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# Doubled patterns are 3-avoidable

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## Abstract

In combinatorics on words, a word  $w$  over an alphabet  $\Sigma$  is said to avoid a pattern  $p$  over an alphabet  $\Delta$  if there is no factor  $f$  of  $w$  such that  $f = h(p)$  where  $h : \Delta^* \rightarrow \Sigma^*$  is a non-erasing morphism. A pattern  $p$  is said to be  $k$ -avoidable if there exists an infinite word over a  $k$ -letter alphabet that avoids  $p$ . A pattern is said to be doubled if no variable occurs only once. Doubled patterns with at most 3 variables and patterns with at least 6 variables are 3-avoidable. We show that doubled patterns with 4 and 5 variables are also 3-avoidable.

**Keywords:** Word; Pattern avoidance.

## 1 Introduction

A pattern  $p$  is a non-empty word over an alphabet  $\Delta = \{A, B, C, \dots\}$  of capital letters called *variables*. An *occurrence* of  $p$  in a word  $w$  is a non-erasing morphism  $h : \Delta^* \rightarrow \Sigma^*$  such that  $h(p)$  is a factor of  $w$ . The avoidability index  $\lambda(p)$  of a pattern  $p$  is the size of the smallest alphabet  $\Sigma$  such that there exists an infinite word  $w$  over  $\Sigma$  containing no occurrence of  $p$ . Bean, Ehrenfeucht, and McNulty [2] and Zimin [13] characterized unavoidable patterns, i.e., such that  $\lambda(p) = \infty$ . We say that a pattern  $p$  is  $t$ -avoidable if  $\lambda(p) \leq t$ . For more informations on pattern avoidability, we refer to Chapter 3 of Lothaire's book [8].

It follows from their characterization that every unavoidable pattern contains a variable that occurs once. Equivalently, every doubled pattern is avoidable. Our result is that :

**Theorem 1.** *Every doubled pattern is 3-avoidable.*

Let  $v(p)$  be the number of distinct variables of the pattern  $p$ . For  $v(p) \leq 3$ , Cassaigne [5] began and I [9] finished the determination of the avoidability index of every

pattern with at most 3 variables. It implies in particular that every avoidable pattern with at most 3 variables is 3-avoidable. Moreover, Bell and Goh [3] obtained that every doubled pattern  $p$  such that  $v(p) \geq 6$  is 3-avoidable.

Therefore, as noticed in the conclusion of [10], there remains to prove Theorem 1 for every pattern  $p$  such that  $4 \leq v(p) \leq 5$ . In this paper, we use both constructions of infinite words and a non-constructive method to settle the cases  $4 \leq v(p) \leq 5$ .

Recently, Blanchet-Sadri and Woodhouse [4] and Ochem and Pinlou [10] independently obtained the following.

**Theorem 2** ([4, 10]). *Let  $p$  be a pattern.*

(a) *If  $p$  has length at least  $3 \times 2^{v(p)-1}$  then  $\lambda(p) \leq 2$ .*

(b) *If  $p$  has length at least  $2^{v(p)}$  then  $\lambda(p) \leq 3$ .*

As noticed in these papers, if  $p$  has length at least  $2^{v(p)}$  then  $p$  contains a doubled pattern as a factor. Thus, Theorem 1 implies Theorem 2.(b).

## 2 Extending the power series method

In this section, we borrow an idea from the entropy compression method to extend the power series method as used by Bell and Goh [3], Rampersad [12], and Blanchet-Sadri and Woodhouse [4].

Let us describe the method. Let  $L \subset \Sigma_m^*$  be a factorial language defined by a set  $F$  of forbidden factors of length at least 2. We denote the factor complexity of  $L$  by  $n_i = |L \cap \Sigma_m^i|$ . We define  $L'$  as the set of words  $w$  such that  $w$  is not in  $L$  and the prefix of length  $|w| - 1$  of  $w$  is in  $L$ . For every forbidden factor  $f \in F$ , we choose a number  $1 \leq s_f \leq |f|$ . Then, for every  $i \geq 1$ , we define an integer  $a_i$  such that

$$a_i \geq \max_{u \in L} \left| \left\{ v \in \Sigma_m^i \mid uv \in L', uv = bf, f \in F, s_f = i \right\} \right|.$$

We consider the formal power series  $P(x) = 1 - mx + \sum_{i \geq 1} a_i x^i$ . If  $P(x)$  has a positive real root  $x_0$ , then  $n_i \geq x_0^{-i}$  for every  $i \geq 0$ .

Let us rewrite that  $P(x_0) = 1 - mx_0 + \sum_{i \geq 1} a_i x_0^i = 0$  as

$$m - \sum_{i \geq 1} a_i x_0^{i-1} = x_0^{-1} \tag{1}$$

Since  $n_0 = 1$ , we will prove by induction that  $\frac{n_i}{n_{i-1}} \geq x_0^{-1}$  in order to obtain that  $n_i \geq x_0^{-i}$  for every  $i \geq 0$ . By using (1), we obtain the base case:  $\frac{n_1}{n_0} = n_1 = m \geq x_0^{-1}$ . Now, for every length  $i \geq 1$ , there are:

- $m^i$  words in  $\Sigma_m^i$ ,
- $n_i$  words in  $L$ ,

- at most  $\sum_{1 \leq j \leq i} n_{i-j} a_j$  words in  $L'$ ,
- $m(m^{i-1} - n_{i-1})$  words in  $\Sigma_m^i \setminus \{L \cup L'\}$ .

This gives  $n_i + \sum_{1 \leq j \leq i} n_j a_{i-j} + m(m^{i-1} - n_{i-1}) \geq m^i$ , that is,  $n_i \geq mn_{i-1} - \sum_{1 \leq j \leq i} n_{i-j} a_j$ .

$$\begin{aligned} \frac{n_i}{n_{i-1}} &\geq m - \sum_{1 \leq j \leq i} a_j \frac{n_{i-j}}{n_{i-1}} \\ &\geq m - \sum_{1 \leq j \leq i} a_j x_0^{j-1} && \text{By induction} \\ &\geq m - \sum_{j \geq 1} a_j x_0^{j-1} \\ &= x_0^{-1} && \text{By (1)} \end{aligned}$$

The power series method used in previous papers [3, 4, 12] corresponds to the special case such that  $s_f = |f|$  for every forbidden factor. Our condition is that  $P(x) = 0$  for some  $x > 0$  whereas the condition in these papers is that every coefficient of the series expansion of  $\frac{1}{P(x)}$  is positive. The two conditions are actually equivalent. The result in [11] concerns series of the form  $S(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \dots$  with real coefficients such that  $a_1 < 0$  and  $a_i \geq 0$  for every  $i \geq 2$ . It states that every coefficient of the series  $1/S(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$  is positive if and only if  $S(x)$  has a positive real root  $x_0$ . Moreover, we have  $b_i \geq x_0^{-i}$  for every  $i \geq 0$ .

The entropy compression method as developed by Gonçalves, Montassier, and Pinlou [6] uses a condition equivalent to  $P(x) = 0$ . The benefit of the present method is that we get an exponential lower bound on the factor complexity. It is not clear whether it is possible to get such a lower bound when using entropy compression for graph coloring, since words have a simpler structure than graphs.

### 3 Applying the method

In this section, we show that some doubled patterns on 4 and 5 variables are 3-avoidable. Given a pattern  $p$ , every occurrence  $f$  of  $p$  is a forbidden factor. With an abuse of notation, we denote by  $|A|$  the length of the image of the variable  $A$  of  $p$  in the occurrence  $f$ . This notation is used to define the length  $s_f$ .

Let us first consider doubled patterns with 4 variables. We begin with patterns of length 9, so that one variable, say  $A$ , appears 3 times. We set  $s_f = |f|$ . Using the obvious upper bound on the number of pattern occurrences, we obtain

$$\begin{aligned} P(x) &= 1 - 3x + \sum_{a,b,c,d \geq 1} 3^{a+b+c+d} x^{3a+2b+2c+2d} \\ &= 1 - 3x + \sum_{a,b,c,d \geq 1} (3x^3)^a (3x^2)^b (3x^2)^c (3x^2)^d \\ &= 1 - 3x + \left( \sum_{a \geq 1} (3x^3)^a \right) \left( \sum_{b \geq 1} (3x^2)^b \right) \left( \sum_{c \geq 1} (3x^2)^c \right) \left( \sum_{d \geq 1} (3x^2)^d \right) \\ &= 1 - 3x + \left( \frac{1}{1-3x^3} - 1 \right) \left( \frac{1}{1-3x^2} - 1 \right) \left( \frac{1}{1-3x^2} - 1 \right) \left( \frac{1}{1-3x^2} - 1 \right) \\ &= 1 - 3x + \left( \frac{1}{1-3x^3} - 1 \right) \left( \frac{1}{1-3x^2} - 1 \right)^3 \\ &= \frac{1-3x-9x^2+24x^3+36x^4-54x^5-108x^6+243x^8+162x^9-243x^{10}}{(1-3x^3)(1-3x^2)^3}. \end{aligned}$$

Then  $P(x)$  admits  $x_0 = 0.3400\dots$  as its smallest positive real root. So, every doubled pattern  $p$  with 4 variables and length 9 is 3-avoidable and there exist at least  $x_0^{-n} > 2.941^n$

ternary words avoiding  $p$ . Notice that for patterns with 4 variables and length at least 10, every term of  $\sum_{a,b,c,d \geq 1} 3^{a+b+c+d} x^{3a+2b+2c+2d}$  in  $P(x)$  gets multiplied by some positive power of  $x$ . Since  $0 < x < 1$ , every term is now smaller than in the previous case. So  $P(x)$  admits a smallest positive real root that is smaller than  $0.3400\dots$ . Thus, these patterns are also 3-avoidable.

Now, we consider patterns with length 8, so that every variable appears exactly twice. If such a pattern has  $ABCD$  as a prefix, then we set  $s_f = \frac{|f|}{2} = |A| + |B| + |C| + |D|$ . So we obtain  $P(x) = 1 - 3x + \sum_{a,b,c,d \geq 1} x^{a+b+c+d} = 1 - 3x + \left(\frac{1}{1-x} - 1\right)^4$ . Then  $P(x)$  admits  $0.3819\dots$  as its smallest positive real root, so that this pattern is 3-avoidable.

Among the remaining patterns, we rule out patterns containing an occurrence of a doubled pattern with at most 3 variables. Also, if one pattern is the reverse of another, then they have the same avoidability index and we consider only one of the two. Thus, there remain the following patterns:  $ABACBD$ ,  $ABACDB$ ,  $ABACDC$ ,  $ABACDCD$ ,  $ABACDBDC$ ,  $ABACDCBD$ ,  $ABCADBDC$ ,  $ABCADCBD$ ,  $ABCADCDB$ , and  $ABCBDADC$ .

Now we consider doubled patterns with 5 variables. Similarly, we rule out every pattern of length at least 11 with the method by setting  $s_f = |f|$ . Then we check that  $P(x) = 1 - 3x + \sum_{a,b,c,d,e \geq 1} 3^{a+b+c+d+e} x^{3a+2b+2c+2d+2e} = 1 - 3x + \left(\frac{1}{1-3x^3} - 1\right) \left(\frac{1}{1-3x^2} - 1\right)^4$  has a positive real root.

We also rule out every pattern of length 10 having  $ABC$  as a prefix. We set  $s_f = |f| - |ABC| = |A| + |B| + |C| + 2|D| + 2|E|$ . Then we check that  $P(x) = 1 - 3x + \sum_{a,b,c,d,e \geq 1} 3^{d+e} x^{a+b+c+2d+2e} = 1 - 3x + \left(\frac{1}{1-x} - 1\right)^3 \left(\frac{1}{1-3x^2} - 1\right)^2$  has a positive real root.

Again, we rule out patterns containing an occurrence of a doubled pattern with at most 4 variables and patterns whose reversed pattern is already considered. Thus, there remain the following patterns:  $ABACBDC$ ,  $ABACBDCD$ ,  $ABACBDCD$ ,  $ABACBDCD$ , and  $ABACBDCD$ .

## 4 Sporadic doubled patterns

In this section, we consider the 10 doubled patterns on 4 and 5 variables whose 3-avoidability has not been obtained in the previous section.

We define the *avoidability exponent*  $AE(p)$  of a pattern  $p$  as the largest real  $x$  such that every  $x$ -free word avoids  $p$ . This notion is not pertinent e.g. for the pattern  $ABWBAXACYCAZBC$  studied by Baker, McNulty, and Taylor [1], since for every  $\epsilon > 0$ , there exists a  $(1 + \epsilon)$ -free word containing an occurrence of that pattern. However,  $AE(p) > 1$  for every doubled pattern. To see that, consider a factor  $A \dots A$  of  $p$ . If an  $x$ -free word contains an occurrence of  $p$ , then the image of this factor is a repetition such that the image of  $A$  cannot be too large compared to the images of the variables occurring between the  $A$ s in  $p$ . We have similar constraints for every variable and this set of constraints becomes unsatisfiable as  $x$  decreases towards 1. We present one way of obtaining the avoidability exponent for a doubled pattern  $p$  of length  $2v(p)$ . We construct the  $v(p) \times v(p)$  matrix  $M$  such that  $M_{i,j}$  is the number of occurrences of the variable  $X_j$  between the two occurrences of the variable  $X_i$ . We compute the largest eigenvalue  $\beta$  of  $M$  and then we

have  $AE(p) = 1 + \frac{1}{\beta+1}$ . For example if  $p = ABACDCBD$ , then we get  $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ ,  $\beta = 1.9403\dots$ , and  $AE(p) = 1 + \frac{1}{\beta+1} = 1.3400\dots$ . The avoidability exponents of the 10 patterns considered in this section range from  $AE(ABCADBDC) = 1.292893219$  to  $AE(ABACBDCD) = 1.381966011$ . For each pattern  $p$  among the 10, we give a uniform morphism  $m : \Sigma_5^* \rightarrow \Sigma_2^*$  such that for every  $\left(\frac{5^+}{4}\right)$ -free word  $w \in \Sigma_5^*$ , we have that  $m(w)$  avoids  $p$ . The proof that  $p$  is avoided follows the method in [9]. Since there exist exponentially many  $\left(\frac{5^+}{4}\right)$ -free words over  $\Sigma_5$  [7], there exist exponentially many binary words avoiding  $p$ .

- $AE(ABACBDCD) = 1.381966011$ , 17-uniform morphism

0  $\mapsto$  00000111101010110  
 1  $\mapsto$  00000110100100110  
 2  $\mapsto$  00000011100110111  
 3  $\mapsto$  00000011010101111  
 4  $\mapsto$  00000011001001011

- $AE(ABACDBDC) = 1.333333333$ , 33-uniform morphism

0  $\mapsto$  00000010110100011111011001010111  
 1  $\mapsto$  000000100110100001111101001010111  
 2  $\mapsto$  00000001011010000111111010010111  
 3  $\mapsto$  000000010011010100011111010010111  
 4  $\mapsto$  000000010011001000001111010010111

- $AE(ABACDCBD) = 1.340090632$ , 28-uniform morphism

0  $\mapsto$  0000101010001110010000111111  
 1  $\mapsto$  0000001111010001101001111111  
 2  $\mapsto$  0000001101000011110100100111  
 3  $\mapsto$  0000001011110000110100111111  
 4  $\mapsto$  0000001010111100100001111111

- $AE(ABCADBDC) = 1.292893219$ , 21-uniform morphism

0  $\mapsto$  0000111011010111111010  
 1  $\mapsto$  000010110100100111101  
 2  $\mapsto$  000001101110100101111  
 3  $\mapsto$  000001101011001111111  
 4  $\mapsto$  000000110111010111111

- $AE(ABCADCBD) = 1.295597743$ , 22-uniform morphism

0  $\mapsto$  0000011011010100011111  
 1  $\mapsto$  0000011010101001001111  
 2  $\mapsto$  0000001101100100111111  
 3  $\mapsto$  0000001010110000111111  
 4  $\mapsto$  0000000110101001110111

- $AE(ABCADCDB) = 1.327621756$ , 26-uniform morphism

0  $\mapsto$  00000011110010101011000111  
 1  $\mapsto$  00000011010111111001011011  
 2  $\mapsto$  00000010011111101001110111  
 3  $\mapsto$  00000001001111110001010111  
 4  $\mapsto$  00000001000111111001010111

- $AE(ABCBDADC) = 1.302775638$ , 33-uniform morphism

0  $\mapsto$  000000101111110011000110011111101  
 1  $\mapsto$  000000101111001000001100111111101  
 2  $\mapsto$  000000011011111001100000100111101  
 3  $\mapsto$  000000011010101011000001001111101  
 4  $\mapsto$  000000010111110010101010011111011

- $AE(ABACBDCED E) = 1.366025404$ , 15-uniform morphism

0  $\mapsto$  001011011110000  
 1  $\mapsto$  001010100111111  
 2  $\mapsto$  000110010011000  
 3  $\mapsto$  000011111111100  
 4  $\mapsto$  000011010101110

- $AE(ABACDBCED E) = 1.302775638$ , 18-uniform morphism

0  $\mapsto$  000010110100100111  
 1  $\mapsto$  000010100111111111  
 2  $\mapsto$  000000110110011111  
 3  $\mapsto$  000000101010101111  
 4  $\mapsto$  000000000111100111

- $AE(ABACDBDECE) = 1.320416579$ , 22-uniform morphism

0  $\mapsto$  0000001111110001011011  
 1  $\mapsto$  0000001111100100110101  
 2  $\mapsto$  0000001111100001101101  
 3  $\mapsto$  0000001111001001011100  
 4  $\mapsto$  0000001111000010101100

## 5 Simultaneous avoidance of doubled patterns

Bell and Goh [3] have also considered the avoidance of multiple patterns simultaneously and ask (question 3) whether there exist an infinite word over a finite alphabet that avoids every doubled pattern. We give a negative answer.

A word  $w$  is  $n$ -splitted if  $|w| \equiv 0 \pmod{n}$  and every factor  $w_i$  such that  $w = w_1 w_2 \dots w_n$  and  $|w_i| = \frac{|w|}{n}$  for  $1 \leq i \leq n$  contains every letter in  $w$ . An  $n$ -splitted pattern is defined similarly. Let us prove by induction on  $k$  that every word  $w \in \Sigma_k^{n^k}$  contains an  $n$ -splitted factor. The assertion is true for  $k = 1$ . Now, if the word  $w \in \Sigma_k^{n^k}$  is not itself  $n$ -splitted, then by definition it must contain a factor  $w_i$  that does not contain every letter of  $w$ . So we have  $w_i \in \Sigma_{k-1}^{n^{k-1}}$ . By induction,  $w_i$  contains an  $n$ -splitted factor, and so does  $w$ .

This implies that for every fixed  $n$ , every infinite word over a finite alphabet contains  $n$ -splitted factors. Moreover, an  $n$ -splitted word is an occurrence of an  $n$ -splitted pattern such that every variable has a distinct image of length 1. So, for every fixed  $n$ , the set of all  $n$ -splitted patterns is not avoidable by an infinite word over a finite alphabet.

Notice that if  $n \geq 2$ , then an  $n$ -splitted word (resp. pattern) contains a 2-splitted word (resp. pattern) and a 2-splitted word (resp. pattern) is doubled.

## 6 Conclusion

Our results answer settles the first of two questions of our previous paper [10]. The second question is whether there exists a finite  $k$  such that every doubled pattern with at least  $k$  variables is 2-avoidable. As already noticed [10], such a  $k$  is at least 5 since, e.g.,  $ABCCBADD$  is not 2-avoidable.

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## References

- [1] K.A. Baker, G.F. McNulty, and W. Taylor. Growth problems for avoidable words, *Theoret. Comput. Sci.* **69** (1989), 319–345.
- [2] D.R. Bean, A. Ehrenfeucht, and G.F. McNulty. Avoidable patterns in strings of symbols. *Pacific J. of Math.* **85** (1979) 261–294.
- [3] J. Bell, T. L. Goh. Exponential lower bounds for the number of words of uniform length avoiding a pattern. *Inform. and Comput.* **205** (2007), 1295-1306.



- [4] F. Blanchet-Sadri, B. Woodhouse. Strict bounds for pattern avoidance. *Theor. Comput. Sci.* **506** (2013), 17–27.
- [5] J. Cassaigne. Motifs évitables et régularité dans les mots. Thèse de Doctorat, Université Paris VI, Juillet 1994.
- [6] D. Gonçalves, M. Montassier, and A. Pinlou. Entropy compression method applied to graph colorings. [arXiv:1406.4380](https://arxiv.org/abs/1406.4380)
- [7] R. Kolpakov and M. Rao: On the number of Dejan words over alphabets of 5, 6, 7, 8, 9 and 10 letters *Theor. Comput. Sci.* **412(46)** (2011), 6507–6516.
- [8] M. Lothaire. Algebraic Combinatorics on Words. *Cambridge Univ. Press* (2002).
- [9] P. Ochem. A generator of morphisms for infinite words. *RAIRO: Theoret. Informatics Appl.* **40** (2006) 427–441.
- [10] P. Ochem and A. Pinlou. Application of entropy compression in pattern avoidance. *Electron. J. Combinatorics.* **21(2)** (2014), #RP2.7.
- [11] D. I. Piotkovskii. On the growth of graded algebras with a small number of defining relations. *Uspekhi Mat. Nauk.* **48:3(291)** (1993), 199–200.
- [12] N. Rampersad. Further applications of a power series method for pattern avoidance. *Electron. J. Combinatorics.* **18(1)** (2011), #P134.
- [13] A.I. Zimin. Blocking sets of terms. *Math. USSR Sbornik* **47(2)** (1984) 353–364. English translation.