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▶ To cite this version:

Pascal Ochem, Matthieu Rosenfeld. Avoidability of Formulas with Two Variables. 20th International Conference on Developments in Language Theory (DLT 2016), Laboratoire de combinatoire et d'informatique mathématique (LaCIM), Université du Québec à Montréal, Jul 2016, Montréal, Canada. pp.344-354, 10.1007/978-3-662-53132-7_28. lirmm-01375829

HAL Id: lirmm-01375829 https://hal-lirmm.ccsd.cnrs.fr/lirmm-01375829v1

Submitted on 3 Oct 2016

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Avoidability of Formulas with two Variables

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Abstract. In combinatorics on words, a word w over an alphabet Σ is said to avoid a pattern p over an alphabet Δ of variables if there is no factor f of w such that f = h(p) where $h : \Delta^* \to \Sigma^*$ is a non-erasing morphism. A pattern p is said to be k-avoidable if there exists an infinite word over a k-letter alphabet that avoids p. We consider the patterns such that at most two variables appear at least twice, or equivalently, the formulas with at most two variables. For each such formula, we determine whether it is 2-avoidable.

Keywords: Word, Pattern avoidance.

1 Introduction

A pattern p is a non-empty finite word over an alphabet $\Delta = \{A, B, C, \ldots\}$ of capital letters called variables. An occurrence of p in a word w is a nonerasing morphism $h: \Delta^* \to \Sigma^*$ such that h(p) is a factor of w. The avoidability index $\lambda(p)$ of a pattern p is the size of the smallest alphabet Σ such that there exists an infinite word over Σ containing no occurrence of p. Bean, Ehrenfeucht, and McNulty [3] and Zimin [11] characterized unavoidable patterns, i.e., such that $\lambda(p) = \infty$. We say that a pattern p is t-avoidable if $\lambda(p) \leq t$. For more informations on pattern avoidability, we refer to Chapter 3 of Lothaire's book [6].

A variable that appears only once in a pattern is said to be *isolated*. Following Cassaigne [4], we associate to a pattern p the *formula* f obtained by replacing every isolated variable in p by a dot. The factors between the dots are called *fragments*.

An occurrence of f in a word w is a non-erasing morphism $h: \Delta^* \to \Sigma^*$ such that the *h*-image of every fragment of f is a factor of w. As for patterns, the avoidability index $\lambda(f)$ of a formula f is the size of the smallest alphabet allowing an infinite word containing no occurrence of p. Clearly, every word avoiding f also avoids p, so $\lambda(p) \leq \lambda(f)$. Recall that an infinite word is *recurrent* if every finite factor appears infinitely many times. If there exists an infinite word over Σ avoiding p, then there there exists an infinite recurrent word over Σ avoiding p. This recurrent word also avoids f, so that $\lambda(p) = \lambda(f)$. Without

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loss of generality, a formula is such that no variable is isolated and no fragment is a factor of another fragment.

Cassaigne [4] began and Ochem [7] finished the determination of the avoidability index of every pattern with at most 3 variables. A *doubled* pattern contains every variable at least twice. Thus, a doubled pattern is a formula with exactly one fragment. Every doubled pattern is 3-avoidable [9]. A formula is said to be *binary* if it has at most 2 variables. In this paper, we determine the avoidability index of every binary formula.

We say that a formula f is *divisible* by a formula f' if f does not avoid f', that is, there is a non-erasing morphism such that the image of any fragment of f' by h is a factor of a fragment of f. If f is divisible by f', then every word avoiding f' also avoids f and thus $\lambda(f) \leq \lambda(f')$. For example, the fact that ABA.AABB is 2-avoidable implies that ABAABB and ABAB.BBAA are 2avoidable. Moreover, the reverse f^R of a formula f satisfies $\lambda(f^R) = \lambda(f)$. See Cassaigne [4] and Clark [5] for more information on formulas and divisibility.

First, we check that every avoidable binary formula is 3-avoidable. Since $\lambda(AA) = 3$, every formula containing a square is 3-avoidable. Then, the only square free avoidable binary formula is ABA.BAB with avoidability index 3 [4]. Thus, we have to distinguish between avoidable binary formulas with avoidability index 2 and 3. A binary formula is minimally 2-avoidable if it is 2-avoidable and is not divisible by any other 2-avoidable binary formula. A binary formula f is maximally 2-unavoidable if it is 2-unavoidable if it is 2-unavoidable if and every other binary formula that is divisible by f is 2-avoidable.

Theorem 1. Up to symmetry, the maximally 2-unavoidable binary formulas are:

- AAB.ABA.ABB.BBA.BAB.BAA
- -AAB.ABBA
- AAB.BBAB
- AAB.BBAA
- -AAB.BABB
- AAB.BABAA
- ABA.ABBA
- AABA.BAAB

Up to symmetry, the minimally 2-avoidable binary formulas are:

- -AA.ABA.ABBA
- -ABA.AABB
- AABA.ABB.BBA
- AA.ABA.BABB
- AA.ABB.BBAB
- AA.ABAB.BB
- AA.ABBA.BAB
- AAB.ABB.BBAA
- AAB.ABBA.BAA
- AABB.ABBA
- ABAB.BABA

- AABA.BABA - AAA
- ABA.BAAB.BAB
- AABA.ABAA.BAB
- AABA.ABAA.BAAB
- -ABAAB

To obtain the 2-unavoidability of the formulas in the first part of Theorem 1, we use a standard backtracking algorithm. In the rest of the paper, we consider the 2-avoidable formulas in the second part of Theorem 1. Fig. 1 gives the maximal length and number of binary words avoiding each maximally 2-unavoidable formula.

We show in Section 3 that the first three of these formulas are avoided by polynomially many binary words only. The proof uses a technical lemma given in Section 2. Then we show in Section 4 that the other formulas are avoided by exponentially many binary words.

Fig. 1: The number and maximal length of binary words avoiding the maximally 2-unavoidable formulas.

	Maximal length of a	Number of binary
Formula	binary word avoiding	words avoiding
	this formula	this formula
AAB.BBAA	22	1428
AAB.ABA.ABB.BBA.BAB.BAA	23	810
AAB.BBAB	23	1662
AABA.BAAB	26	2124
AAB.ABBA	30	1684
AAB.BABAA	42	71002
AAB.BABB	69	9252
ABA.ABBA	90	31572

2 The Useful Lemma

Let us define the following words:

- $-b_2$ is the fixed point of $0 \mapsto 01$, $1 \mapsto 10$.
- $-b_3$ is the fixed point of $0 \mapsto 012$, $1 \mapsto 02$, $2 \mapsto 1$.
- b_4 is the fixed point of $0 \mapsto 01$, $1 \mapsto 03$, $2 \mapsto 21$, $3 \mapsto 23$.
- b_5 is the fixed point of $0 \mapsto 01$, $1 \mapsto 23$, $2 \mapsto 4$, $3 \mapsto 21$, $4 \mapsto 0$.

Let w and w' be infinite (right infinite or bi-infinite) words. We say that w and w' are equivalent if they have the same set of finite factors. We write $w \sim w'$ if w and w' are equivalent. A famous result of Thue [10] can be stated as follows:

Theorem 2. [10] Every bi-infinite ternary word avoiding 010, 212, and squares is equivalent to b_3 .

Given an alphabet Σ and forbidden structures S, we say that a finite set W of infinite words over Σ essentially avoids S if every word in W avoids S and every bi-infinite words over Σ avoiding S is equivalent to one of the words in S. If W contains only one word w, we denote the set W by w instead of $\{w\}$. Then we can restate Theorem 2: b_3 essentially avoids 010, 212, and squares

The results in the next section involve b_3 . We have tried without success to prove them by using Theorem 2. We need the following stronger property of b_3 :

Lemma 3. b_3 essentially avoids 010, 212, XX with $1 \leq |X| \leq 3$, and 2YY with $|Y| \geq 4$.

Proof. We start by checking by computer that b_3 has the same set of factors of length 100 as every bi-infinite ternary word avoiding 010, 212, XX with $1 \leq |X| \leq 3$, and 2YY with $|Y| \geq 4$. The set of the forbidden factors of b_3 of length at most 4 is $F = \{00, 11, 22, 010, 212, 0202, 2020, 1021, 1201\}$. To finish the proof, we use Theorem 2 and we suppose for contradiction that w is a bi-infinite ternary word that contains a large square MM and avoids both F and large factors of the form 2YY.

- Case M = 0N. Then w contains MM = 0N0N. Since $00 \in F$ and 2YY is forbidden, w contains 10N0N. Since $\{11,010\} \subset F$, w contains 210N0N. If N = P1, then w contains 210P10P1, which contains 2YY with Y =10P. So N = P2 and w contains 210P20P2. If P = Q1, then w contains 210Q120Q12. Since $\{11,212\} \subset F$, the factor Q12 implies that Q = R0and w contains 210R0120R012. Moreover, since $\{00, 1201\} \subset F$, the factor 120R implies that R = 2S and w contains 2102S01202S012. Then there is no possible prefix letter for S: 0 gives 2020, 1 gives 1021, and 2 gives 22. This rules out the case P = Q1. So P = Q0 and w contains 210Q020Q02. The factor Q020Q implies that Q = 1R1, so that w contains 2101R10201R102. Since $\{11,010\} \subset F$, the factor 01R implies that R = 2S, so that w contains 21012S102012S102. The only possible right extension with respect to F of 102 is 102012. So w contains 21012S102012S102012, which contains 2YYwith Y = S102012.
- Case M = 1N. Then w contains MM = 1N1N. In order to avoid 11 and 2YY, w must contain 01N1N. If N = P0, then w contains 01P01P0. So w contains the large square 01P01P and this case is covered by the previous item. So N = P2 and w contains 01P21P2. Then there is no possible prefix letter for P: 0 gives 010, 1 gives 11, and 2 gives 212.
- Case M = 2N. Then w contains MM = 2N2N. If N = P1, then w contains 2P12P1. This factor cannot extend to 2P12P12, since this is 2YY with Y = P12. So w contains 2P12P10. Then there is no possible suffix letter for P: 0 gives 010, 1 gives 11, and 2 gives 212. This rules out the case N = P1. So N = P0 and w contains 2P02P0. This factor cannot extend to 02P02P0, since this contains the large square 02P02P and this case is

covered by the first item. Thus w contains 12P02P0. If P = Q1, then w contains 12Q102Q10. Since $\{22, 1021\} \subset F$, the factor 102Q implies that Q = 0R, so that w contains 120R1020R10. Then there is no possible prefix letter for R: 0 gives 00, 1 gives 1201, and 2 gives 0202. This rules out the case P = Q1. So P = Q2 and w contains 12Q202Q20. The factor Q202 implies that Q = R1 and w contains 12R1202R120. Since $\{00, 1201\} \subset F$, w contains 12R1202R1202, which contains 2YY with Y = R1202.

3 Formulas Avoided by Few Binary Words

The first three 2-avoidable formulas in Theorem 1 are not avoided by exponentially many binary words:

- $\{g_x(b_3), g_y(b_3), g_z(b_3), g_{\overline{z}}(b_3)\}$ essentially avoids AA.ABA.ABBA.
- $\{g_x(b_3), g_t(b_3)\}$ essentially avoids ABA.AABB.
- $-g_x(b_3)$ essentially avoids AABA.ABB.BBA.

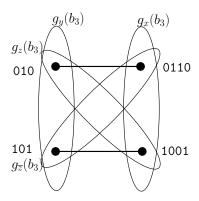
The words avoiding these formulas are morphic images of b_3 by the morphisms given below. Let \overline{w} denote the word obtained from the (finite or bi-infinite) binary word w by exchanging 0 and 1. Obviously, if w avoids a given formula, then so does \overline{w} . A (bi-infinite) binary word w is *self-complementary* if $w \sim \overline{w}$. The words $g_x(b_3)$, $g_y(b_3)$, and $g_t(b_3)$ are self-complementary. Since the frequency of 0 in $g_z(b_3)$ is $\frac{5}{9}$, $g_z(b_3)$ is not self-complementary. Then $g_{\overline{z}}$ is obtained from g_z by exchanging 0 and 1, so that $g_{\overline{z}}(b_3) = \overline{g_z(b_3)}$.

$g_x(0) = 01110,$	$g_y(0) = 0111,$	$g_z(0) = 0001,$	$g_t(0) = 01011011010,$
$g_x(1) = 0110,$	$g_y(1) = 01,$	$g_z(1) = 001,$	$g_t(1) = 01011010,$
$g_x(2) = 0.$	$g_{y}(2) = 00.$	$g_z(2) = 11.$	$g_t(2) = 010.$

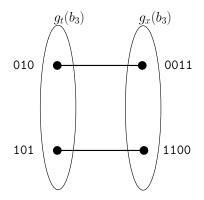
To prove the avoidability, we have implemented Cassaigne's algorithm that decides, under mild assumptions, whether a morphic word avoids a formula [4]. For the first two formulas, we have to explain how the long enough binary words split into 4 or 2 distinct incompatible types. A similar phenomenon has been described for AABB.ABBA [8].

First, consider any infinite binary word w avoiding AA.ABA.ABBA. A computer check shows by backtracking that w must contain the factor 01110001110. In particular, w contains 00. Thus, w cannot contain both 010 and 0110, since it would produce an occurrence of AA.ABA.ABBA. Moreover, a computer check shows by backtracking that w cannot avoid both 010 and 0110. So, w must contain either 010 or 0110 (this is an exclusive or). Similarly, w must contain either 101 or 1001. There are thus at most 4 possibilities for w, depending on which subset of {010,0110,101,1001} appears among the factors of w, see Figure 2a.

Now, consider any infinite binary word w avoiding ABA.AABB. Notice that w cannot contain both 010 and 0011. Also, a computer check shows by back-tracking that w cannot avoid both 010 and 1100. By symmetry, there are thus at most 2 possibilities for w, depending on which subset of {010,0011,101,1100} appears among the factors of w, see Figure 2b.



(a) The four bi-infinite binary words avoiding *AA.ABA.ABBA*.



(b) The two bi-infinite binary words avoiding *ABA*.*AABB*.



Let us first prove that $g_y(b_3)$ essentially avoids AA.ABA.ABBA, 0110, and 1001. We check that the set of prolongable binary words of length 100 avoiding AA.ABA.ABBA, 0110, and 1001 is exactly the set of factors of length 100 of $g_y(b_3)$. Using Cassaigne's notion of circular morphism [4], this is sufficient to prove that every bi-infinite binary word of this type is the g_y -image of some biinfinite ternary word w_3 . It also ensures that w_3 and b_3 have the same set of small factors. Suppose for contradiction that $w_3 \neq b_3$. By Lemma 3, w_3 contains 2YY. Then w_3 contains 2YYa with $a \in \Sigma_3$. Notice that 0 is a prefix of the g_y -image of every letter. So $g_y(w_3)$ contains $g_y(2YYa) = 000U0U0V$ with $U, V \in \Sigma_3^+$, which contains an occurrence of AA.ABA.ABBA with A = 0 and B = 0U. This shows that $w_3 \sim b_3$, and thus $g_y(w_3) \sim g_y(b_3)$. Thus $g_y(b_3)$ essentially avoids AA.ABA.ABBA, 0110, and 1001. The argument is similar for the other types and we only detail the final contradiction:

- Since 1 is a suffix of the g_z -image of every letter, $g_z(2YY) = 11U1U1$ contains an occurrence of AA.ABA.ABBA with A = 1 and B = 1U.
- Since 010 is a prefix and a suffix of the g_t -image of every letter, $g_t(u2YY) = V010010010U010010U010$ contains an occurrence of ABA.AABB with A = 010 and B = 010U010.
- Since 0 is a prefix and a suffix of the g_x -image of every letter, $g_x(u2YYa) = V000U00U00W$ contains an occurrence of AABA.AABBA with A = 0 and B = 0U0. Therefore, $g_x(u2YYa)$ contains an occurrence of AA.ABBA.ABBA, ABA.AABB, and AABA.ABB.BBA.

4 Formulas Avoided by Exponentially Many Binary Words

The other 2-avoidable formulas in Theorem 1 are avoided by exponentially many binary words. For every such formula f, we give below a uniform morphism g that

maps every ternary square free word to a binary word avoiding f. If possible, we simultaneously avoid the reverse formula f^R of f. We also avoid large squares. Let SQ_t denote the pattern corresponding to squares of period at least t, that is, $SQ_1 = AA$, $SQ_2 = ABAB$, $SQ_3 = ABCABC$, and so on. The morphism g produces words avoiding SQ_t with t as small as possible.

- AA.ABA.BABB is avoided with its reverse by the following 22-uniform morphism which also avoids SQ_6 :

Notice that $\{AA.ABA.BABB, AA.ABA.BBAB, SQ_5\}$ is 2-unavoidable. However, $\{AA.ABA.BABB, SQ_4\}$ is 2-avoidable:

- AA.ABB.BBAB is avoided with its reverse, 60-uniform morphism, avoids SQ_{11} :

Notice that $\{AA.ABB.BBAB, SQ_{10}\}$ is 2-unavoidable. - AA.ABAB.BB is self-reverse, 11-uniform morphism, avoids SQ_4 :

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\begin{array}{l} 0 \mapsto 00100110111 \\ 1 \mapsto 00100110001 \\ 2 \mapsto 00100011011 \end{array}
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-AA.ABBA.BAB is self-reverse, 30-uniform morphism, avoids SQ_6 :

-AAB.ABB.BBAA is self-reverse, 30-uniform morphism, avoids SQ_5 :

 $\begin{array}{c} 0 \mapsto 000100101110100010110111011101\\ 1 \mapsto 000100101101110100010111011101\\ 2 \mapsto 0001000100010111011101100001 \end{array}$

-AAB.ABBA.BAA is self-reverse, 38-uniform morphism, avoids SQ_5 :

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- AABB.ABBA is unavoidable with its reverse, 193-uniform morphism, avoids $SQ_{16}\colon$

Previous papers [7,8] have considered a 102-uniform morphism to avoid AABB.ABBA and SQ_{27} . No infinite binary word avoids AABB.ABBA and SQ_{15} .

- ABAB.BABA is self-reverse, 50-uniform morphism, avoids SQ_3 , see [7]:

Notice that a binary word avoiding ABAB.BABA and SQ_3 contains only the squares 00, 11, and 0101 (or 00, 11, and 1010).

- AABA.BABA: A case analysis of the small factors shows that a recurrent binary word avoids AABA.BABA, ABAA.ABAB, and SQ_3 if and only if it contains only the squares 00, 11, and 0101 (or 00, 11, and 1010). We thus obtain the same morphism as for ABAB.BABA.
- -AAA is self-reverse, 32-uniform morphism, avoids SQ_4 :

 $\begin{array}{l} 0\mapsto 0010100110110100101100100110101\\ 1\mapsto 0010100110110010110100100110101\\ 2\mapsto 00100101101001001101101001011011\end{array}$

-ABA.BAAB.BAB is self-reverse, 10-uniform morphism, avoids SQ_3 :

 $\begin{array}{l} 0 \mapsto 0001110101 \\ 1 \mapsto 0001011101 \\ 2 \mapsto 0001010111 \end{array}$

-AABA.ABAA.BAB is self-reverse, 57-uniform morphism, avoids SQ_6 :

-AABA.ABAA.BAAB is self-reverse, 30-uniform morphism, avoids SQ_3 :

 $\begin{array}{l} 0 \mapsto 00010111000111010000101011101 \\ 1 \mapsto 00010111000111010001010101110101 \\ 2 \mapsto 00010111000101011101000011101 \end{array}$

- ABAAB is avoided with its reverse, 10-uniform morphism, avoids SQ_3 , see [7]:

$0\mapsto0001110101$
$1\mapsto0000111101$
$2\mapsto 0000101111$

For every q-uniform morphism g above, we say that a binary word is an sqf-g-image if it is the g-image of a ternary square free word. Let us show that for every minimally 2-avoidable formula f and corresponding morphism g, every sqf-g-image avoids f.

We start by checking that every morphism is synchronizing, that is, for every letters $a, b, c \in \Sigma_3$, the factor g(a) only appears as a prefix or a suffix in g(bc).

For every morphism g, the sqf-g-images are claimed to avoid SQ_t with 2t < q. Let us prove that SQ_t is avoided. We first check exhaustively that the sqf-gimages contain no square uu such that $t \leq |u| < 2q - 1$. Now suppose for contradiction that an sqf-g-image contains a square uu with $|u| \ge 2q - 1$. The condition $|u| \ge 2q - 1$ implies that u contains a factor g(a) with $a \in \Sigma_3$. This factor q(a) only appears as the *g*-image of the letter *a* because *g* is synchronizing. Thus the distance between any two factors u in an sqf-q-image is a multiple of q. Since uu is a factor of an sqf-g-image, we have $q \mid |u|$. Also, the center of the square uu cannot lie between the g-images of two consecutive letters, since otherwise there would be a square in the pre-image. The only remaining possibility is that the ternary square free word contains a factor aXbXc with $a, b, c \in \Sigma_3$ and $X \in \Sigma_3^+$ such that g(aXbXc) = bsYpsYpe contains the square uu = sYpsYp, where g(X) = Y, g(a) = bs, g(b) = ps, g(c) = pe. Then, we also have $a \neq b$ and $b \neq c$ since aXbXc is square free. Then *abc* is square free and g(abc) = bspspe contains a square with period |s| + |p| = |g(a)| = q. This is a contradiction since the sqf -g-images contain no square with period q.

Notice that f is not square free, since the only avoidable square free binary formula is ABA.BAB, which is not 2-avoidable. Now, we distinguish two kinds of formula. A formula is *easy* if every appearing variable is contained in at least one square. Every potential occurrence of an easy formula then satisfies |A| < tand |B| < t since SQ_t is avoided. The longest fragment of every easy formula has length 4. So, to check that the sqf-g-images avoids an easy formula, it is sufficient to consider the set of factors of the sqf-g-images with length at most 4(t-1).

A tough formula is such that one of the variables is not contained in any square. The tough formulas have been named so that this variable is B. The tough formulas are ABA.BAAB.BAB, ABAAB, AABA.ABAA.BAAAB, and AABA.ABAA.BAAB. As before, every potential occurrence of a tough formula satisfies |A| < t since SQ_t is avoided. Suppose for contradiction that $|B| \ge 2q-1$. By previous discussion, the distance between any two occurrences of B in an sqf-g-image is a multiple of q. The case of ABA.BAAB.BAB can be settled as follows. The factor BAAB implies that $q \mid |BAA|$ and the factor BAB implies that $q \mid |BAA|$ and the factor BAB implies that $q \mid |A|$, which contradicts |A| < t. For the other formulas, only one fragment contains B twice. This fragment is said to

be *important*. Since |A| < t, the important fragment is a repetition which is "almost" a square. The important fragment is **BAB** for AABA.ABAA.BAAB, **BAAB** for AABA.ABAA.BAAB, and **ABAAB** for AABA.ABAA.BAAB, and **ABAAB** for ABAAB. Informally, this almost square implies a factor aXbXc in the ternary pre-image, such that |a| = |c| = 1 and $1 \leq |b| \leq 2$. If |X| is small, then |B| is small and we check exhaustively that there exists no small occurrence of f. If |X| is large, there would exist a ternary square free factor aYbYc with |Y| small, such that g(aYbYc) contains the important fragment of an occurrence of f.

5 Concluding Remarks

From our results, every minimally 2-avoidable binary formula, and thus every 2-avoidable binary formula, is avoided by some morphic image of b_3 .

What can we forbid so that there exists only few infinite avoiding words? The known examples from the literature [1,2,10] are:

- one pattern and two factors:
 - b_3 essentially avoids AA, 010, and 212.
 - A morphic image of b_5 essentially avoids AA, 010, and 020.
 - A morphic image of b_5 essentially avoids AA, 121, and 212.
 - b_2 essentially avoids ABABA, 000, and 111.
- two patterns: b_2 essentially avoids ABABA and AAA.
- one formula over three variables: b_4 and two words from b_4 obtained by letter permutation essentially avoid AB.AC.BA.BC.CA.

Now we can extend this list:

- one formula over two variables:
 - $g_x(b_3)$ essentially avoids AAB.BAA.BBAB.
 - $\{g_x(b_3), g_t(b_3)\}$ essentially avoids *ABA*.*AABB*.
 - $\{g_x(b_3), g_y(b_3), g_z(b_3), g_{\overline{z}}(b_3)\}$ essentially avoids AA.ABA.ABBA.
- one pattern over three variables: ABACAABB (same as ABA.AABB).

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