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# Aperiodic tilings and entropy

Bruno DURAND, Guilhem GAMARD, Anaël GRANDJEAN

August 27, 2014

- 1 Introduction
- 2 J. Kari & K. Culik's tileset
- 3 Aperiodicity
- 4 Positive entropy
- 5 Conclusion

# Wang tiles

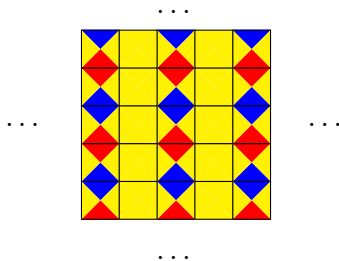
## Wang tiles

- Finite set of colors
- Alphabet = colored squares
- Adjacent borders have matching colors

$$c = \{\text{red, yellow, blue}\}$$

$$\Sigma = \left\{ \begin{array}{c} \text{blue} \\ \text{red} \\ \text{yellow} \end{array}, \begin{array}{c} \text{red} \\ \text{blue} \\ \text{yellow} \end{array}, \begin{array}{c} \text{yellow} \\ \text{yellow} \\ \text{yellow} \end{array} \right\}$$

$$\Sigma' = \left\{ \begin{array}{c} \text{blue} \\ \text{red} \\ \text{yellow} \end{array} \right\}$$



# Aperiodic tilesets

## Definition

A set of tiles is **aperiodic** when:

- it can cover the plane;
- it cannot cover the plane periodically.

Cover such that adjacent borders have matching colors

# The history of small aperiodic tilesets

## Self-similar

1964	R. Berger	> 20,000 tiles
1966	D. Knuth	96 tiles
1971	R. Robinson	52 tiles
1974	R. Penrose	32 tiles
1986	R. Ammann, B. Grünbaum, G. Shephard	16 tiles

## **Not** self-similar

1996	J. Kari	14 tiles
1996	K. Culik	13 tiles

# Our result

## Theorem

*The Kari-Culik tileset has positive entropy.*

Intuitively:

- Description of a  $n \times n$ -square takes  $\Omega(n^2)$  bits
- It contains dense “random” bits

Note: entropy zero was conjectured.

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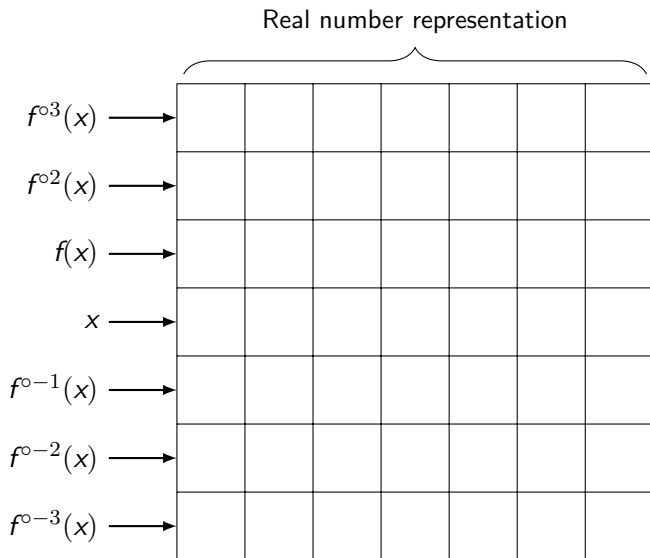
## A function with aperiodic orbits

Consider this function:

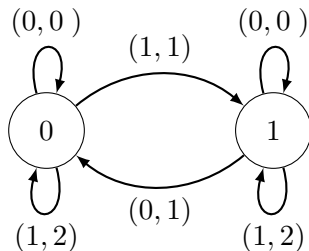
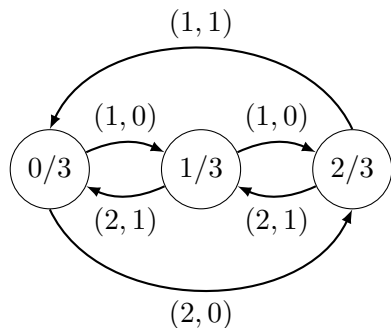
$$f: \left[\frac{1}{3}; 2\right] \rightarrow \left[\frac{1}{3}; 2\right]$$
$$x \mapsto \begin{cases} 2x & \text{if } x \leq 1 \\ x/3 & \text{if } x \geq 1 \end{cases}$$

- Its orbits, i.e. sequences  $u_x = (f^{on}(x))_{n \in \mathbb{N}}$ , are aperiodic
- Its orbits are also dense in  $\left[\frac{1}{3}; 2\right]$

# The general idea

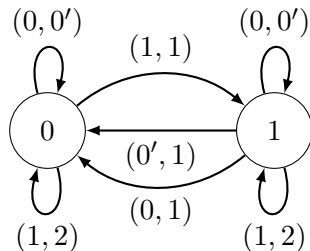
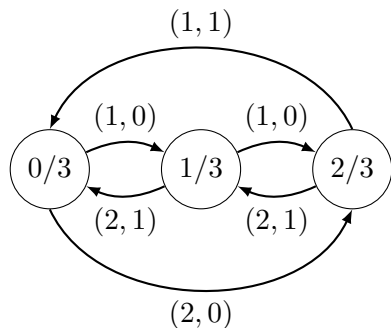


## Multiplications done by transducers



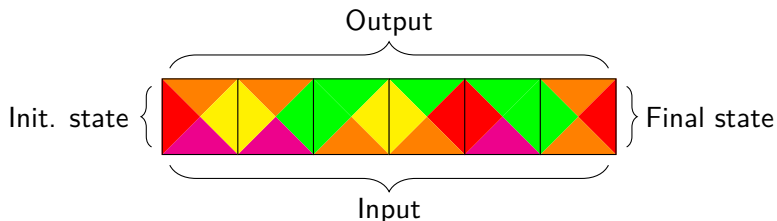
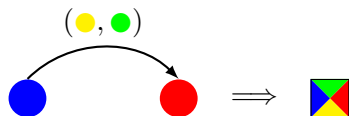
- $(2, 0)$  stands for “read 2, write 0”

## Multiplications done by transducers



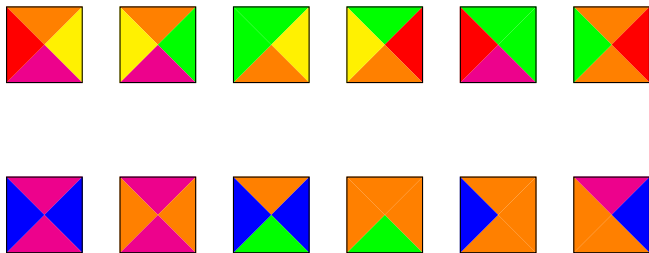
- $(2, 0)$  stands for “read 2, write 0”

## From transducers to tile sets

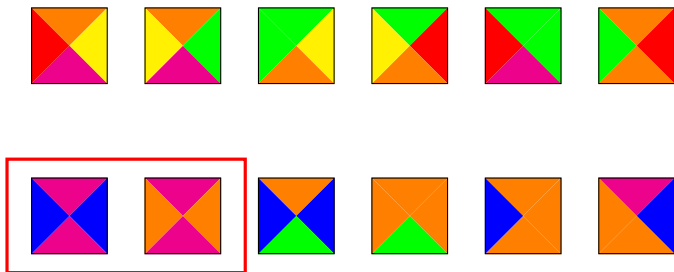


- States of  $M_{1/3}$ ,  $M_2$ : disjoint colors
- One line = one run

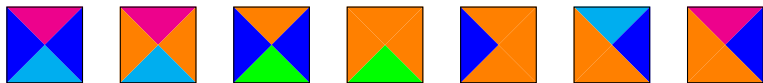
# An aperiodic set of tiles



# An aperiodic set of tiles



# An aperiodic set of tiles





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## Lines and averages

### Definition

The *average* of a sequence  $(u_n)$  is:

$$\text{avg}(u) = \lim_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{k=-n}^n u_k$$

Tilings: average of a line = average of northern sides

# Aperiodicity

Theorem (J. Kari and K. Culik, 1996)

*The Kari-Culik tileset is aperiodic.*

Sketch of proof.

- Suppose there is a periodic tiling. Then each line has an average. The averages are periodic: contradiction.
- To tile the plane, start from  $\dots 11111111 \dots$  and run the transducers forever.



## Encoding real numbers in bi-infinite sequences

$$S_x(n) = \lfloor (n+1)x \rfloor - \lfloor nx \rfloor$$

$$S_{1/2} = \dots 01 01 01 01 01 01 01 01 \dots$$

$$S_{7/5} = \dots 2112111 2112111 2112111 \dots$$

$$S_{\pi/3} = \dots 21111111111111111111111121111111111111111111111112 \dots$$

- $S_x$  is on alphabet  $\{\lfloor x \rfloor, \lceil x \rceil\}$
- The average of values of  $S_x$  is  $x$

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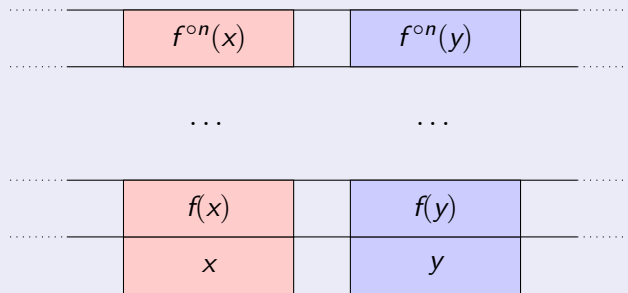
# Each line has an average

## Lemma

*In any tiling, each line has an average.*

## Sketch of proof.

Consider a line without an average.



Density  $\implies \exists n$  s.t.  $f^{on}(x) < 1 < f^{on}(y)$



# Entropy

$C(n)$  = the number of  $n \times n$ -squares found in any tiling

## Definition

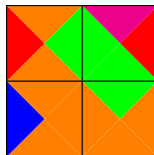
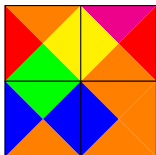
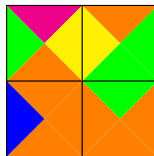
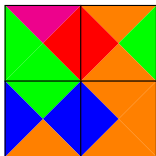
We call the **entropy** the following quantity:

$$E = \lim_{n \rightarrow \infty} \frac{\log C(n)}{n^2}$$

- Classical definition in dynamical systems
- With 13 tiles, if  $C(n) \sim 13^{\epsilon n^2}$ , then  $E = \epsilon$

## Substitutive pairs

are pairs of distinct patterns with the same borders.

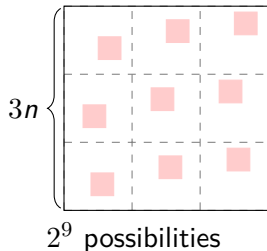
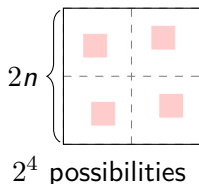
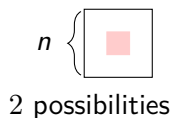





# Substitutive pairs generate entropy

## Lemma

*If a substitutive square is found in any  $n \times n$ -square of any tiling, then the entropy of the tiles is positive.*



 = substitutive square

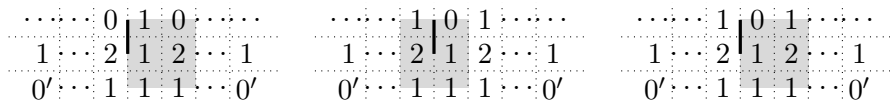
# Substitutive pairs appear often (1/2)

## Lemma

*Whenever a pattern  $0111^\alpha 0$  occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

## Sketch of proof.

Case analysis. □



Middle case

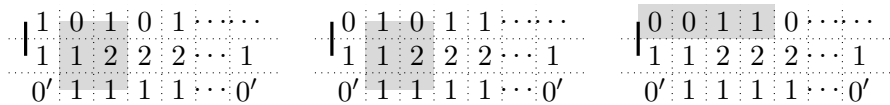
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Leftmost case

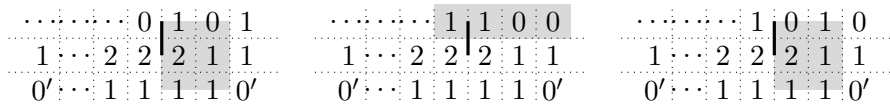
# Substitutive pairs appear often (1/2)

## Lemma

*Whenever a pattern  $0111^\alpha 0$  occurs on a line of tiles, there is a substitutive square intersecting this pattern.*

## Sketch of proof.

Case analysis. □



Rightmost case

## Substitutive pairs appear often (2/2)

### Lemma

*In any line with an average  $\in ]\frac{3}{4}; \frac{9}{10}[$ , a pattern of the form  $0111^\alpha 0$  appears regularly.*

### Sketch of proof.

If there are no “0” regularly, then the average is 1.

If there are no “111” regularly, then the average is  $\leq \frac{3}{4}$ . □

- All orbits regularly meet the interval  $] \frac{3}{4}; \frac{9}{10}[$
- Hence substitutive squares appear often enough

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# Many thanks for your attention!

## Our result

- The entropy of the Kari-Culik tileset is positive
- The Kari-Culik-tilings are not all self-similar

## Open problems

- Characterize the language of words which can appear on K.C.'s lines?
- Is there a tileset working the same way, but with 0 entropy?
- Is there a sub-shift of finite type  $A$ , with positive entropy, such that any subshift of finite type  $\subset A$  also has positive entropy?