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# More on square-free words obtained from prefixes by permutations

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## Abstract

An infinite square-free word  $w$  over the alphabet  $\Sigma_3 = \{0, 1, 2\}$  is said to have a  $k$ -stem  $\sigma$  if  $|\sigma| = k$  and  $w = \sigma w_1 w_2 \dots$  where for each  $i$ , there exists a permutation  $\pi_i$  of  $\Sigma_3$  which extended to a morphism gives  $w_i = \pi_i(\sigma)$ . Harju proved that there exists an infinite  $k$ -stem word for  $k = 1, 2, 3, 9$  and  $13 \leq k \leq 19$ , but not for  $4 \leq k \leq 8$  and  $10 \leq k \leq 12$ . He asked whether  $k$ -stem words exist for each  $k \geq 20$ . We give a positive answer to this question. Currie has found another construction that answers Harju's question.

## 1 Introduction

An infinite square-free word  $w$  over the alphabet  $\Sigma_3 = \{0, 1, 2\}$  is said to have a  $k$ -stem  $\sigma$  if  $|\sigma| = k$  and  $w = \sigma w_1 w_2 \dots$  where for each  $i$ , there exists a permutation  $\pi_i$  of  $\Sigma_3$  which extended to a morphism gives  $w_i = \pi_i(\sigma)$ . Harju [3] proved that there exists an infinite  $k$ -stem word for  $k = 1, 2, 3, 9$  and  $13 \leq k \leq 19$ , but not for  $4 \leq k \leq 8$  and  $10 \leq k \leq 12$  and asked whether  $k$ -stem words exist for each  $k \geq 20$ . We construct  $k$ -stem words for each  $20 \leq k \leq 10000$  in Section 3 and for every  $k \geq 23$  in Section 4. Currie [2] has found another construction that answers Harju's question.

Let  $t = 012021012102012021020121012\dots$  denote the fixed point of the morphism  $0 \mapsto 012, 1 \mapsto 02, 2 \mapsto 1$ . By definition,  $t$  contains neither 010 nor 212 as a factor. Harju [3] also asked whether  $t$  has a  $k$ -stem factorization for some  $k \geq 3$ . We give a negative answer in Section 2. This result has also been obtained by Harju and Müller [4].

## 2 $k$ -stem factorization of $t$

**Theorem 1** *No suffix of  $t$  admits a  $k$ -stem factorization for any  $k \geq 3$ .*

**Proof.** By previous results [3], we only need to consider the cases  $k = 9$  and  $k \geq 13$ .

A computer check shows that no factor  $f$  of  $t$  of length 18 is such that the suffix of length 9 of  $f$  is a permutation of the prefix of length 9 of  $f$ . This rules out the case  $k = 9$ .

A computer check shows that every factor  $f$  of  $t$  of length 12 contains a factor  $a0a$  with  $a \in \Sigma_3$ . By symmetry, it also contains a factor  $b2b$  with  $b \in \Sigma_3$ . Remember that 010 and 212 are not factors of  $t$ . A permutation of  $\Sigma_3$  mapping 0 to 1 (resp. mapping 2 to 1) cannot be applied to  $f$ , since it would produce a factor  $c1c$  with  $c \in \Sigma_3$  that cannot appear in  $t$ . There remain two possible permutations, namely the identity and the permutation swapping 0 and 2, but an infinite square free word cannot be obtained by a concatenation of only two distinct factors. This rules out the case  $k \geq 13$ .

### 3 $k$ -stem words for $20 \leq k \leq 10000$

**Theorem 2** *There exist  $k$ -stem words for every  $20 \leq k \leq 10000$ .*

**Proof.** Let  $\pi$  be the permutation (012). We say that a morphism  $h : \Sigma_3^* \rightarrow \Sigma_3^*$  is circular if  $h(1) = \pi(h(0))$  and  $h(2) = \pi(h(1))$ . For every  $20 \leq k \leq 10000$ , we found a word  $w_k$  such that  $|w_k| = c_k \times k$  and the circular morphism  $m$  defined by  $m(0) = w_k$  is square-free. We have  $c_k = 8$  for  $20 \leq k \leq 22$  and  $c_k = 1$  for  $23 \leq k \leq 10000$ . Square-freeness is checked using the result of Crochemore [1] that a uniform morphism  $h$  is square-free if and only if the  $h$ -images of square-free words of length 3 are square-free. Since we consider circular morphisms, we only need to check the images of 010 and 012.

These are our words  $w_k$  for  $20 \leq k \leq 22$ , where  $|w_k| = 8k$ .

```

w20 = 012102010210121021201020121012010201202120102120210201021012
      021201020120212012101202101210212021020121012021201210120102
      1202101210212021020102120102012021201210
w21 = 012021020102120102012021012010201210201021201210212021012021201
      021012010201210201021012021020102120102012102120121012021012102
      120102101210201210120210201202120102120210
w22 = 012021020102120210201202101201020121012010212012102120210121021201
      021012010201210120102101202102010212021020121021201210120212012102
      12010210121020102101202102012021201020120210

```

Consider now the case  $k \geq 23$ , where  $|w_k| = k$ . Let  $t' = 012021020121012 \dots$  denote the infinite suffix of  $t$  obtained from  $t$  by deleting the first 12 letters. To speed up the search of a suitable  $w_k$ , we impose that  $w_k = pr120210$  where  $p$  is the prefix of length  $k - 22$  of  $t'$  and  $r$  belongs to the set  $S$  of size 13 below, except that  $r = 2102010210121020$  for  $k = 26$ .

$S = \{0102120121020102, 0120102120121020, 0212012101201020, 1012010212012102, 1021201021012102, 1201020121020102, 1202120121020102,$

1210120212012102, 1210201202120102, 1210201210120102, 2010210121020102,  
2102120121020102, 2120102101201020}

## 4 $k$ -stem words for large $k$

**Theorem 3** *There exist  $k$ -stem words for every  $k \geq 1$  except for  $4 \leq k \leq 8$  and  $10 \leq k \leq 12$*

**Proof.** Consider the following morphism  $d$ , having two possible images for each letter: one image of length 17 and one image of length 18.

$$\begin{aligned} 0 &\mapsto \begin{cases} 01202120102120210 \\ 012021020102120210 \end{cases} \\ 1 &\mapsto \begin{cases} 12010201210201021 \\ 120102101210201021 \end{cases} \\ 2 &\mapsto \begin{cases} 20121012021012102 \\ 201210212021012102 \end{cases} \end{aligned}$$

Again, using the result of Crochemore [1],  $d$  is shown to be square-free by checking that the  $d$ -images of square-free words of length  $\max(3, \lceil \frac{18-3}{17} \rceil) = 3$  are square-free. Since the restriction of  $d$  to images of length 17 (resp. 18) is circular, we only need to check the images of 010 and 012 are square-free. For each of the factors 010 and 012, we actually have  $2^3$  images to check since each of the letters can be mapped either to its image of length 17 or 18.

If  $m$  is a square-free circular morphism, then for every  $d$ -image  $w_0$  of  $m(0)$ , the circular morphism defined by  $0 \mapsto w_0$  is square-free. This means that given a  $k$ -uniform square-free circular morphism, we can construct a  $k'$ -uniform square-free circular morphism for every  $k'$  such that  $17k \leq k' \leq 18k$ .

Now we prove that there exist  $k$ -uniform square-free circular morphisms for every  $k \geq 23$ . We start with the cases  $k \in [23, 10000]$  which are proved in the previous section. They imply the cases  $k \in \bigcup_{23 \leq p \leq 10000} [17p, 18p]$ , i.e.,  $k \in [391, 180000]$ . We then obtain every  $k \geq 23$  by induction.

## 5 Concluding remarks

We have proved that there exist infinite square-free ternary words with a  $k$ -stem factorization for every  $k$  except  $4 \leq k \leq 8$  and  $10 \leq k \leq 12$ . We conjecture that there exist  $k$ -stem words of the form  $w_k = pr120210$  described in the proof of Theorem 2 for every  $k \geq 23$ , rather than  $23 \leq k \leq 10000$ . Before we found the morphism of the proof of Theorem 3, we pushed the verification to up to 10000 in order to find a way to prove this conjecture, but the proof of Theorem 3 only requires a verification for  $23 \leq k \leq 390$ .

From the proof of Theorem 3, we see that the number of  $k$ -uniform square-free circular morphisms is exponential in  $k$ , at least about  $\binom{2k/35}{k/35} \approx 2^{2k/35}$ . We conjecture the following:

**Conjecture 4** *The growth rate of ternary words defining a square-free circular morphism exists and is equal to the growth rate  $1.3017\dots$  of ternary square-free words.*

See Shur [5] for more information on the growth rate of ternary square-free words.

## References

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