Filling the Complexity Gaps for Colouring Planar and Bounded Degree Graphs

Konrad K. Dabrowski, François Dross, Matthew Johnson, Daniël Paulusma

Filling the Complexity Gaps for Colouring Planar and Bounded Degree Graphs[*]

Konrad K. Dabrowski[1], François Dross[2], Matthew Johnson[1] and
Daniël Paulusma[1]

[1] Department of Computer Science, Durham University,
Lower Mountjoy, South Road, Durham DH1 3LE, United Kingdom
{konrad.dabrowski,matthew.johnson2,daniel.paulusma}@durham.ac.uk
[2] Université Côte d'Azur, I3S, CNRS, Inria, France francois.dross@inria.fr

Abstract. A colouring of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ for every $uv \in E$. A k-regular list assignment of G is a function L with domain V such that for every $u \in V$, $L(u)$ is a subset of $\{1, 2, \ldots\}$ of size k. A colouring c of G respects a k-regular list assignment L of G if $c(u) \in L(u)$ for every $u \in V$. A graph G is k-choosable if for every k-regular list assignment L of G, there exists a colouring of G that respects L. We may also ask if for a given k-regular list assignment L of a given graph G, there exists a colouring of G that respects L. This yields the k-REGULAR LIST COLOURING problem. For $k \in \{3, 4\}$ we determine a family of classes $\mathcal{G}$ of planar graphs, such that either k-REGULAR LIST COLOURING is NP-complete for instances (G, L) with $G \in \mathcal{G}$, or every $G \in \mathcal{G}$ is k-choosable. By using known examples of non-3-choosable and non-4-choosable graphs, this enables us to classify the complexity of k-REGULAR LIST COLOURING restricted to planar graphs, planar bipartite graphs, planar triangle-free graphs and to planar graphs with no 4-cycles and no 5-cycles. We also classify the complexity of k-REGULAR LIST COLOURING and a number of related colouring problems for graphs with bounded maximum degree.

Keywords: list colouring, choosability, planar graphs, maximum degree

1 Introduction

A *colouring* of a graph is a labelling of its vertices such that adjacent vertices do not have the same label. We call these labels *colours*. Graph colouring problems are central to the study of combinatorial algorithms and they have many theoretical and practical applications. A typical problem asks whether a colouring exists under certain constraints, or how difficult it is to find such a colouring. For example, in the LIST COLOURING problem, a graph is given where each vertex

has a list of colours and one wants to know if the vertices can be coloured using only colours in their lists. The Choosability problem asks whether such list colourings are guaranteed to exist whenever all the lists have a certain size k (if so, then the graph is said to be k-choosable). In fact, an enormous variety of colouring problems can be defined and there is now a vast collection of literature on this subject. For longer introductions to the type of problems we consider we refer to two recent surveys [7, 18].

In this paper, we are concerned with the computational complexity of colouring problems (we give formal definitions of these problems in Section 2). For many such problems, the complexity is well understood in the case where we allow every graph as input, so it is natural to consider problems with restricted inputs in order to increase our understanding of their hardness. Our two main objectives are related to this aim. They are:

1. to exploit structural results in order to obtain complexity results;
2. to fill a number of complexity gaps in order to obtain complete complexity classifications.

The graph classes we consider are the classes of planar graphs and of graphs with bounded maximum degree. As such our paper consists of two main parts.

In the first (Section 3), we study planar graphs and consider a natural variant of the List Colouring problem, closely related to the Choosability problem. As we will discuss later in more detail, there exist many results for the latter problem restricted to planar graphs (see for example [1, 5, 11, 20, 22, 23, 25–28, 32–34, 36–38]). Most of these results are structural. They either show that every graph of some subclass $\mathcal{G}$ of planar graphs is k-choosable for some small value of k, or construct a concrete example of a non-k-choosable graph that belongs to $\mathcal{G}$. When not every graph in $\mathcal{G}$ is k-choosable, it is natural to ask if the vertices of a given graph from $\mathcal{G}$ can be coloured in polynomial time using only colours from their lists for some given list assignment L, which assigns a list of size k to each vertex. For $k \in \{3, 4\}$, we prove two general theorems, which give us a family of subclasses $\mathcal{G}$ of planar graphs, such that either this problem is NP-complete for instances (G, L) with $G \in \mathcal{G}$ or every graph in $\mathcal{G}$ is k-choosable. We will then combine known structural choosability results with these theorems to obtain NP-hardness results for this problem on planar graphs and on a number of subclasses of planar graphs. This enables us to fill some complexity gaps in order to obtain a complete classification of the computational complexity of the problem for these graph classes.

Some of the known results we use in the first part of our paper are for planar graphs with bounded degree. In the second part of the paper (Section 4), we combine these and other old and new results to fill some more complexity gaps and obtain complete complexity classifications for a number of colouring problems on graphs with bounded maximum degree.

We first introduce necessary definitions and terminology in Section 2.

2 Preliminaries

A *colouring* of a graph $G = (V, E)$ is a function $c : V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ whenever $uv \in E$. We say that $c(u)$ is the *colour* of u. For a positive integer k, if $1 \leq c(u) \leq k$ for all $u \in V$, then c is a *k-colouring* of G. We say that G is *k-colourable* if a k-colouring of G exists. The COLOURING problem is to decide whether a graph G is k-colourable for some given integer k. If k is fixed, that is, not part of the input, we obtain the k-COLOURING problem.

A *list assignment* of a graph $G = (V, E)$ is a function L with domain V such that for each vertex $u \in V$, $L(u)$ is a subset of $\{1, 2, \ldots\}$. This set is called the *list* of *admissible* colours for u. If $L(u) \subseteq \{1, \ldots, k\}$ for each $u \in V$, then L is a *k-list assignment*. The *size* of a list assignment L is the maximum list size $|L(u)|$ over all vertices $u \in V$. A colouring c *respects* L if $c(u) \in L(u)$ for all $u \in V$. Given a graph G with a k-list assignment L, the LIST COLOURING problem is to decide whether G has a colouring that respects L. If k is fixed, then we have the LIST k-COLOURING problem. Fixing the size of L to be at most ℓ gives the ℓ-LIST COLOURING problem. We say that a list assignment L of a graph $G = (V, E)$ is *ℓ-regular* if, for all $u \in V$, $L(u)$ contains exactly ℓ colours. This gives us the following problem, which is one focus of this paper. It is defined for each integer $\ell \geq 1$ (note that ℓ is *fixed*, that is, ℓ is not part of the input).


ℓ-REGULAR LIST COLOURING
    Instance: a graph G with an ℓ-regular list assignment L.
    Question: does G have a colouring that respects L?

A *k-precolouring* of a graph $G = (V, E)$ is a function $c_W : W \to \{1, 2, \ldots, k\}$ for some subset $W \subseteq V$. A k-colouring c of G is an *extension* of a k-precolouring c_W of G if $c(v) = c_W(v)$ for each $v \in W$. Given a graph G with a precolouring c_W, the PRECOLOURING EXTENSION problem is to decide whether G has a k-colouring that extends c_W. If k is fixed, we obtain the k-PRECOLOURING EXTENSION problem. The relationships amongst the problems introduced are shown in Fig. 1.

For an integer $\ell \geq 1$, a graph $G = (V, E)$ is *ℓ-choosable* if, for every ℓ-regular list assignment L of G, there exists a colouring that respects L. The corresponding decision problem is the CHOOSABILITY problem. If ℓ is fixed, we obtain the ℓ-CHOOSABILITY problem.

We emphasize that ℓ-REGULAR LIST COLOURING and ℓ-CHOOSABILITY are two fundamentally different problems. For the former we must decide whether there exists a colouring that respects a *particular* ℓ-regular list assignment. For the latter we must decide whether or not *every* ℓ-regular list assignment has a colouring that respects it. As we will see later, this difference also becomes clear from a complexity point of view: for some graph classes ℓ-REGULAR LIST COLOURING is computationally easier than ℓ-CHOOSABILITY, whereas, perhaps more surprisingly, for other graph classes, the reverse holds.

For two vertex-disjoint graphs G and H, we let $G + H$ denote the disjoint union $(V(G) \cup V(H), E(G) \cup E(H))$, and kG denote the disjoint union of k copies of G. If G is a graph containing an edge e or a vertex v then $G - e$ and $G - v$
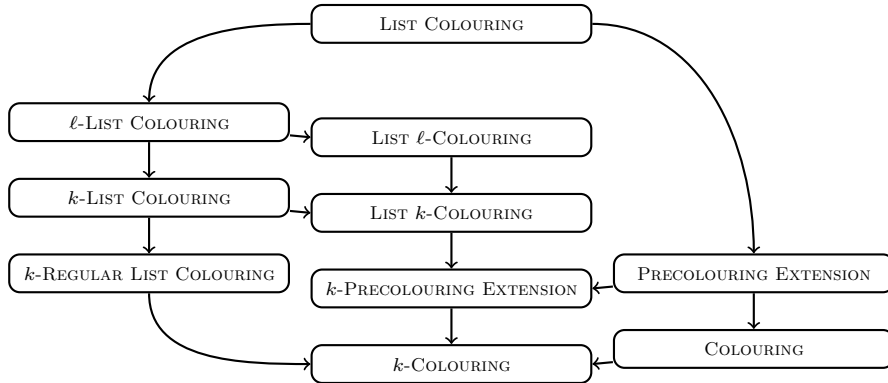
Fig. 1. Relationships between COLOURING and its variants. An arrow from one problem to another indicates that the latter is (equivalent to) a special case of the former; k and ℓ are any two arbitrary integers for which $\ell \geq k$. For instance, k-COLOURING is a special case of k-REGULAR LIST COLOURING. This can be seen by giving the list $L(u) = \{1, \ldots, k\}$ to each vertex u in an instance graph of COLOURING. We recall that in the ℓ-LIST COLOURING problem, the size of the lists is *at most* ℓ. We also observe that ℓ-REGULAR LIST COLOURING and k-REGULAR LIST COLOURING are not comparable for any $k \neq \ell$.

denote the graphs obtained from G by deleting e or v, respectively. If G' is a subgraph of G then $G - G'$ denotes the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(G')$.

We let C_n, K_n and P_n denote the cycle, complete graph and path on n vertices, respectively. A wheel is a cycle with an extra vertex added that is adjacent to all other vertices. The wheel on n vertices is denoted W_n; note that $W_4 = K_4$. A graph on at least three vertices is *2-connected* if it is connected and there is no vertex whose removal disconnects it.

For a set of graphs $\mathcal{H}$, a graph G is $\mathcal{H}$-*free* if G contains no induced subgraph isomorphic to a graph in $\mathcal{H}$, whereas G is $\mathcal{H}$-*subgraph-free* if it contains no subgraph isomorphic to a graph in $\mathcal{H}$. The *girth* of a graph is the length of its shortest cycle.

3 Planar Graphs

3.1 Known Results for Planar Graphs

We start with a classical result observed by Erdős et al. [14] and Vizing [30].

Theorem 1 ([14, 30]). 2-LIST COLOURING *is polynomial-time solvable.*

Garey et al. proved the following result, which is in contrast to the fact that every planar graph is 4-colourable by the Four Colour Theorem [2].

4

Theorem 2 ([16]). 3-Colouring *is* NP-*complete for planar graphs of maximum degree* 4.

Next we present results found by several authors on the existence of k-choosable graphs for various graph classes.

Theorem 3. *The following statements hold for k-choosability:*

  (i) *Every planar graph is* 5-*choosable* [27].
 (ii) *There exists a planar graph that is not* 4-*choosable* [32].
(iii) *Every planar triangle-free graph is* 4-*choosable* [22].
 (iv) *Every planar graph with no* 4-*cycles is* 4-*choosable* [23].
  (v) *There exists a planar triangle-free graph that is not* 3-*choosable* [33].
 (vi) *There exists a planar graph with no* 4-*cycles, no* 5-*cycles and no intersecting triangles that is not* 3-*choosable* [26].
(vii) *Every planar bipartite graph is* 3-*choosable* [1].

We note that smaller examples of graphs than were used in the original proofs have been found for Theorems 3.(ii) [20], 3.(v) [25] and 3.(vi) [38] and that Theorem 3.(vi) strengthens a result of Voigt [34]. We recall that Thomassen [28] first showed that every planar graph of girth at least 5 is 3-choosable, and that a number of results have since been obtained on 3-choosability of planar graphs in which certain cycles are forbidden; see, for example, [5, 11, 36, 37].

We will also use the following result of Chlebík and Chlebíková.

Theorem 4 ([6]). List Colouring *is* NP-*complete for* 3-*regular planar bipartite graphs that have a list assignment in which each list is one of* $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ *and all the neighbours of each vertex with three colours in its list have two colours in their lists.*

3.2 New Results for Planar Graphs

Theorems 1–3 have a number of immediate consequences for the complexity of ℓ-Regular List Colouring when restricted to planar graphs. For instance, Theorem 2 implies that 3-Regular List Colouring is NP-complete for planar graphs, whereas Theorem 3.(i) shows that 5-Regular List Colouring is polynomial-time solvable on this graph class. In fact, we notice a complexity jump from being NP-complete to being trivial (that is, the answer is always yes) when ℓ changes from 3 to 5. It is a natural question to determine the complexity for the missing case $\ell = 4$.

In this section we settle this missing case and also present a number of new hardness results for ℓ-Regular List Colouring restricted to various subclasses of planar graphs. At the end of this section we show how to combine the known results with our new ones to obtain a number of dichotomy results (Corollaries 2–5). We deduce some of our new results from two more general theorems, namely Theorems 5 and 7, which we state below, but which we prove in Sections 5.1 and 5.2, respectively.

Theorem 5. *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then 4-*REGULAR LIST COLOURING *is* NP-*complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 4-choosable.*

Note that the class of $\mathcal{H}$-subgraph-free graphs is contained in the class of $\mathcal{H}$-free graphs. Hence, whenever a problem is NP-complete for $\mathcal{H}$-subgraph-free graphs, it is also NP-complete for $\mathcal{H}$-free graphs.

An alternative formulation of Theorem 5 is that, for every finite set $\mathcal{H}$ of 2-connected planar graphs, either every pair (G, L), where G is a planar $\mathcal{H}$-subgraph-free graph, is a yes-instance of 4-REGULAR LIST COLOURING, or 4-REGULAR LIST COLOURING is NP-complete for planar $\mathcal{H}$-subgraph-free graphs. Results of the same flavour were proved for 3-COLOURING, ACYCLIC 3-COLOURING, $(1, 0)$-COLOURING and C_{2p+1}-COLOURING restricted to classes of planar graphs by Esperet et al. [15]. To give a more recent example of such a result, Dross, Montassier and Pinlou [9] proved that either every triangle-free planar graph is near-bipartite – that is, can be decomposed into an independent set and a forest – or recognizing near-bipartite triangle-free planar graphs is NP-complete. By using their construction and the existence of a planar graph that is not near-bipartite (for instance K_4), Bonamy et al. [3, see arXiv version] observed that the problem of recognizing near-bipartite graphs is NP-complete on planar graphs.

We will exploit the power of Theorem 5 by combining it with Theorem 3. For instance, combining Theorem 5 with Theorem 3.(ii) yields that 4-REGULAR LIST COLOURING is NP-complete for planar graphs. We are not aware of any paper proving this result although it seems to have been known: the result is mentioned by Thomassen [29] without proof, and Dvořák and Thomas [12] mistakenly attribute it to [20]. We can sharpen this result as follows. By the Four Colour Theorem [2], every planar graph is 4-colourable, that is, has a colouring respecting the list assignment that assigns list of colours $\{1, 2, 3, 4\}$ to every vertex of the graph. In contrast, Voigt and Wirth [35] showed the existence of a planar graph that does not allow a colouring respecting some specific list assignment L, in which each list contains four distinct colours from $\{1, 2, 3, 4, 5\}$. We use their example to prove the following result in Section 5.1.

Theorem 6. 4-REGULAR LIST COLOURING *is* NP-*complete for planar graphs even if every list contains four colours from $\{1, 2, 3, 4, 5\}$.*

Theorem 5 has further applications and can also be combined with other results from the literature. For instance, consider the non-4-choosable planar graph H from the proof of Theorem 1.7 in [20]. It can be observed that H is W_p-subgraph-free for all $p \geq 8$. Wheels are 2-connected and planar. Hence, if $\mathcal{H}$ is any finite set of wheels on at least eight vertices, then 4-REGULAR LIST COLOURING is NP-complete for planar $\mathcal{H}$-subgraph-free graphs.

Our basic idea for proving Theorem 5 is similar to the proof technique used in [9,15]. We pick a minimal counterexample H with list assignment L. We select an "appropriate" edge $e = uv$ and consider the graph $F' = F - e$. We reduce from an appropriate colouring problem restricted to planar graphs and use copies

of F' as a gadget to ensure that we can enforce a regular list assignment. The proof of the next theorem also uses this idea.

Theorem 7. *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then 3-*REGULAR LIST COLOURING *is* NP-*complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 3-choosable.*

Just like Theorem 5, Theorem 7 has a number of applications. For instance, if we let $\mathcal{H} = \{K_3\}$ then Theorem 7, combined with Theorem 3.(v), leads to the following result (Dvořák and Kawarabayashi also briefly mentioned how to obtain this result in their paper [10] but do not provide a full proof).

Corollary 1. 3-REGULAR LIST COLOURING *is* NP-*complete for planar triangle-free graphs.*

Theorem 7 can also be used for other classes of graphs. For example, let $\mathcal{H}$ be a finite set of graphs, each of which includes a 2-connected graph on at least five vertices as a subgraph. Let $\mathcal{I}$ be the set of these 2-connected graphs. The graph K_4 is a planar $\mathcal{I}$-subgraph-free graph that is not 3-choosable (since it is not 3-colourable). Therefore, Theorem 7 implies that 3-REGULAR LIST COLOURING is NP-complete for planar $\mathcal{H}$-subgraph-free graphs. We can obtain more hardness results by taking some other planar graph that is not 3-choosable, such as a wheel on an even number of vertices. Also, if we let $\mathcal{H} = \{C_4, C_5\}$ we can use Theorem 7 by combining it with Theorem 3.(vi) to find that 3-REGULAR LIST COLOURING is NP-complete for planar graphs with no 4-cycles and no 5-cycles. We strengthen this result as follows (see Section 5.3 for the proof).

Theorem 8. 3-REGULAR LIST COLOURING *is* NP-*complete for planar graphs with no 4-cycles, no 5-cycles and no intersecting triangles.*

Theorem 6, Corollary 1 and Theorem 8 can be seen as strengthenings of Theorems 3.(ii), 3.(v) and 3.(vi), respectively. Moreover, they complement Theorem 2, which implies that 3-LIST COLOURING is NP-complete for planar graphs, and a result of Kratochvíl [21] that, for planar bipartite graphs, 3-PRECOLOURING EXTENSION is NP-complete. Theorem 6 and Corollary 1 also complement results of Gutner [20] who showed that 3-CHOOSABILITY and 4-CHOOSABILITY are Π_2^p-complete for planar triangle-free graphs and planar graphs, respectively. However, we emphasize that, for special graph classes, it is not necessarily the case that ℓ-CHOOSABILITY is computationally harder than ℓ-REGULAR LIST COLOURING. For instance, contrast the fact that CHOOSABILITY is polynomial-time solvable on $3P_1$-free graphs [17] with our next result, which we prove in Section 5.4.

Theorem 9. 3-REGULAR LIST COLOURING *is* NP-*complete for $(3P_1, P_1 + P_2)$-free graphs.*

3.3 Closing Complexity Gaps for Planar Graphs

Our new results, combined with known results, close a number of complexity gaps for the ℓ-REGULAR LIST COLOURING problem. Combining Theorem 6 with Theorems 1, 2 and 3.(i) gives us Corollary 2. Combining Theorem 8 with Theorems 1 and 3.(iv) gives us Corollary 3. Combining Corollary 1 with Theorems 1 and 3.(iii) gives us Corollary 4, whereas Theorems 1 and 3.(vii) imply Corollary 5.

Corollary 2. *Let ℓ be a positive integer. Then ℓ-REGULAR LIST COLOURING, restricted to planar graphs, is* NP-*complete if $\ell \in \{3, 4\}$ and polynomial-time solvable otherwise.*

Corollary 3. *Let ℓ be a positive integer. Then ℓ-REGULAR LIST COLOURING, restricted to planar graphs with no 4-cycles and no 5-cycles and no intersecting triangles, is* NP-*complete if $\ell = 3$ and polynomial-time solvable otherwise (even if we allow intersecting triangles and 5-cycles).*

Corollary 4. *Let ℓ be a positive integer. Then ℓ-REGULAR LIST COLOURING, restricted to planar triangle-free graphs, is* NP-*complete if $\ell = 3$ and polynomial-time solvable otherwise.*

Corollary 5. *Let ℓ be a positive integer. Then ℓ-REGULAR LIST COLOURING, restricted to planar bipartite graphs, is polynomial-time solvable.*

4 Bounded Degree Graphs

4.1 Known Results for Bounded Degree Graphs

First we present a result of Kratochvíl and Tuza [22].

Theorem 10 ([22]). LIST COLOURING *is polynomial-time solvable on graphs of maximum degree at most 2.*

Brooks' Theorem [4] states that every graph G with maximum degree d has a d-colouring unless G is a complete graph or a cycle with an odd number of vertices. The next result of Vizing [31] generalizes Brooks' Theorem to list colourings.

Theorem 11 ([31]). *Let d be a positive integer. Let $G = (V, E)$ be a connected graph of maximum degree at most d and let L be a d-regular list assignment for G. If G is not a cycle or a complete graph then G has a colouring that respects L.*

And we need two other results of Chlebík and Chlebíková [6].

Theorem 12 ([6]). PRECOLOURING EXTENSION *is polynomial-time solvable on graphs of maximum degree 3.*

Theorem 13 ([6]). *Let k be a positive integer. Then k-PRECOLOURING EXTENSION *is polynomial-time solvable for graphs of maximum degree at most k.*

4.2 Closing Complexity Gaps for Bounded Degree Graphs

We have the following two classifications, which we obtain by combining previously known results with some small observations.

Corollary 6. *Let d be a positive integer. The following two statements hold for graphs of maximum degree at most d.*

(i) LIST COLOURING *is* NP*-complete if $d \geq 3$ and polynomial-time solvable if $d \leq 2$.*

(ii) PRECOLOURING EXTENSION *and* COLOURING *are* NP*-complete if $d \geq 4$ and polynomial-time solvable if $d \leq 3$.*

Proof. We first consider (i). If $d \geq 3$, we use Theorem 4. If $d \leq 2$, we use Theorem 10. We now consider (ii). If $d \geq 4$, we use Theorem 2. If $d \leq 3$, we use Theorem 12. □

Corollary 7. *Let d and k be two positive integers. The following two statements hold for graphs of maximum degree at most d.*

(i) k*-*LIST COLOURING *and* LIST k*-*COLOURING *are* NP*-complete if $k \geq 3$ and $d \geq 3$ and polynomial-time solvable otherwise.*

(ii) k*-*REGULAR LIST COLOURING *and* k*-*PRECOLOURING EXTENSION *are* NP*-complete if $k \geq 3$ and $d \geq k+1$ and polynomial-time solvable otherwise.*

Proof. We first consider (i). If $k \geq 3$ and $d \geq 3$, we use Theorem 4. If $k \leq 2$ or $d \leq 2$, we use Theorems 1 or 10, respectively.

We now consider (ii). We start with the hardness cases and so let $k \geq 3$ and $d \geq k+1$.

First consider k-PRECOLOURING EXTENSION. Theorem 2 implies that 3-COLOURING is NP-complete for graphs of maximum degree at most d for all $d \geq 4$. The $k = 3$ case follows immediately from this result. Suppose $k \geq 4$ and $d \geq k+1$. Consider a graph G of maximum degree 4. For each vertex v, we add $k-3$ new vertices $x_1^v, \ldots, x_{k-3}^v$ and edges $vx_1^v, \ldots, vx_{k-3}^v$. Let G' be the resulting graph. Note that G' has maximum degree at most $4 + k - 3 = k+1 \leq d$. We define a precolouring c on the newly added vertices by assigning colour $i + 3$ to each x_i^v. Then G' has a k-colouring extending c if and only if G has a 3-colouring.

Now consider k-REGULAR LIST COLOURING. The $k = 3$ case follows immediately from Theorem 2. Suppose $k \geq 4$ and $d \geq k+1$. Consider a graph G of maximum degree 4. We define the list $L(v) = \{1, \ldots, k\}$ for each vertex $v \in V(G)$. For each vertex v, we add $k-3$ new vertices $x_1^v, \ldots, x_{k-3}^v$ and edges $vx_1^v, \ldots, vx_{k-3}^v$. We define the list $L(x_i^v) = \{i, k+1, k+2, \ldots, 2k-1\}$ for each x_i^v. For each vertex x_i^v, we also add k new vertices $w_1(x_i^v), \ldots, w_k(x_i^v)$ and edges such that $x_i^v, w_1(x_i^v), \ldots, w_k(x_i^v)$ form a clique (on $k+1$ vertices). We define the list $L(w_j(x_i^v)) = \{k+1, \ldots, 2k\}$ for each $w_j(x_i^v)$. Let G' be the resulting graph. Note that G' has maximum degree at most $k+1$ and that the resulting list assignment L is a k-regular list assignment of G'. Then G' has a k-colouring respecting L if and only if G has a 3-colouring.

We continue with the polynomial-time solvable cases. If $k \leq 2$, the result follows from Theorem 1. Now suppose that $k \geq 3$ and $d \leq k$. The result for k-PRECOLOURING EXTENSION follows from Theorem 13. We now consider the k-REGULAR LIST COLOURING problem. Let (G, L) be an instance. Since k-REGULAR LIST COLOURING can be solved component-wise, we may assume that G is connected. If G is a cycle, we use Theorem 10. If G is a complete graph, then we use the folklore result that LIST COLOURING is polynomial-time solvable on complete graphs (see [19] for a proof). In all other cases we apply Theorem 11. □

Note that Corollary 7 does not contain a dichotomy for k-COLOURING restricted to graphs of maximum degree at most d. A full classification of this problem is open, but a number of results are known. Molloy and Reed [24] classified the complexity for all pairs (k, d) for sufficiently large d. Emden-Weinert et al. [13] proved that k-COLOURING is NP-complete for graphs of maximum degree at most $k + \lceil \sqrt{k} \rceil - 1$. It follows from Brooks' Theorem [4] that for every integer $k \geq 1$, k-COLOURING is polynomial-time solvable for graphs of maximum degree k. Combining this observation with the result of [13] means that the smallest open case is when $k = 5$ and $d = 6$.

5   Proofs

In Section 5.1 we prove Theorems 5 and 6. In Section 5.2 we prove Theorem 7, whereas in Section 5.3 we prove Theorem 8, and in Section 5.4 we prove Theorem 9.

5.1   The Proofs of Theorems 5 and 6

We need an additional result.

Theorem 14. *For every integer $p \geq 3$, 3-LIST COLOURING is NP-complete for planar graphs of girth at least p that have a list assignment in which each list is one of $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$.*

Proof. By Theorem 4, LIST COLOURING is NP-complete for 3-regular planar bipartite graphs that have a list assignment in which each list is one of $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$ and all the neighbours of each vertex with three colours in its list have two colours in their lists. We modify the hardness construction as follows. Note that for each edge at least one of the incident vertices has a list of size 2. We replace each edge by a path on an odd number of edges in such a way that the girth of the graph obtained is at least p. The new vertices on the path are all given the same list of size 2, identical to the list on one or other of the end-vertices. It is readily seen that these modifications do not affect whether or not the graph can be coloured. □

We are now ready to prove Theorem 5, which we restate below.

Theorem 5 (restated). *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then 4-Regular List Colouring is NP-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 4-choosable.*

Proof. The problem is readily seen to be in NP. Let F be a planar $\mathcal{H}$-subgraph-free graph with a 4-regular list assignment L such that F has no colouring respecting L. We may assume that F is minimal (with respect to the subgraph relation). In particular, this means that F is connected. Let r be the length of a longest cycle in any graph of $\mathcal{H}$. We reduce from the problem of 3-List Colouring restricted to planar graphs of girth at least $r + 1$ in which each vertex has list $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ or $\{1, 2, 3\}$. This problem is NP-complete by Theorem 14. Let a graph G and list assignment L_G be an instance of this problem. We will construct a planar $\mathcal{H}$-subgraph-free graph G' with a 4-regular list assignment L' such that G has a colouring that respects L_G if and only if G' has a colouring that respects L'.

If every pair of adjacent vertices in F has the same list, then the problem of finding a colouring that respects L is just the problem of finding a 4-colouring which, by the Four Colour Theorem [2], we know is possible. Thus we may assume that, on the contrary, there is an edge $e = uv$ such that $|L(u) \cap L(v)| \leq 3$. Let $F' = F - e$. Then, by minimality, F' has at least one colouring respecting L, and moreover, for any colouring of F' that respects L, u and v are coloured alike (otherwise we would have a colouring of F that respects L). Let T be the set of possible colours that can be used on u and v in colourings of F' that respect L and let $t = |T|$. As $T \subseteq L(u) \cap L(v)$, we have $1 \leq t \leq 3$. Up to renaming the colours in L, we can build copies of F' with 4-regular list assignments such that

(i) the set T is any given list of colours of size t, and
(ii) the vertex corresponding to u has any given list of 4 colours containing T.

We will implicitly make use of this several times in the remainder of the proof.

We say that a vertex w in G is a *bivertex* or *trivertex* if $|L_G(w)|$ is 2 or 3, respectively. We construct a planar $\mathcal{H}$-subgraph-free graph G' and a 4-regular list assignment L' as follows.

First suppose that $t = 1$. For each bivertex w in G, we do as follows. We add two copies of F' to G, which we label $F_1(w)$ and $F_2(w)$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w for $i \in \{1, 2\}$ and we set $U^w = \{u_1^w, u_2^w\}$. We add the edges wu_1^w and wu_2^w. We give list assignments to the vertices of $F_1(w)$ and $F_2(w)$ such that $T = \{4\}$ for F_1 and $T = \{5\}$ for F_2. We let $L'(w) = L_G(w) \cup \{4, 5\}$. For each trivertex w in G, we do as follows. We add one copy of F' to G, which we label $F_1(w)$. The vertex in $F_1(w)$ corresponding to u is labelled u_1^w and we set $U^w = \{u_1^w\}$. We add the edge wu_1^w. We give list assignments to vertices of $F_1(w)$ such that $T = \{4\}$ for F_1. We let $L'(w) = L_G(w) \cup \{4\}$. This completes the construction of G' and L' when $t = 1$.

Now suppose that $t = 2$. Let $s = r$ if r is even and $s = r + 1$ if r is odd (so s is even in both cases). For each bivertex w in G, we do as follows. We

add a copy of F' to G, which we label $F_1(w)$, and identify the vertex in $F_1(w)$ corresponding to u with w. We give list assignments to vertices of $F_1(w)$ such that $T = L_G(w)$ and $L'(w) = L_G(w) \cup \{4, 5\}$. For each trivertex w in G, we do as follows. We add s copies of F' to G which we label $F_i(w)$, $1 \leq i \leq s$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w. Let $U^w = \{u_i^w \mid 1 \leq i \leq s\}$. Add edges such that the union of w and U^w induces a cycle on $s + 1$ vertices. For all $1 \leq i \leq s$, we give list assignments to vertices of $F_i(w)$ such that $T = \{4, 5\}$. We let $L'(w) = \{1, 2, 3, 4\}$. This completes the construction of G' and L' when $t = 2$.

Now suppose that $t = 3$. For each bivertex w in G, we do as follows. We add two copies of F' to G which we label $F_1(w)$ and $F_2(w)$, such that for $i \in \{1, 2\}$, the vertex in $F_i(w)$ corresponding to u is identified with w. We give list assignments to vertices of $F_1(w)$ and $F_2(w)$ such that $T = L_G(w) \cup \{4\}$ for $F_1(w)$, $T = L_G(w) \cup \{5\}$ for $F_2(w)$ and $L'(w) = L_G(w) \cup \{4, 5\}$. For each trivertex w in G, we do as follows. We add a copy of F' to G which we label $F_1(w)$, such that the vertex in $F_1(w)$ corresponding to u is identified with w. We give list assignments to the vertices of $F_1(w)$ such that $T = \{1, 2, 3\}$ and $L'(w) = \{1, 2, 3, 4\}$. This completes the construction of G' and L' when $t = 3$.

Note that G' is planar. Suppose that there is a subgraph H in G' that is isomorphic to a graph of $\mathcal{H}$. Since F is $\mathcal{H}$-subgraph-free, and since F' is obtained from F by removing one edge, F' is also $\mathcal{H}$-subgraph-free. Therefore for all w, H is not fully contained in any $F_i(w)$. Since H is 2-connected and since for all w only one vertex of any $F_i(w)$ has a neighbour outside of $F_i(w)$, we find that H has at most one vertex in each $F_i(w)$. In particular, H cannot contain any vertex of any $F_i(w)$ in which the vertex corresponding to u has been attached to w (as opposed to being identified with w); this includes the case when the union of w and U^w induces a cycle on $s + 1$ vertices. Hence we have found that H is a subgraph of G, which contradicts the fact that G has girth at least $r + 1$. Therefore G' is $\mathcal{H}$-subgraph-free.

Note that in any colouring of G' that respects L', each copy of F' must be coloured such that the vertices corresponding to u and v have the same colour, which must be one of the colours from the corresponding set T. If $t = 1$ and w is a trivertex, this means that the unique neighbour of w in U^w must be coloured with colour 4, so w cannot be coloured with colour 4. Similarly, if $t = 1$ and w is a bivertex or $t = 2$ and w is a trivertex then the two neighbours of w in U^w must be coloured with colours 4 and 5, so w cannot be coloured with colours 4 or 5. If $t = 2$ and w is a bivertex or $t = 3$ and w is a trivertex then w belongs to a copy of F' with $T = L_G(w)$, so w cannot have colour 4 or 5. If $t = 3$ and w is a bivertex then w belongs to two copies of F', one with $T = L_G(w) \cup \{4\}$ and one with $T = L_G(w) \cup \{5\}$. Therefore, w must be coloured with a colour from the intersection of these two sets, that is it must be coloured with a colour from $L_G(w)$. Therefore none of the vertices of G can be coloured 4 or 5. Thus the problem of finding a colouring of G' that respects L' is equivalent to the problem of finding a colouring of G that respects L_G. This completes the proof. $\qquad\square$

We will now prove Theorem 6.

Theorem 6 (restated). 4-REGULAR LIST COLOURING *is* NP*-complete for planar graphs even if every list contains four colours from* $\{1, 2, 3, 4, 5\}$.

Proof. Recall that there exists a planar graph F^* with a 4-regular list assignment L in which each list $L(u)$ contains four colours from $\{1, 2, 3, 4, 5\}$ such that F^* has no colouring respecting L [35]. We may assume without loss of generality that F^* is minimal (with respect to the subgraph relation) and use F^* as the graph F in the proof of Theorem 5. This means that we give each vertex of the graph G' in the proof of Theorem 5 a list of colours from $\{1, 2, 3, 4, 5\}$. The result follows. □

5.2 The Proof of Theorem 7

Theorem 7 (restated). *Let $\mathcal{H}$ be a finite set of 2-connected planar graphs. Then* 3-REGULAR LIST COLOURING *is* NP*-complete for planar $\mathcal{H}$-subgraph-free graphs if there exists a planar $\mathcal{H}$-subgraph-free graph that is not 3-choosable.*

Proof. The problem is readily seen to be in NP. Every graph in $\mathcal{H}$ is 2-connected, and therefore contains a cycle. Let r be the length of a longest cycle in any graph of $\mathcal{H}$. By assumption, there exists a planar graph F and 3-regular list assignment L such that F is $\mathcal{H}$-subgraph-free and has no colouring respecting L. We may assume that F is minimal by removing edges and vertices until any further removal would give a graph with a colouring respecting L. In particular, we note that F is connected.

We distinguish two cases.

Case 1. $L(v)$ is the same for every vertex v in F.
Then we may assume without loss of generality that $L(v) = \{1, 2, 3\}$ for all v. We reduce from 3-COLOURING which is NP-complete even for planar graphs by Theorem 2. Let G be a planar graph. We will construct a planar $\mathcal{H}$-subgraph-free graph G' as follows.

Let $e = uv$ be an edge of F. Let $F' = F - e$. Then, by minimality, F' has at least one colouring respecting L, which must be a 3-colouring as every list consists of the colours $1, 2, 3$. For every 3-colouring c of F', it holds that $c(u) = c(v)$ (otherwise c would be a colouring of F that respects L). Moreover, since we can permute the colours, there is such a colouring c that colours u (and thus v) with colour i for each $i \in \{1, 2, 3\}$. Note that in F' the vertices u and v must be at distance at least 2 from each other.

Let $s = \left\lceil \frac{r}{6} \right\rceil$. Assume that the vertices of G are ordered. For each edge $xy \in E(G)$ with $x < y$, we do the following:

(i) delete xy;
(ii) add s copies of F' labelled $F_1(xy), \ldots, F_s(xy)$ and, for $1 \leq i \leq s$, let u_i^{xy} and v_i^{xy} be the vertices in $F_i(xy)$ corresponding to u and v;
(iii) identify x with u_1^{xy} and, for $1 \leq i \leq s-1$, identify v_i^{xy} with u_{i+1}^{xy};
(iv) add an edge from v_s^{xy} to y.

13

Let G' be the obtained graph and note that G' is planar. Every cycle in G' that is not contained in a copy of F' has length at least $3(2s+1) \geq r+1$, since it corresponds to a cycle in G and in which every edge has been replaced by s successive copies of F' plus an edge.

Suppose that there is a subgraph H in G' that is isomorphic to a graph of $\mathcal{H}$. Since F is $\mathcal{H}$-subgraph-free, and since F' is obtained from F by removing one edge, F' must also be $\mathcal{H}$-subgraph-free. Therefore H is not fully contained in a copy of F'. Since H is 2-connected, this implies that there is a cycle in H that is not fully contained in a copy of F'. By definition, this cycle has length at most r, a contradiction.

Suppose the graph G' has a 3-colouring c. For each copy of F', the vertices corresponding to u and v must be coloured the same. For all edges xy with $x < y$ in G, there is a vertex in G' coloured the same as x in G that is adjacent to y in G', so $c(x) \neq c(y)$. Therefore c restricted to $V(G)$ is a 3-colouring of G.

On the other hand, suppose the graph G has a 3-colouring c. We can extend this 3-colouring to G' by doing the following: for all edges xy with $x < y$ in G, colour every $F_i(xy)$ in such a way that the vertex corresponding to u and the vertex corresponding to v have colour $c(x)$. This leads to a 3-colouring of G'.

Case 2. F contains two vertices u and v with $L(u) \neq L(v)$.
As F is connected, we assume without loss of generality that u and v are adjacent; let $e = uv$.

We reduce from the problem of 3-LIST COLOURING restricted to planar graphs of girth at least $r+1$ in which each vertex has list $\{1,2\}$, $\{1,3\}$, $\{2,3\}$ or $\{1,2,3\}$. This problem is NP-complete by Theorem 14. Let a graph G and list assignment L_G be an instance of this problem. We will construct a planar $\mathcal{H}$-subgraph-free graph G' with a 3-regular list assignment L' such that G has a colouring that respects L_G if and only if G' has a colouring that respects L'.

We define $F' = F - e$. Then, by minimality, F' has at least one colouring respecting L, and moreover, for any colouring of F' that respects L, u and v are coloured alike (otherwise we would have a colouring of F that respects L). Let T be the set of possible colours that can be used on u and v in colourings of F' that respect L and let $t = |T|$. As $T \subseteq L(u) \cap L(v)$, we have $1 \leq t \leq 2$. Let us assume, without loss of generality, that $T \subseteq \{4,5\}$ and that $4 \in T$.

We say that a vertex w in G is a bivertex or trivertex if $|L(w)|$ is 2 or 3, respectively. We construct a planar $\mathcal{H}$-free graph G'.

First suppose that $t = 1$. For each bivertex w in G, we do as follows. We add a copy of F' to G which we label $F(w)$. The vertex in $F(w)$ corresponding to u is labelled u^w and we set $U^w = \{u^w\}$. We add the edge wu^w. This completes the construction of G' when $t = 1$.

Now suppose that $t = 2$. Let $s = r$ if r is even and $s = r+1$ if r is odd (so s is even in both cases). For each bivertex w in G, we add s copies of F' to G which we label $F_i(w)$, $1 \leq i \leq s$. The vertex in $F_i(w)$ corresponding to u is labelled u_i^w. Let $U^w = \{u_i^w \mid 1 \leq i \leq s\}$. Add edges such that, for each bivertex w in G, the union of w and U^w induces a cycle on $s+1$ vertices. This completes the construction of G' when $t = 2$.

Note that G' is planar, since it is made of planar graphs (G and copies of F') connected in a way that does not obstruct planarity. Suppose that there is a subgraph H in G' that is isomorphic to a graph of $\mathcal{H}$. Since F is $\mathcal{H}$-subgraph-free, and since F' is obtained from F by removing one edge, F' is also $\mathcal{H}$-subgraph-free. Therefore for all w, H is not fully contained in $F(w)$. Since H is 2-connected and since for all w, only one vertex of $F(w)$ has a neighbour outside of $F(w)$, we find that H has at most one vertex in each $F(w)$. This means that H is a subgraph of G, which contradicts the fact that G has girth at least $r + 1$. Therefore G' is $\mathcal{H}$-subgraph-free.

Now we define a list assignment L'. We give the vertices of each copy of F' the same lists as their corresponding vertices in F, and for each bivertex w in G, we define $L'(w) = L_G(w) \cup \{4\}$, and for each trivertex w in G, we define $L'(w) = L_G(w)$. This gives us the 3-regular list assignment L' of G'.

The graph $G' - G$ has a colouring that respects (the restriction of) L' and we notice that in such a colouring each copy of F' must be coloured in such a way that, for each bivertex w in G, the t vertices of U^w that are adjacent to w are coloured with the t colours of T. So one of the neighbours of w in U^w must be coloured 4. Thus the problem of finding a colouring of G' that respects L' is equivalent to the problem of finding a colouring of G that respects L_G. The proof is complete. □

5.3 The Proof of Theorem 8

Theorem 8 is not quite implied by Theorem 7. However, we can adapt the proof of Theorem 7 to prove Theorem 8, which we restate below.

Theorem 8 (restated). 3-REGULAR LIST COLOURING *is* NP-*complete for planar graphs with no 4-cycles, no 5-cycles and no intersecting triangles.*

Proof. Let B be the graph on five vertices with two triangles sharing exactly one vertex (this graph is known as the *butterfly*). Consider the previous proof with $\mathcal{H} = \{C_4, C_5, B\}$. Note that the only problem is that B is not 2-connected.

In [26] an example is given of a graph with no 4-cycle, no 5-cycle and no intersecting triangles that is not 3-choosable. We omit the details of this construction, as the existence of such a graph is sufficient for our proof. Hence, let I be the example of [26], with L a 3-regular list assignment such that there is no colouring of I respecting L. We remove edges and vertices from I until any further removal would give a graph with a colouring respecting L. This leads to a connected graph F. There are no four vertices in I with the same list inducing a connected subgraph, so there are no such four vertices in F. Therefore in F there is an edge uv such that $L(u) \neq L(v)$ (otherwise F would have at most three vertices, and thus would be 3-choosable).

Therefore we can skip Case 1 and directly adapt the proof in Case 2. Note that the only thing to prove is that in this case G' does not contain a subgraph isomorphic to B. Suppose H is such a subgraph. Since F' is B-subgraph-free, H cannot be fully contained in $F(w)$ for any w. Since no vertex is in two different $F(w)$ and no vertex of $F(w)$ has two adjacent neighbours outside $F(w)$,

this implies that there is a triangle in G, which is impossible since G has girth at least 6. □

5.4 The Proof of Theorem 9

The proof is obtained by a modification of the NP-hardness construction of 3-LIST COLOURING for $(3P_1, P_1+P_2)$-free graphs from [19]. Recall that we included this result in our paper to illustrate that k-CHOOSABILITY and k-REGULAR LIST COLOURING can have different complexities when restricted to special graph classes. Indeed, since CHOOSABILITY is polynomial-time solvable on $3P_1$-free graphs [17], Theorem 9 shows that k-CHOOSABILITY may even be easier than k-REGULAR LIST COLOURING.

Theorem 9 (restated). 3-REGULAR LIST COLOURING *is* NP-*complete for* $(3P_1, P_1 + P_2)$-*free graphs.*

Proof. The problem is readily seen to belong to NP. Golovach et al. [17] showed that 3-LIST COLOURING is NP-complete for $(3P_1, P_1 + P_2)$-free graphs in which every vertex has a list of size 2 or 3. Let G be such an instance. We add three new vertices s, t, u to G. We make s, t, u adjacent to each other and to each original vertex of G. This results in a $(3P_1, P_1 + P_2)$-free graph G'. We take three new colours $1, 2, 3$ and set $L(s) = L(t) = L(u) = \{1, 2, 3\}$. This forces colour 1 to be used to colour one of s, t or u. Then all that remains is to add colour 1 to the list of every vertex of G that has a list of size 2. □

6 Conclusions

As well as presenting some dichotomies for a number of colouring problems for graphs with bounded maximum degree, we have given several dichotomies for the k-REGULAR LIST COLOURING problem restricted to subclasses of planar graphs. In particular we showed NP-hardness of the cases $k = 3$ and $k = 4$ restricted to planar $\mathcal{H}$-subgraph-free graphs for several sets $\mathcal{H}$ of 2-connected planar graphs. Our method implies that for such sets $\mathcal{H}$ it suffices to find a counterexample to 3-choosability or to 4-choosability, respectively. It is a natural to ask whether we can determine the complexity of 3-REGULAR LIST COLOURING and 4-REGULAR LIST COLOURING for any class of planar $\mathcal{H}$-subgraph-free graphs. However, we point out that even when restricting $\mathcal{H}$ to be a finite set of 2-connected planar graphs, this would be very hard (and beyond the scope of this paper) as it would require solving several long-standing conjectures in the literature. For example, when $\mathcal{H} = \{C_4, C_5, C_6\}$, Montassier [25] conjectured that every planar $\mathcal{H}$-subgraph-free graph is 3-choosable.

A drawback of our method is that we need the set of graphs $\mathcal{H}$ to be 2-connected. If we forbid a set $\mathcal{H}$ of graphs that are not 2-connected, the distinction between polynomial-time solvable and NP-complete cases is not clear, and both cases may occur even if we forbid only one graph. We illustrate this below with an example.

Example. Let $\mathcal{H}$ contain only the star $K_{1,r}$ for some $r \geq 2$. Note that $K_{1,r}$-subgraph-free graphs are exactly those graphs that have maximum degree at most $r - 1$. Hence, if $r = 3$, then 3-REGULAR LIST COLOURING is polynomial-time solvable due to Theorem 10. However, there exist larger values of r for which the problem is NP-complete. In order to see this we adapt the proof of Theorem 7. The hardness reductions in this proof multiply the maximum degree of our instances by some constant d that is at most the maximum degree of the no-instance F. By Theorems 2 and 4, the problems we reduce from are NP-complete even for graphs with maximum degree at most 4. Hence, we have proven the following: if $\mathcal{H}$ is a finite set of 2-connected planar graphs and F is a non-3-choosable planar $\mathcal{H}$-subgraph-free graph with maximum degree d, then 3-REGULAR LIST COLOURING is NP-complete on planar $\mathcal{H}$-subgraph-free graphs with maximum degree at most $4d$. We can take $F = K_4$ to deduce that 3-REGULAR LIST COLOURING is NP-complete on planar $K_{1,13}$-subgraph-free graphs.

References

1. N. Alon and M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992) 125–134.
2. K. Appel and W. Haken, Every planar map is four colorable, Contemporary Mathematics 89, AMS Bookstore, 1989.
3. M. Bonamy, K. K. Dabrowski, C. Feghali, M. Johnson, and D. Paulusma, Recognizing graphs close to bipartite graphs, Proc. MFCS 2017, LIPIcs 83, 70:1–70:14, Full version: arXiv:1707.09817.
4. R. L. Brooks, On colouring the nodes of a network, Mathematical Proceedings of the Cambridge Philosophical Society 37 (1941) 194–197.
5. M. Chen, M. Montassier and A. Raspaud, Some structural properties of planar graphs and their applications to 3-choosability, Discrete Mathematics 312 (2012) 362–373.
6. M. Chlebík and J. Chlebíková, Hard coloring problems in low degree planar bipartite graphs, Discrete Applied Mathematics 154 (2006) 1960–1965.
7. M. Chudnovsky, Coloring graphs with forbidden induced subgraphs, Proc. ICM 2014 vol IV, 291–302.
8. K.K. Dabrowski, F. Dross, M. Johnson and D. Paulusma, Filling the complexity gaps for colouring planar and bounded degree graphs, Proc. IWOCA 2015, LNCS 9538, 100–111.
9. F. Dross, M. Montassier, and A. Pinlou, Partitioning a triangle-free planar graph into a forest and a forest of bounded degree, European Journal of Combinatorics 66 (2017) 81–94.
10. Z. Dvořák and K. Kawarabayashi, List-coloring embedded graphs, Proc. SODA 2013, 1004–1012.
11. Z. Dvořák, B. Lidický and R. Škrekovski, Planar graphs without 3-, 7-, and 8-cycles are 3-choosable, Discrete Mathematics 309 (2009) 5899–5904.

12. Z. Dvořák and R. Thomas, List-coloring apex-minor-free graphs, Manuscript, arXiv:1401.1399.
13. T. Emden-Weinert, S. Hougardy and B. Kreuter, Uniquely colourable graphs and the hardness of colouring graphs of large girth, Combinatorics, Probability & Computing 7 (1998) 375–386.
14. P. Erdős, A. L. Rubin, and H. Taylor, Choosability in graphs, Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), Congress. Numer., XXVI, Winnipeg, Man., 1980, Utilitas Math., pp. 125–157.
15. L. Esperet, M. Montassier, P. Ochem, and A. Pinlou, A complexity dichotomy for the coloring of sparse graphs, Journal of Graph Theory 73 (2013), 85–102.
16. M.R. Garey, D.S. Johnson, and L.J. Stockmeyer, Some simplified NP-complete graph problems, Proc. STOC 1974, 47–63.
17. P.A. Golovach, P. Heggernes, P. van 't Hof and D. Paulusma, Choosability on H-free graphs, Information Processing Letters 113 (2013) 107–110.
18. P.A. Golovach, M. Johnson, D. Paulusma, and J. Song, A survey on the computational complexity of colouring graphs with forbidden subgraphs, Journal of Graph Theory 84 (2017) 331–363.
19. P.A. Golovach, D. Paulusma and J. Song, Closing complexity gaps for coloring problems on H-free graphs, Information and Computation 237 (2014) 204–214.
20. S. Gutner, The complexity of planar graph choosability, Discrete Mathematics 159 (1996) 119–130.
21. J. Kratochvíl, Precoloring extension with fixed color bound. Acta Mathematica Universitatis Comenianae 62 (1993) 139–153.
22. J. Kratochvíl and Z. Tuza, Algorithmic complexity of list colourings, Discrete Applied Mathematics 50 (1994) 297–302.
23. P.C.B. Lam, B. Xu and J. Liu, The 4-choosability of plane graphs without 4-cycles, Journal of Combinatorial Theory, Series B 76 (1999) 117–126.
24. M. Molloy and B. Reed, Colouring graphs when the number of colours is almost the maximum degree, Journal of Combinatorial Theory, Series B 109 (2014) 134–195.
25. M. Montassier, A note on the not 3-choosability of some families of planar graphs, Information Processing Letters 99 (2006) 68–71.
26. M. Montassier, A. Raspaud and W. Wang, Bordeaux 3-color conjecture and 3-choosability, Discrete Mathematics 306 (2006) 573–579.
27. C. Thomassen, Every planar graph is 5-choosable, Journal of Combinatorial Theory, Series B 62 (1994) 180–181.
28. C. Thomassen, 3-List-coloring planar graphs of girth 5, Journal of Combinatorial Theory, Series B 64 (1995) 101–107.
29. C. Thomassen, Color-critical graphs on a fixed surface, Journal of Combinatorial Theory, Series B 70 (1997) 67–100.
30. V.G. Vizing, Coloring the vertices of a graph in prescribed colors, in Diskret. Analiz., no. 29, Metody Diskret. Anal. v. Teorii Kodov i Shem 101 (1976) 3–10.
31. V.G. Vizing, Vertex colorings with given colors, Diskret. Analiz. 29 (1976) 3–10.
32. M. Voigt, List colourings of planar graphs, Discrete Mathematics 120 (1993) 215–219.
33. M. Voigt, A not 3-choosable planar graph without 3-cycles, Discrete Mathematics 146 (1995) 325–328.
34. M. Voigt, A non-3-choosable planar graph without cycles of length 4 and 5, Discrete Mathematics 307 (2007) 1013–1015.
35. M. Voigt and B. Wirth, On 3-colorable non-4-choosable planar graphs, Journal of Graph Theory 24 (1997) 233-235.

36. Y. Wang, H. Lu and M. Chen, Planar graphs without cycles of length 4, 5, 8, or 9 are 3-choosable. Discrete Mathematics 310 (2010) 147–158.
37. Y. Wang, H. Lu and M. Chen, Planar graphs without cycles of length 4, 7, 8, or 9 are 3-choosable. Discrete Applied Mathematics 159 (2011) 232–239.
38. D-Q. Wang, Y-P. Wen and K-L. Wang, A smaller planar graph without 4-, 5-cycles and intersecting triangles that is not 3-choosable, Information Processing Letters 108 (2008) 87–89.