The avoidability index $\lambda(p)$ of a pattern $p$ is the size of the smallest alphabet $\Sigma$ such that there exists an infinite word $w$ over $\Sigma$ containing no occurrence of $p$. Bean, Ehrenfeucht, and McNulty [2] and Zimin [13] characterized unavoidable patterns, i.e., such that $\lambda(p) = \infty$. We say that a pattern $p$ is $t$-avoidable if $\lambda(p) \leq t$. For more informations on pattern avoidability, we refer to Chapter 3 of Lothaire’s book [8].

It follows from their characterization that every unavoidable pattern contains a variable that occurs once. Equivalently, every doubled pattern is avoidable. Our result is that:

Theorem 1. Every doubled pattern is 3-avoidable.

Let $v(p)$ be the number of distinct variables of the pattern $p$. For $v(p) \leq 3$, Cassaigne [5] began and I [9] finished the determination of the avoidability index of every
pattern with at most 3 variables. It implies in particular that every avoidable pattern with at most 3 variables is 3-avoidable. Moreover, Bell and Goh [3] obtained that every doubled pattern \( p \) such that \( v(p) \geq 6 \) is 3-avoidable.

Therefore, as noticed in the conclusion of [10], there remains to prove Theorem 1 for every pattern \( p \) such that \( 4 \leq v(p) \leq 5 \). In this paper, we use both constructions of infinite words and a non-constructive method to settle the cases \( 4 \leq v(p) \leq 5 \).

Recently, Blanchet-Sadri and Woodhouse [4] and Ochem and Pinlou [10] independently obtained the following.

**Theorem 2** ([4, 10]). Let \( p \) be a pattern.

(a) If \( p \) has length at least \( 3 \times 2^{v(p)-1} \) then \( \lambda(p) \leq 2 \).

(b) If \( p \) has length at least \( 2^v(p) \) then \( \lambda(p) \leq 3 \).

As noticed in these papers, if \( p \) has length at least \( 2^v(p) \) then \( p \) contains a doubled pattern as a factor. Thus, Theorem 1 implies Theorem 2.(b).

## 2 Extending the power series method

In this section, we borrow an idea from the entropy compression method to extend the power series method as used by Bell and Goh [3], Rampersad [12], and Blanchet-Sadri and Woodhouse [4].

Let us describe the method. Let \( L \subset \Sigma_m^* \) be a factorial language defined by a set \( F \) of forbidden factors of length at least 2. We denote the factor complexity of \( L \) by \( n_i = L \cap \Sigma_i^m \). We define \( L' \) as the set of words \( w \) such that \( w \) is not in \( L \) and the prefix of length \( |w| - 1 \) of \( w \) is in \( L \). For every forbidden factor \( f \in F \), we choose a number \( 1 \leq s_f \leq |f| \). Then, for every \( i \geq 1 \), we define an integer \( a_i \) such that

\[
a_i \geq \max_{u \in L} \left| \{ v \in \Sigma_i^m \mid uv \in L', uv = bf, f \in F, s_f = i \} \right|.
\]

We consider the formal power series \( P(x) = 1 - mx + \sum_{i \geq 1} a_i x^i \). If \( P(x) \) has a positive real root \( x_0 \), then \( n_i \geq x_0^{-i} \) for every \( i \geq 0 \).

Let us rewrite that \( P(x_0) = 1 - mx_0 + \sum_{i \geq 1} a_i x_0^i = 0 \) as

\[
m - \sum_{i \geq 1} a_i x_0^{i-1} = x_0^{-1}
\]

Since \( n_0 = 1 \), we will prove by induction that \( \frac{m}{n_{i-1}} \geq x_0^{-1} \) in order to obtain that \( n_i \geq x_0^{-i} \) for every \( i \geq 0 \). By using (1), we obtain the base case: \( \frac{n_1}{n_0} = n_1 = m \geq x_0^{-1} \). Now, for every length \( i \geq 1 \), there are:

- \( m^i \) words in \( \Sigma_i^m \),
- \( n_i \) words in \( L \),
• at most $\sum_{1 \leq j \leq i} n_{i-j}a_j$ words in $L'$,
• $m(m^{i-1} - n_{i-1})$ words in $\Sigma_m \setminus \{L \cup L'\}$.

This gives $n_i + \sum_{1 \leq j \leq i} n_ja_{i-j} + m(m^{i-1} - n_{i-1}) \geq m^i$, that is, $n_i \geq mn_{i-1} - \sum_{1 \leq j \leq i} n_{i-j}a_j$:

\[
\begin{align*}
\frac{n_i}{n_{i-1}} &\geq m - \sum_{1 \leq j \leq i} a_jn_{i-j} \\
&\geq m - \sum_{1 \leq j \leq i} a_jx_0^{-1} \quad \text{By induction} \\
&\geq m - \sum_{j \geq 1} a_jx_0^{-1} \\
&= x_0^{-1} \quad \text{By (1)}
\end{align*}
\]

The power series method used in previous papers [3, 4, 12] corresponds to the special case such that $s_f = |f|$ for every forbidden factor. Our condition is that $P(x) = 0$ for some $x > 0$ whereas the condition in these papers is that every coefficient of the series expansion of $\frac{1}{P(x)}$ is positive. The two conditions are actually equivalent. The result in [11] concerns series of the form $S(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \ldots$ with real coefficients such that $a_1 < 0$ and $a_i \geq 0$ for every $i \geq 2$. It states that every coefficient of the series $1/S(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots$ is positive if and only if $S(x)$ has a positive real root $x_0$. Moreover, we have $b_i \geq x_0^{-i}$ for every $i \geq 0$.

The entropy compression method as developed by Gonçalves, Montassier, and Pinlou [6] uses a condition equivalent to $P(x) = 0$. The benefit of the present method is that we get an exponential lower bound on the factor complexity. It is not clear whether it is possible to get such a lower bound when using entropy compression for graph coloring, since words have a simpler structure than graphs.

3 Applying the method

In this section, we show that some doubled patterns on 4 and 5 variables are 3-avoidable. Given a pattern $p$, every occurrence $f$ of $p$ is a forbidden factor. With an abuse of notation, we denote by $|A|$ the length of the image of the variable $A$ of $p$ in the occurrence $f$. This notation is used to define the length $s_f$.

Let us first consider doubled patterns with 4 variables. We begin with patterns of length 9, so that one variable, say $A$, appears 3 times. We set $s_f = |f|$. Using the obvious upper bound on the number of pattern occurrences, we obtain

\[
P(x) = 1 - 3x + \sum_{a,b,c,d \geq 1} 3^{a+b+c+d}x^{3a+2b+2c+2d}
\]

\[
= 1 - 3x + \sum_{a,b,c,d \geq 1} (3x^3)^a (3x^2)^b (3x^2)^c (3x^2)^d
\]

\[
= 1 - 3x + (\sum_{a \geq 1} (3x^3)^a) (\sum_{b \geq 1} (3x^2)^b) (\sum_{c \geq 1} (3x^2)^c) (\sum_{d \geq 1} (3x^2)^d)
\]

\[
= 1 - 3x + (\frac{1}{1 - 3x} - 1) (\frac{1}{1 - 3x} - 1) (\frac{1}{1 - 3x} - 1) (\frac{1}{1 - 3x} - 1)
\]

\[
= 1 - 3x + (\frac{1}{1 - 3x} - 1) (\frac{1}{1 - 3x} - 1)^3
\]

\[
= \frac{1 - 3x - 9x^2 + 24x^3 + 36x^4 + 54x^5 - 108x^6 + 243x^7 + 162x^8 - 243x^9}{(1 - 3x)^3(1 - 3x^3)^3}
\]

Then $P(x)$ admits $x_0 = 0.3400\ldots$ as its smallest positive real root. So, every doubled pattern $p$ with 4 variables and length 9 is 3-avoidable and there exist at least $x_0^n > 2.941^n$
ternary words avoiding \( p \). Notice that for patterns with 4 variables and length at least 10, every term of \( \sum_{a,b,c,d \geq 1} 3^{a+b+c+d} x^{3a+2b+2c+2d} \) in \( P(x) \) gets multiplied by some positive power of \( x \). Since \( 0 < x < 1 \), every term is now smaller than in the previous case. So \( P(x) \) admits a smallest positive real root that is smaller than 0.340... Thus, these patterns are also 3-avoidable.

Now, we consider patterns with length 8, so that every variable appears exactly twice. If such a pattern has \( ABCD \) as a prefix, then we set \( s_f = \frac{|f|}{2} = |A| + |B| + |C| + |D| \). So we obtain \( P(x) = 1 - 3x + \sum_{a,b,c,d \geq 1} x^{a+b+c+d} \) as a prefix. We set \( s_f = \frac{|f|}{2} = |A| + |B| + |C| + |D| \). So we obtain \( P(x) = 1 - 3x + \sum_{a,b,c,d \geq 1} x^{a+b+c+d} = 1 - 3x + \left( \frac{1}{1-x} \right)^4 \). Then \( P(x) \) admits 0.3819... as its smallest positive real root, so that this pattern is 3-avoidable.

Among the remaining patterns, we rule out patterns containing an occurrence of a doubled pattern with at most 3 variables. Also, if one pattern is the reverse of another, then they have the same avoidability index and we consider only one of the two. Thus, there remain the following patterns: \( ABACBD, ABACBD, ABCDCBD, ABCDCDB, ABCDABC, ABCBDABC, \) and \( ABCBDABC \).

Now we consider doubled patterns with 5 variables. Similarly, we rule out every pattern of length at least 11 with the method by setting \( s_f = |f| \). Then we check that \( P(x) = 1 - 3x + \sum_{a,b,c,d,e \geq 1} x^{a+b+c+d+e} \) has a positive real root.

We also rule out every pattern of length 10 having \( ABC \) as a prefix. We set \( s_f = |f| = |ABC| = |A| + |B| + |C| + 2|D| + 2|E| \). Then we check that \( P(x) = 1 - 3x + \sum_{a,b,c,d,e \geq 1} x^{a+b+c+d+2e} = 1 - 3x + \left( \frac{1}{1-x} \right)^3 \left( \frac{1}{1-x} \right)^2 \) has a positive real root.

Again, we rule out patterns containing an occurrence of a doubled pattern with at most 4 variables and patterns whose reversed pattern is already considered. Thus, there remain the following patterns: \( ABACBDCEDE, ABACBDCEDE, \) and \( ABACDBDECE \).

## 4 Sporadic doubled patterns

In this section, we consider the 10 doubled patterns on 4 and 5 variables whose 3-avoidability has not been obtained in the previous section.

We define the avoidability exponent \( AE(p) \) of a pattern \( p \) as the largest real \( x \) such that every \( x \)-free word avoids \( p \). This notion is not pertinent e.g. for the pattern \( ABW BAX ACY CAZ BC \) studied by Baker, McNulty, and Taylor [1], since for every \( \epsilon > 0 \), there exists a \( (1+\epsilon) \)-free word containing an occurrence of that pattern. However, \( AE(p) > 1 \) for every doubled pattern. To see that, consider a factor \( A \ldots A \) of \( p \). If an \( x \)-free word contains an occurrence of \( p \), then the image of this factor is a repetition such that the image of \( A \) cannot be too large compared to the images of the variables occurring between the \( A \)s in \( p \). We have similar constraints for every variable and this set of constraints becomes unsatisfiable as \( x \) decreases towards 1. We present one way of obtaining the avoidability exponent for a doubled pattern \( p \) of length \( 2v(p) \). We construct the \( v(p) \times v(p) \) matrix \( M \) such that \( M_{i,j} \) is the number of occurrences of the variable \( X_j \) between the two occurrences of the variable \( X_i \). We compute the largest eigenvalue \( \beta \) of \( M \) and then we
have $AE(p) = 1 + \frac{1}{\beta+1}$. For example if $p = ABACDCBD$, then we get $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, $\beta = 1.9403 \ldots$, and $AE(p) = 1 + \frac{1}{\beta+1} = 1.3400 \ldots$. The avoidability exponents of the 10 patterns considered in this section range from $AE(ABCA DBD) = 1.292893219$ to $AE(ABACBDCD) = 1.381966011$. For each pattern $p$ among the 10, we give a uniform morphism $m : \Sigma_5^* \to \Sigma_2^*$ such that for every $(\frac{5}{4})$-free word $w \in \Sigma_5^*$, we have that $m(w)$ avoids $p$. The proof that $p$ is avoided follows the method in [9]. Since there exist exponentially many $(\frac{5}{4})$-free words over $\Sigma_5$ [7], there exist exponentially many binary words avoiding $p$.

- $AE(ABACBDCD) = 1.381966011$, 17-uniform morphism
  
  $0 \mapsto 000001111101010110$
  $1 \mapsto 00000110100100110$
  $2 \mapsto 00000011100110111$
  $3 \mapsto 00000011010101111$
  $4 \mapsto 0000001100100110$

- $AE(ABACDBDC) = 1.333333333$, 33-uniform morphism
  
  $0 \mapsto 0000001011010001111110100101111$
  $1 \mapsto 00000010011010000111110100101111$
  $2 \mapsto 0000000101101000011111010010111$
  $3 \mapsto 0000000101101000011111010010111$
  $4 \mapsto 0000000101101000011111010010111$

- $AE(ABACDCBD) = 1.340090632$, 28-uniform morphism
  
  $0 \mapsto 00000101010001110010001111101100101111$
  $1 \mapsto 0000001111101001100111111110100011111$
  $2 \mapsto 0000000111110100111111111010010011111$
  $3 \mapsto 00000001111101001100111111010010011111$
  $4 \mapsto 00000001111101001100111111010010011111$

- $AE(ABCADBDC) = 1.292893219$, 21-uniform morphism
  
  $0 \mapsto 00000111101101011111101$
  $1 \mapsto 00000101101001111101011$
  $2 \mapsto 00000101111010011111111$
  $3 \mapsto 00000110111010011111111$
  $4 \mapsto 00000110111010011111111$
• $AE(ABCADCBD) = 1.295597743$, 22-uniform morphism
  
  0 $\mapsto$ 0000011011010100100111111
  1 $\mapsto$ 0000011010101001001111111
  2 $\mapsto$ 000001101001111111001111111
  3 $\mapsto$ 0000001010111101110111111
  4 $\mapsto$ 0000000111011011110111111

• $AE(ABCADCDB) = 1.327621756$, 26-uniform morphism
  
  0 $\mapsto$ 000000111111100110101011001111111
  1 $\mapsto$ 0000001011111001001111111
  2 $\mapsto$ 0000000111111101100111111
  3 $\mapsto$ 000000010111111111000101111
  4 $\mapsto$ 000000010101111111101111111

• $AE(ABCBDADC) = 1.302775638$, 33-uniform morphism
  
  0 $\mapsto$ 0000010111111001101111111
  1 $\mapsto$ 00000101111101001111111
  2 $\mapsto$ 0000001101111111111101111
  3 $\mapsto$ 0000000110101010111111111
  4 $\mapsto$ 0000000101111111111111111

• $AE(ABACBDCEDE) = 1.366025404$, 15-uniform morphism
  
  0 $\mapsto$ 001011011110000
  1 $\mapsto$ 001010100111111
  2 $\mapsto$ 000110010011000
  3 $\mapsto$ 000011111111100
  4 $\mapsto$ 000011010101110

• $AE(ABACBDCEDE) = 1.302775638$, 18-uniform morphism
  
  0 $\mapsto$ 000010110100111
  1 $\mapsto$ 000010100111111
  2 $\mapsto$ 0000001101100111
  3 $\mapsto$ 000000101010111
  4 $\mapsto$ 00000001001111011

• $AE(ABACDBCEDE) = 1.320416579$, 22-uniform morphism
  
  0 $\mapsto$ 0000110111100110100111
  1 $\mapsto$ 000010100111111
  2 $\mapsto$ 000001011110111
  3 $\mapsto$ 000000101010111
  4 $\mapsto$ 0000000111111101111
5 Simultaneous avoidance of doubled patterns

Bell and Goh [3] have also considered the avoidance of multiple patterns simultaneously and ask (question 3) whether there exist an infinite word over a finite alphabet that avoids every doubled pattern. We give a negative answer.

A word $w$ is $n$-splitted if $|w| \equiv 0 \pmod{n}$ and every factor $w_i$ such that $w = w_1 w_2 \ldots w_n$ and $|w_i| = \frac{|w|}{n}$ for $1 \leq i \leq n$ contains every letter in $w$. An $n$-splitted pattern is defined similarly. Let us prove by induction on $k$ that every word $w \in \Sigma_k^{n^k}$ contains an $n$-splitted factor. The assertion is true for $k = 1$. Now, if the word $w \in \Sigma_k^{n^k}$ is not itself $n$-splitted, then by definition it must contain a factor $w_i$ that does not contain every letter of $w$. So we have $w_i \in \Sigma_k^{n^{k-1}}$. By induction, $w_i$ contains an $n$-splitted factor, and so does $w$.

This implies that for every fixed $n$, every infinite word over a finite alphabet contains $n$-splitted factors. Moreover, an $n$-splitted word is an occurrence of an $n$-splitted pattern such that every variable has a distinct image of length 1. So, for every fixed $n$, the set of all $n$-splitted patterns is not avoidable by an infinite word over a finite alphabet.

Notice that if $n \geq 2$, then an $n$-splitted word (resp. pattern) contains a 2-splitted word (resp. pattern) and a 2-splitted word (resp. pattern) is doubled.

6 Conclusion

Our results answer settles the first of two questions of our previous paper [10]. The second question is whether there exists a finite $k$ such that every doubled pattern with at least $k$ variables is 2-avoidable. As already noticed [10], such a $k$ is at least 5 since, e.g., $ABCCBADD$ is not 2-avoidable.

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References


