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Doubled patterns are 3-avoidable

Pascal Ochem

LIRMM, Université de Montpellier, CNRS
Montpellier, France
ochem@lirmm.fr

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Abstract

In combinatorics on words, a word w over an alphabet Σ is said to avoid a pattern p over an alphabet Δ if there is no factor f of w such that $f = h(p)$ where $h : \Delta^* \rightarrow \Sigma^*$ is a non-erasing morphism. A pattern p is said to be k -avoidable if there exists an infinite word over a k -letter alphabet that avoids p . A pattern is said to be doubled if no variable occurs only once. Doubled patterns with at most 3 variables and patterns with at least 6 variables are 3-avoidable. We show that doubled patterns with 4 and 5 variables are also 3-avoidable.

Keywords: Word; Pattern avoidance.

1 Introduction

A pattern p is a non-empty word over an alphabet $\Delta = \{A, B, C, \dots\}$ of capital letters called *variables*. An *occurrence* of p in a word w is a non-erasing morphism $h : \Delta^* \rightarrow \Sigma^*$ such that $h(p)$ is a factor of w . The avoidability index $\lambda(p)$ of a pattern p is the size of the smallest alphabet Σ such that there exists an infinite word w over Σ containing no occurrence of p . Bean, Ehrenfeucht, and McNulty [2] and Zimin [13] characterized unavoidable patterns, i.e., such that $\lambda(p) = \infty$. We say that a pattern p is t -avoidable if $\lambda(p) \leq t$. For more informations on pattern avoidability, we refer to Chapter 3 of Lothaire's book [8].

It follows from their characterization that every unavoidable pattern contains a variable that occurs once. Equivalently, every doubled pattern is avoidable. Our result is that :

Theorem 1. *Every doubled pattern is 3-avoidable.*

Let $v(p)$ be the number of distinct variables of the pattern p . For $v(p) \leq 3$, Cassaigne [5] began and I [9] finished the determination of the avoidability index of every

pattern with at most 3 variables. It implies in particular that every avoidable pattern with at most 3 variables is 3-avoidable. Moreover, Bell and Goh [3] obtained that every doubled pattern p such that $v(p) \geq 6$ is 3-avoidable.

Therefore, as noticed in the conclusion of [10], there remains to prove Theorem 1 for every pattern p such that $4 \leq v(p) \leq 5$. In this paper, we use both constructions of infinite words and a non-constructive method to settle the cases $4 \leq v(p) \leq 5$.

Recently, Blanchet-Sadri and Woodhouse [4] and Ochem and Pinlou [10] independently obtained the following.

Theorem 2 ([4, 10]). *Let p be a pattern.*

(a) *If p has length at least $3 \times 2^{v(p)-1}$ then $\lambda(p) \leq 2$.*

(b) *If p has length at least $2^{v(p)}$ then $\lambda(p) \leq 3$.*

As noticed in these papers, if p has length at least $2^{v(p)}$ then p contains a doubled pattern as a factor. Thus, Theorem 1 implies Theorem 2.(b).

2 Extending the power series method

In this section, we borrow an idea from the entropy compression method to extend the power series method as used by Bell and Goh [3], Rampersad [12], and Blanchet-Sadri and Woodhouse [4].

Let us describe the method. Let $L \subset \Sigma_m^*$ be a factorial language defined by a set F of forbidden factors of length at least 2. We denote the factor complexity of L by $n_i = |L \cap \Sigma_m^i|$. We define L' as the set of words w such that w is not in L and the prefix of length $|w| - 1$ of w is in L . For every forbidden factor $f \in F$, we choose a number $1 \leq s_f \leq |f|$. Then, for every $i \geq 1$, we define an integer a_i such that

$$a_i \geq \max_{u \in L} \left| \left\{ v \in \Sigma_m^i \mid uv \in L', uv = bf, f \in F, s_f = i \right\} \right|.$$

We consider the formal power series $P(x) = 1 - mx + \sum_{i \geq 1} a_i x^i$. If $P(x)$ has a positive real root x_0 , then $n_i \geq x_0^{-i}$ for every $i \geq 0$.

Let us rewrite that $P(x_0) = 1 - mx_0 + \sum_{i \geq 1} a_i x_0^i = 0$ as

$$m - \sum_{i \geq 1} a_i x_0^{i-1} = x_0^{-1} \tag{1}$$

Since $n_0 = 1$, we will prove by induction that $\frac{n_i}{n_{i-1}} \geq x_0^{-1}$ in order to obtain that $n_i \geq x_0^{-i}$ for every $i \geq 0$. By using (1), we obtain the base case: $\frac{n_1}{n_0} = n_1 = m \geq x_0^{-1}$. Now, for every length $i \geq 1$, there are:

- m^i words in Σ_m^i ,
- n_i words in L ,

- at most $\sum_{1 \leq j \leq i} n_{i-j} a_j$ words in L' ,
- $m(m^{i-1} - n_{i-1})$ words in $\Sigma_m^i \setminus \{L \cup L'\}$.

This gives $n_i + \sum_{1 \leq j \leq i} n_j a_{i-j} + m(m^{i-1} - n_{i-1}) \geq m^i$, that is, $n_i \geq mn_{i-1} - \sum_{1 \leq j \leq i} n_{i-j} a_j$.

$$\begin{aligned}
 \frac{n_i}{n_{i-1}} &\geq m - \sum_{1 \leq j \leq i} a_j \frac{n_{i-j}}{n_{i-1}} \\
 &\geq m - \sum_{1 \leq j \leq i} a_j x_0^{j-1} && \text{By induction} \\
 &\geq m - \sum_{j \geq 1} a_j x_0^{j-1} \\
 &= x_0^{-1} && \text{By (1)}
 \end{aligned}$$

The power series method used in previous papers [3, 4, 12] corresponds to the special case such that $s_f = |f|$ for every forbidden factor. Our condition is that $P(x) = 0$ for some $x > 0$ whereas the condition in these papers is that every coefficient of the series expansion of $\frac{1}{P(x)}$ is positive. The two conditions are actually equivalent. The result in [11] concerns series of the form $S(x) = 1 + a_1x + a_2x^2 + a_3x^3 + \dots$ with real coefficients such that $a_1 < 0$ and $a_i \geq 0$ for every $i \geq 2$. It states that every coefficient of the series $1/S(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$ is positive if and only if $S(x)$ has a positive real root x_0 . Moreover, we have $b_i \geq x_0^{-i}$ for every $i \geq 0$.

The entropy compression method as developed by Gonçalves, Montassier, and Pinlou [6] uses a condition equivalent to $P(x) = 0$. The benefit of the present method is that we get an exponential lower bound on the factor complexity. It is not clear whether it is possible to get such a lower bound when using entropy compression for graph coloring, since words have a simpler structure than graphs.

3 Applying the method

In this section, we show that some doubled patterns on 4 and 5 variables are 3-avoidable. Given a pattern p , every occurrence f of p is a forbidden factor. With an abuse of notation, we denote by $|A|$ the length of the image of the variable A of p in the occurrence f . This notation is used to define the length s_f .

Let us first consider doubled patterns with 4 variables. We begin with patterns of length 9, so that one variable, say A , appears 3 times. We set $s_f = |f|$. Using the obvious upper bound on the number of pattern occurrences, we obtain

$$\begin{aligned}
 P(x) &= 1 - 3x + \sum_{a,b,c,d \geq 1} 3^{a+b+c+d} x^{3a+2b+2c+2d} \\
 &= 1 - 3x + \sum_{a,b,c,d \geq 1} (3x^3)^a (3x^2)^b (3x^2)^c (3x^2)^d \\
 &= 1 - 3x + \left(\sum_{a \geq 1} (3x^3)^a \right) \left(\sum_{b \geq 1} (3x^2)^b \right) \left(\sum_{c \geq 1} (3x^2)^c \right) \left(\sum_{d \geq 1} (3x^2)^d \right) \\
 &= 1 - 3x + \left(\frac{1}{1-3x^3} - 1 \right) \left(\frac{1}{1-3x^2} - 1 \right) \left(\frac{1}{1-3x^2} - 1 \right) \left(\frac{1}{1-3x^2} - 1 \right) \\
 &= 1 - 3x + \left(\frac{1}{1-3x^3} - 1 \right) \left(\frac{1}{1-3x^2} - 1 \right)^3 \\
 &= \frac{1-3x-9x^2+24x^3+36x^4-54x^5-108x^6+243x^8+162x^9-243x^{10}}{(1-3x^3)(1-3x^2)^3}.
 \end{aligned}$$

Then $P(x)$ admits $x_0 = 0.3400\dots$ as its smallest positive real root. So, every doubled pattern p with 4 variables and length 9 is 3-avoidable and there exist at least $x_0^{-n} > 2.941^n$

ternary words avoiding p . Notice that for patterns with 4 variables and length at least 10, every term of $\sum_{a,b,c,d \geq 1} 3^{a+b+c+d} x^{3a+2b+2c+2d}$ in $P(x)$ gets multiplied by some positive power of x . Since $0 < x < 1$, every term is now smaller than in the previous case. So $P(x)$ admits a smallest positive real root that is smaller than $0.3400\dots$. Thus, these patterns are also 3-avoidable.

Now, we consider patterns with length 8, so that every variable appears exactly twice. If such a pattern has $ABCD$ as a prefix, then we set $s_f = \frac{|f|}{2} = |A| + |B| + |C| + |D|$. So we obtain $P(x) = 1 - 3x + \sum_{a,b,c,d \geq 1} x^{a+b+c+d} = 1 - 3x + \left(\frac{1}{1-x} - 1\right)^4$. Then $P(x)$ admits $0.3819\dots$ as its smallest positive real root, so that this pattern is 3-avoidable.

Among the remaining patterns, we rule out patterns containing an occurrence of a doubled pattern with at most 3 variables. Also, if one pattern is the reverse of another, then they have the same avoidability index and we consider only one of the two. Thus, there remain the following patterns: $ABACBD CD$, $ABACDBDC$, $ABACDCBD$, $ABCADBDC$, $ABCADCBD$, $ABCADCDB$, and $ABCBDADC$.

Now we consider doubled patterns with 5 variables. Similarly, we rule out every pattern of length at least 11 with the method by setting $s_f = |f|$. Then we check that $P(x) = 1 - 3x + \sum_{a,b,c,d,e \geq 1} 3^{a+b+c+d+e} x^{3a+2b+2c+2d+2e} = 1 - 3x + \left(\frac{1}{1-3x^3} - 1\right) \left(\frac{1}{1-3x^2} - 1\right)^4$ has a positive real root.

We also rule out every pattern of length 10 having ABC as a prefix. We set $s_f = |f| - |ABC| = |A| + |B| + |C| + 2|D| + 2|E|$. Then we check that $P(x) = 1 - 3x + \sum_{a,b,c,d,e \geq 1} 3^{d+e} x^{a+b+c+2d+2e} = 1 - 3x + \left(\frac{1}{1-x} - 1\right)^3 \left(\frac{1}{1-3x^2} - 1\right)^2$ has a positive real root.

Again, we rule out patterns containing an occurrence of a doubled pattern with at most 4 variables and patterns whose reversed pattern is already considered. Thus, there remain the following patterns: $ABACBDC EDE$, $ABACDBC EDE$, and $ABACDBDECE$.

4 Sporadic doubled patterns

In this section, we consider the 10 doubled patterns on 4 and 5 variables whose 3-avoidability has not been obtained in the previous section.

We define the *avoidability exponent* $AE(p)$ of a pattern p as the largest real x such that every x -free word avoids p . This notion is not pertinent e.g. for the pattern $ABWBAXACYCAZBC$ studied by Baker, McNulty, and Taylor [1], since for every $\epsilon > 0$, there exists a $(1 + \epsilon)$ -free word containing an occurrence of that pattern. However, $AE(p) > 1$ for every doubled pattern. To see that, consider a factor $A\dots A$ of p . If an x -free word contains an occurrence of p , then the image of this factor is a repetition such that the image of A cannot be too large compared to the images of the variables occurring between the A s in p . We have similar constraints for every variable and this set of constraints becomes unsatisfiable as x decreases towards 1. We present one way of obtaining the avoidability exponent for a doubled pattern p of length $2v(p)$. We construct the $v(p) \times v(p)$ matrix M such that $M_{i,j}$ is the number of occurrences of the variable X_j between the two occurrences of the variable X_i . We compute the largest eigenvalue β of M and then we

have $AE(p) = 1 + \frac{1}{\beta+1}$. For example if $p = ABACDCBD$, then we get $M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$, $\beta = 1.9403\dots$, and $AE(p) = 1 + \frac{1}{\beta+1} = 1.3400\dots$. The avoidability exponents of the 10 patterns considered in this section range from $AE(ABCADBDC) = 1.292893219$ to $AE(ABACBDCD) = 1.381966011$. For each pattern p among the 10, we give a uniform morphism $m : \Sigma_5^* \rightarrow \Sigma_2^*$ such that for every $\left(\frac{5^+}{4}\right)$ -free word $w \in \Sigma_5^*$, we have that $m(w)$ avoids p . The proof that p is avoided follows the method in [9]. Since there exist exponentially many $\left(\frac{5^+}{4}\right)$ -free words over Σ_5 [7], there exist exponentially many binary words avoiding p .

- $AE(ABACBDCD) = 1.381966011$, 17-uniform morphism

0 \mapsto 00000111101010110
 1 \mapsto 00000110100100110
 2 \mapsto 00000011100110111
 3 \mapsto 00000011010101111
 4 \mapsto 00000011001001011

- $AE(ABACDBDC) = 1.333333333$, 33-uniform morphism

0 \mapsto 000000101101000111111011001010111
 1 \mapsto 000000100110100001111101001010111
 2 \mapsto 000000010110100001111110100101111
 3 \mapsto 000000010011010100011111010010111
 4 \mapsto 000000010011001000001111010010111

- $AE(ABACDCBD) = 1.340090632$, 28-uniform morphism

0 \mapsto 0000101010001110010000111111
 1 \mapsto 0000001111010001101001111111
 2 \mapsto 0000001101000011110100100111
 3 \mapsto 0000001011110000110100111111
 4 \mapsto 0000001010111100100001111111

- $AE(ABCADBDC) = 1.292893219$, 21-uniform morphism

0 \mapsto 0000111011010111111010
 1 \mapsto 0000101101001001111101
 2 \mapsto 000001101110100101111
 3 \mapsto 000001101011001111111
 4 \mapsto 000000110111010111111

- $AE(ABCADCBD) = 1.295597743$, 22-uniform morphism

0 \mapsto 0000011011010100011111
 1 \mapsto 0000011010101001001111
 2 \mapsto 0000001101100100111111
 3 \mapsto 0000001010110000111111
 4 \mapsto 0000000110101001110111

- $AE(ABCADCDB) = 1.327621756$, 26-uniform morphism

0 \mapsto 00000011110010101011000111
 1 \mapsto 00000011010111111001011011
 2 \mapsto 00000010011111101001110111
 3 \mapsto 00000001001111110001010111
 4 \mapsto 00000001000111111001010111

- $AE(ABCBDADC) = 1.302775638$, 33-uniform morphism

0 \mapsto 000000101111110011000110011111101
 1 \mapsto 000000101111001000001100111111101
 2 \mapsto 000000011011111001100000100111101
 3 \mapsto 000000011010101011000001001111101
 4 \mapsto 000000010111110010101010011111011

- $AE(ABACBDCED E) = 1.366025404$, 15-uniform morphism

0 \mapsto 001011011110000
 1 \mapsto 001010100111111
 2 \mapsto 000110010011000
 3 \mapsto 000011111111100
 4 \mapsto 000011010101110

- $AE(ABACDBCED E) = 1.302775638$, 18-uniform morphism

0 \mapsto 000010110100100111
 1 \mapsto 000010100111111111
 2 \mapsto 000000110110011111
 3 \mapsto 000000101010101111
 4 \mapsto 000000000111100111

- $AE(ABACDBDECE) = 1.320416579$, 22-uniform morphism

0 \mapsto 0000001111110001011011
 1 \mapsto 0000001111100100110101
 2 \mapsto 0000001111100001101101
 3 \mapsto 0000001111001001011100
 4 \mapsto 0000001111000010101100

5 Simultaneous avoidance of doubled patterns

Bell and Goh [3] have also considered the avoidance of multiple patterns simultaneously and ask (question 3) whether there exist an infinite word over a finite alphabet that avoids every doubled pattern. We give a negative answer.

A word w is n -splitted if $|w| \equiv 0 \pmod{n}$ and every factor w_i such that $w = w_1 w_2 \dots w_n$ and $|w_i| = \frac{|w|}{n}$ for $1 \leq i \leq n$ contains every letter in w . An n -splitted pattern is defined similarly. Let us prove by induction on k that every word $w \in \Sigma_k^{n^k}$ contains an n -splitted factor. The assertion is true for $k = 1$. Now, if the word $w \in \Sigma_k^{n^k}$ is not itself n -splitted, then by definition it must contain a factor w_i that does not contain every letter of w . So we have $w_i \in \Sigma_{k-1}^{n^{k-1}}$. By induction, w_i contains an n -splitted factor, and so does w .

This implies that for every fixed n , every infinite word over a finite alphabet contains n -splitted factors. Moreover, an n -splitted word is an occurrence of an n -splitted pattern such that every variable has a distinct image of length 1. So, for every fixed n , the set of all n -splitted patterns is not avoidable by an infinite word over a finite alphabet.

Notice that if $n \geq 2$, then an n -splitted word (resp. pattern) contains a 2-splitted word (resp. pattern) and a 2-splitted word (resp. pattern) is doubled.

6 Conclusion

Our results answer settles the first of two questions of our previous paper [10]. The second question is whether there exists a finite k such that every doubled pattern with at least k variables is 2-avoidable. As already noticed [10], such a k is at least 5 since, e.g., $ABCCBADD$ is not 2-avoidable.

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