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On the complexity of computing the k -restricted edge-connectivity of a graph^{*}

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Abstract. The k -restricted edge-connectivity of a graph G , denoted by $\lambda_k(G)$, is defined as the minimum size of an edge set whose removal leaves exactly two connected components each containing at least k vertices. This graph invariant, which can be seen as a generalization of a minimum edge-cut, has been extensively studied from a combinatorial point of view. However, very little is known about the complexity of computing $\lambda_k(G)$. Very recently, in the parameterized complexity community the notion of *good edge separation* of a graph has been defined, which happens to be essentially the same as the k -restricted edge-connectivity. Motivated by the relevance of this invariant from both combinatorial and algorithmic points of view, in this article we initiate a systematic study of its computational complexity, with special emphasis on its parameterized complexity for several choices of the parameters. We provide a number of NP-hardness and W[1]-hardness results, as well as FPT-algorithms.

Keywords: Graph cut; k -restricted edge-connectivity; Good edge separation; Parameterized complexity; FPT-algorithm; Polynomial kernel.

1 Introduction

Motivation. The k -restricted edge-connectivity is a graph invariant that has been widely studied in the literature from a combinatorial point of view [1, 4, 17, 22, 32, 33]. Since the classical edge-connectivity may not suffice to measure accurately how connected a graph is after deleting some edges, Esfahanian and Hakimi [16] proposed in 1988 the notion of *restricted edge-connectivity*. Given a graph G , a non-empty set $S \subseteq E(G)$ is an *edge-cut* if $G - S$ has at least two connected components. An edge-cut S is called a *restricted edge-cut* if there are no isolated vertices in $G - S$. The *restricted edge-connectivity* $\lambda'(G)$ is the minimum cardinality over all *restricted edge-cuts* S .

Inspired by the above definition, Fàbrega and Fiol [17] proposed in 1994 the notion of *k -restricted edge-connectivity*, where k is a positive integer, generalizing this notion. An edge-cut S is called a *k -restricted edge-cut* if every component

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of $G - S$ has at least k vertices. Assuming that G has k -restricted edge-cuts, the k -restricted edge-connectivity of G , denoted by $\lambda_k(G)$, is defined as the minimum cardinality over all k -restricted edge-cuts of G , i.e.,

$$\lambda_k(G) = \min\{|S| : S \subseteq E(G) \text{ is a } k\text{-restricted edge-cut}\}.$$

Note that for any graph G , $\lambda_1(G)$ is the size of a minimum edge-cut, and $\lambda_2(G) = \lambda'(G)$. A connected graph G is called λ_k -connected if $\lambda_k(G)$ exists. Let $[X, Y]$ denote the set of edges between two disjoint vertex sets $X, Y \subseteq V(G)$, and let \bar{X} denote the complement $\bar{X} = V(G) \setminus X$ of vertex set X . It is clear that for any k -restricted cut $[X, \bar{X}]$ of size $\lambda_k(G)$, the graph $G - [X, \bar{X}]$ has exactly two connected components.

Very recently, Chitnis *et al.* [6] defined the notion of *good edge separation* for algorithmic purposes. For two positive integers k and ℓ , a partition (X, \bar{X}) of the vertex set of a connected graph G is called a (k, ℓ) -good edge separation if $|X|, |\bar{X}| > k$, $|[X, \bar{X}]| \leq \ell$, and both $G[X]$ and $G[\bar{X}]$ are connected. That is, it holds that $\lambda_k(G) \leq \ell$ if and only if G admits a $(k - 1, \ell)$ -good edge separation. Thus both notions, which have been defined independently and for which there existed no connection so far, are essentially the same.

Good edge separations turned out to be very useful for designing parameterized algorithms for cut problems [6], by using a technique known as *recursive understanding*, which basically consists in breaking up the input graph into highly connected pieces in which the considered problem can be *efficiently* solved. It should be mentioned that Kawarabayashi and Thorup [23] had defined before a very similar notion for *vertex-cuts* and introduced the idea of recursive understanding. This technique has also been subsequently used in [10, 24, 29].

Very little is known about the complexity of computing the k -restricted edge-connectivity of a graph, in spite of its extensive study in combinatorics. In this article we initiate a systematic analysis on this topic, with special emphasis on the parameterized complexity of the problem. In a nutshell, the main idea is to identify relevant parameters of the input of some problem, and study how the running time of an algorithm solving the problem depends on the chosen parameters. See [8, 13, 18, 28] for introductory textbooks to this area.

Our results. We consider the following two problems concerning the k -restricted edge-connectivity of a graph.

EXISTENTIAL RESTRICTED EDGE-CONNECTIVITY (EREC)

Instance: A graph $G = (V, E)$ and a positive integer k .

Question: Is G λ_k -connected ?

RESTRICTED EDGE-CONNECTIVITY (REC)

Instance: A connected graph $G = (V, E)$ and a positive integer k .

Output: $\lambda_k(G)$, or a correct report that G is not λ_k -connected.

The latter problem can be seen as a generalization of computing a MINIMUM CUT in a graph, which is polynomial-time solvable [30]. In Section 2 we prove that it is NP-hard, even restricted to λ_k -connected graphs. In Section 3 we study the parameterized complexity of the REC problem. More precisely, given a connected graph G and two integers k and ℓ , we consider the problem of determining whether $\lambda_k(G) \leq \ell$. Existing results concerning good edge separations imply that the problem is FPT when parameterized by k and ℓ . We prove that it is W[1]-hard when parameterized by k , and that it is FPT when parameterized by ℓ but unlikely to admit polynomial kernels. Moreover, we prove that the EREC problem is FPT when parameterized by k . Finally, in Section 4 we also consider the maximum degree Δ of the input graph as a parameter, and we prove that the EREC problem remains NP-complete in graphs with $\Delta \leq 5$, and that the REC problem is FPT when parameterized by k and Δ . Note that this implies, in particular, that the REC problem parameterized by k is FPT in graphs of bounded degree. Table 1 below summarizes the results of this article.

Problem	Classical complexity	Parameterized complexity with parameter			
		$k + \ell$	k	ℓ	$k + \Delta$
Is G λ_k -connected ?	<small>NPc</small> , even if $\Delta \leq 5$ (Thm 7)	★	<small>FPT</small> (Thm 4)	★	<small>FPT</small> (Thm 8)
$\lambda_k(G) \leq \ell$?	<small>NPh</small> , even if G is λ_k -connected (Thm 1)	<small>FPT</small> (Thm 2, by [6])	<small>W[1]</small> -hard (Thm 3)	<small>FPT</small> (Thm 5) No poly kernels (Thm 6)	<small>FPT</small> (Thm 8)

Table 1. Summary of our results, where Δ denotes the maximum degree of the input graph G , and NPc (resp. NPh) stands for NP-complete (resp. NP-hard). The symbol ‘★’ denotes that the problem is not defined for that parameter.

Further research. Some open questions are determining the existence of polynomial kernels for the REC problem with parameters $k + \ell$ or $k + \Delta$, speeding-up the FPT algorithm of Theorem 5 (which is quite inefficient), improving the bound on the maximum degree in Theorem 7, and studying the (parameterized) complexity of the REC problem in planar graphs and other sparse graph classes.

Notation. We use standard graph-theoretic notation; see for instance [11]. For a graph G , let $\Delta(G)$ denote its maximum degree, and for a vertex v , its degree in G is denoted by $d_G(v)$. If $S \subseteq V(G)$, we define $G - S = G[V(G) \setminus S]$, and if $S \subseteq E(G)$, we define $G - S = (V(G), E(G) \setminus S)$. Unless stated otherwise, throughout the article n denotes the number of vertices of the input graph of the problem under consideration. We will always assume that the input graphs are connected.

2 Preliminary results

Clearly, any connected graph G is λ_1 -connected, and $\lambda_1(G)$ can be computed in polynomial time by a MINIMUM CUT algorithm (cf. [30]). However, for $k \geq 2$, there exist infinitely many connected graphs which are not λ_k -connected, such as the graphs containing a cut vertex u such that every component of $G - u$ has at most $k - 1$ vertices (these graphs are called *flowers* in the literature [4], and correspond exactly to stars when $k = 2$). Moreover, the EREC problem is hard. Indeed, given a graph G , if n is even and $k = n/2$, by [14, Theorem 2.2] it is NP-complete to determine whether G contains two vertex-disjoint connected subgraphs of order $n/2$ each. We can summarize this discussion as follows.

Remark 1. *The EREC problem is NP-complete.*

In Section 4 we will strengthen the above hardness result to the case where the maximum degree of the input graph is at most 5.

Note that Remark 1 implies that the REC problem is NP-hard. Furthermore, even if the input graph G is guaranteed to be λ_k -connected, computing $\lambda_k(G)$ remains hard, as shown by the following theorem.

Theorem 1. *The REC problem is NP-hard restricted to λ_k -connected graphs.*

PROOF: We prove it for n even and $k = n/2$. The reduction is from the MINIMUM BISECTION problem¹ restricted to connected 3-regular graphs, which is known to be NP-hard [2]. Given a 3-regular connected graph G with even number of vertices as instance of MINIMUM BISECTION, we construct from it an instance G' of REC by adding two non-adjacent universal vertices v_1 and v_2 . Note that G' is $\lambda_{n/2}$ -connected, since any bipartition of $V(G')$ containing v_1 and v_2 in different parts induces two connected subgraphs.

We claim that v_1 and v_2 should necessarily belong to different connected subgraphs in any optimal solution in G' . Indeed, let (V_1, V_2) be a bipartition of $V(G)$ such that $|[V_1, V_2]| = \lambda_{n/2}(G')$, and assume for contradiction that $v_1, v_2 \in V_1$. Since G is connected, there is a vertex $u \in V_2$ with at least one neighbor in $V_1 \setminus \{v_1, v_2\}$. Let $V'_1 := V_1 \cup \{u\} \setminus \{v_2\}$ and $V'_2 := V_2 \cup \{v_2\} \setminus \{u\}$, and note that both $G[V'_1]$ and $G[V'_2]$ are connected. Since u has at least one neighbor in $V_1 \setminus \{v_1, v_2\}$, G is 3-regular, and v_1 and v_2 are non-adjacent and adjacent to all other vertices of G' , it can be checked that $|[V'_1, V'_2]| \leq |[V_1, V_2]| - 1 = \lambda_{n/2}(G') - 1$, contradicting the definition of $\lambda_{n/2}(G')$.

Therefore, solving the REC problem in G' corresponds exactly to solving the MINIMUM BISECTION problem in G , concluding the proof. \square

¹ Given a graph G with even number of vertices, the MINIMUM BISECTION problem consists in partitioning $V(G)$ into two equally-sized parts minimizing the number of edges with one endpoint in each part.

3 A parameterized analysis

The NP-hardness results of the previous section naturally lead to considering parameterized versions of the problem. In this section we consider the following three distinct parameterizations.

PARAMETERIZED RESTRICTED EDGE-CONNECTIVITY (**p**-REC)

Instance: A connected graph $G = (V, E)$ and two integers k and ℓ .

Parameter 1: The integers k and ℓ .

Parameter 2: The integer k .

Parameter 3: The integer ℓ .

Question: $\lambda_k(G) \leq \ell$?

As mentioned in the introduction, determining whether $\lambda_k(G) \leq \ell$ corresponds exactly to determining whether G admits a $(k-1, \ell)$ -good edge separation. This latter problem has been recently shown to be solvable in time $2^{O(\min\{k, \ell\} \log(k+\ell))} \cdot n^3 \log n$ by Chitnis *et al.* [6, Lemma II.2].

Theorem 2 (Chitnis *et al.* [6]). *The p-REC problem is FPT when parameterized by both k and ℓ .*

We would like to note that any improvement on the running time of the algorithm behind Theorem 2 would answer an open question raised in [7, 9], and would have direct consequences and improve the algorithms described in [6, 10, 24].

As pointed out in [16, 21], the **p**-REC problem can be solved in time $O^*(n^{2k})$. Roughly, the idea is to guess two sets of k vertices inducing a connected subgraph, contract them into two vertices s and t , and then call a polynomial-time MINIMUM CUT algorithm between s and t (cf. [30]). In other words, it is in XP when parameterized by k . The following theorem shows that this is essentially the best algorithm we can hope for when the parameter is only k . Indeed, since the blow-up of the parameter in the reduction is linear, the fact that k -CLIQUE cannot be solved in time $f(k) \cdot n^{o(k)}$ unless an unlikely collapse occurs in parameterized complexity theory [5] implies that the **p**-REC problem cannot be solved in time $f(k) \cdot n^{o(k)}$ either.

Theorem 3. *The p-REC problem is W[1]-hard when parameterized by k .*

PROOF: We reduce from k -CLIQUE, which is known to be W[1]-hard [13]. The parameterized reduction is the same as the one given by Downey *et al.* in [12, Theorem 2] to show the W[1]-hardness of the CUTTING k VERTICES FROM A GRAPH problem, only the analysis changes.

Let $G = (V, E)$ be an n -vertex graph for which we wish to determine whether it has a k -clique. We construct a graph G' as follows:

- (1) We start with a clique C of size n^3 and n representative vertices corresponding bijectively with the vertices of G .

- (2) Every representative vertex v is connected to $n^2 - d_G(v)$ arbitrary vertices of C .
- (3) If $uv \in E(G)$ then $uv \in E(G')$.

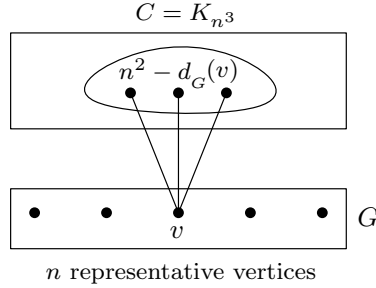


Fig. 1. Illustration of the graph G' in the proof of Theorem 3.

See Fig. 1 for an illustration of G' . Consider $\ell = kn^2 - 2\binom{k}{2}$ and take $k \leq n/2$. We claim that G has a k -clique if and only if G' is a YES-instance of **p-REC**.

Suppose first that $K \subseteq V(G)$ is a k -clique in G . Obviously, K is connected in G' and has k vertices. On the other hand, $G' - K$ is also connected with at least $n^3 - |K| > k$ vertices. Finally, it is straightforward to check that $|[K, V(G') \setminus K]| = kn^2 - 2\binom{k}{2} = \ell$.

In the other direction, suppose G' has a k -restricted edge-cut with at most ℓ edges, i.e., there exists $K \subseteq V(G')$ such that $G'[K]$ and $G' - K$ are connected, $|K| \geq k$, $|V(G') \setminus K| \geq k$, and $|[K, V(G') \setminus K]| \leq \ell$. Two cases need to be distinguished.

Case 1. $C \cap K = \emptyset$. Then every vertex of K must be a representative vertex. Hence $|[K, V(G') \setminus K]| = |K|n^2 - 2|E(G'[K])|$ since every representative vertex has degree n^2 . As by hypothesis $|[K, V(G') \setminus K]| \leq \ell = kn^2 - 2\binom{k}{2}$, it follows that $|E(G'[K])| = \binom{k}{2}$, hence K must be a k -clique.

Case 2. $C \cap K \neq \emptyset$. Note that for every bipartition (C_1, C_2) of C we have that $|[C_1, C_2]| \geq n^3 - 1 > \ell$. Now suppose $C \cap (V(G') \setminus K) \neq \emptyset$ and consider the bipartition $C_1 = C \cap K$ and $C_2 = C \cap (V(G') \setminus K)$ of C . Then $|[K, V(G') \setminus K]| \geq |[C_1, C_2]| \geq n^3 - 1 > \ell$, a contradiction. Therefore, we have that $C \cap (V(G') \setminus K) = \emptyset$. The proof concludes by applying Case 1 to $V(G') \setminus K$ instead of K . \square

In contrast to Theorem 3 above, we now prove that the EREC problem (which is NP-complete by Remark 1) is FPT when parameterized by k . The proof uses the technique of *splitters* introduced by Naor *et al.* [27], which has also been recently used for designing parameterized algorithms in [6, 10, 24]. Our main tool is the following lemma.

Lemma 1 (Chitnis *et al.* [6]). *There exists an algorithm that given a set U of size n and two integers $a, b \in [0, n]$, outputs in time $2^{O(\min\{a, b\} \cdot \log(a+b))} \cdot n \log n$ a set $\mathcal{F} \subseteq 2^U$ with $|\mathcal{F}| = 2^{O(\min\{a, b\} \cdot \log(a+b))} \cdot \log n$ such that for every two sets $A, B \subseteq U$, where $A \cap B = \emptyset$, $|A| \leq a$, and $|B| \leq b$, there exists a set $S \in \mathcal{F}$ with $A \subseteq S$ and $B \cap S = \emptyset$.*

Theorem 4. *The EREC problem is FPT when parameterized by k . More precisely, it can be solved by an algorithm running in time $2^{O(k \log k)} \cdot n^2 \log n$.*

PROOF: We use the easy property that G is λ_k -connected if and only if G contains two vertex-disjoint trees T_1 and T_2 such that $|V(T_1)| \geq k$ and $|V(T_2)| \geq k$. In order to apply Lemma 1, we take $U = V(G)$ and $a = b = k$, obtaining in time $k^{O(k)} \cdot n \log n$ the desired family \mathcal{F} of subsets of vertices of G . Now, if such trees T_1 and T_2 exist, then necessarily there exists a set $S \in \mathcal{F}$ such that $V(T_1) \subseteq S$ and $V(T_2) \cap S = \emptyset$. Therefore, in order to determine whether G is λ_k -connected or not, it suffices to check, for each set $S \in \mathcal{F}$, whether both $G[S]$ and $G - S$ contain a connected component with at least k vertices. (Note that for each such set $S \in \mathcal{F}$, this can be done in linear time.) Indeed, if such a set S exists, then clearly G is λ_k -connected. Otherwise, by the property of the family \mathcal{F} , G does not contain two disjoint trees T_1 and T_2 of size k each, and therefore G is not λ_k -connected. \square

Concerning the parameterized complexity of the **p**-REC problem, in view of Theorems 2 and 3, it just remains to settle the case when the parameter is ℓ only. The following theorem provides an answer to this question. We will need the following result, which is a reformulation of [31, Corollary 1].

Lemma 2 (van Bevern *et al.* [31]). *Given a graph G on n vertices and an integer ℓ , determining whether $\lambda_{\lfloor n/2 \rfloor}(G) \leq \ell$ can be solved in time $f(\ell) \cdot n^{11}$ for some explicit function f depending only on ℓ .*

Theorem 5. *The **p**-REC problem is FPT when parameterized by ℓ .*

PROOF: If $k \leq \ell$, we solve the problem using the FPT algorithm given by Theorem 2 with parameter $k + \ell \leq 2\ell$. Otherwise, in the case where $k > \ell$, we proceed to Turing-reduce the problem to the particular case where $k = \lfloor n/2 \rfloor$, which is FPT with parameter ℓ by Lemma 2 above, as follows. For each vertex v of the n -vertex input graph G , and for each integer p with $0 \leq p \leq n - 2k$, let G_v^p be the graph obtained from G by adding a clique K_p on p vertices and all the edges between vertex v and the vertices in K_p .

Claim. It holds that $\lambda_k(G) \leq \ell$ if and only if there exist a vertex $v \in V(G)$ and an integer p with $0 \leq p \leq n - 2k$ such that $\lambda_{\lfloor \frac{n+p}{2} \rfloor}(G_v^p) \leq \ell$.

PROOF: Assume first that $\lambda_k(G) \leq \ell$. Let (X, \bar{X}) be a partition of $V(G)$ achieving $\lambda_k(G)$, where we assume without loss of generality that $X \geq \bar{X}$, let $p = |X| - |\bar{X}|$, and let v be any vertex in \bar{X} . Then $p \leq (n - k) - k = n - 2k$ and by construction it holds that $\lambda_{\lfloor \frac{n+p}{2} \rfloor}(G_v^p) \leq \lambda_k(G) \leq \ell$.

Conversely, suppose that there exist a vertex $v \in V(G)$ and an integer p with $0 \leq p \leq n - 2k$ such that $\lambda_{\lfloor \frac{n+p}{2} \rfloor}(G_v^p) \leq \ell$. Let (X, \bar{X}) be a partition of $V(G_v^p)$ achieving $\lambda_{\lfloor \frac{n+p}{2} \rfloor}(G_v^p)$, and assume without loss of generality that $v \in \bar{X}$. We claim that the clique K_p is entirely contained in \bar{X} . Indeed, suppose for contradiction that $K_p \cap X \neq \emptyset$, and let $K = K_p \cap X$. Since $G_v^p[X]$ is connected

and $v \in \overline{X}$, necessarily $X = K$. We distinguish two cases. If $p < k$, then $|X| = |K| \leq p - 1 \leq k - 2 \leq n/2 - 2 < \lfloor \frac{n+p}{2} \rfloor$, contradicting the definition of $\lambda_{\lfloor \frac{n+p}{2} \rfloor}(G_v^p)$. Otherwise, if $p \geq k$, then we use that the number of edges in $[X, \overline{X}]$ is at least the minimum cut of the clique of size $p+1$ induced by $K_p \cup \{v\}$, which is equal to p . That is, $|[X, \overline{X}]| \geq p \geq k > \ell$, contradicting the hypothesis that (X, \overline{X}) is a partition of $V(G_v^p)$ achieving $\lambda_{\lfloor \frac{n+p}{2} \rfloor}(G_v^p) \leq \ell$.

We now claim that the partition of $V(G)$ given by $(X, \overline{X} \setminus K_p)$ defines a k -restricted edge-cut of G with at most ℓ edges, concluding the proof. First note that both $G[X]$ and $G[\overline{X} \setminus K_p]$ are connected, since both $G_v^p[X]$ and $G_v^p[\overline{X}]$ are connected by hypothesis, and the removal of K_p from $G_v^p[\overline{X}]$ clearly preserves connectivity. On the other hand, we have that $|X| \geq \lfloor \frac{n+p}{2} \rfloor \geq \lfloor \frac{n}{2} \rfloor \geq k$ and $|\overline{X} \setminus K_p| = |\overline{X}| - p \geq \lfloor \frac{n+p}{2} \rfloor - p \geq \frac{n-p-1}{2} \geq \frac{n-(n-2k)-1}{2} = k - \frac{1}{2}$, and since both $|\overline{X} \setminus K_p|$ and k are integers, the latter inequality implies that $|\overline{X} \setminus K_p| \geq k$. Finally, since $K_p \subseteq \overline{X}$, it holds that $|[X, \overline{X} \setminus K_p]| = |[X, \overline{X}]| \leq \ell$. \diamond

Note that the above claim yields the theorem, as it implies that the problem of deciding whether $\lambda_k(G) \leq \ell$ can be solved by invoking $O(n^2)$ times the FPT algorithm given by Lemma 2. \square

To complement Theorem 5, in the next theorem we prove that the **p**-REC problem does not admit polynomial kernels when parameterized by ℓ , unless $\text{coNP} \subseteq \text{NP/poly}$.

Theorem 6. *Unless $\text{coNP} \subseteq \text{NP/poly}$, the **p**-REC problem does not admit polynomial kernels when parameterized by ℓ .*

PROOF: The proof is strongly inspired by the one given by van Bevern *et al.* [31, Theorem 3] to prove that the MINIMUM BISECTION problem does not admit polynomial kernels, which in turn resembles the proof given by Garey *et al.* [20] to prove the NP-hardness of MINIMUM BISECTION. The main difference with respect to the proof given in [31] is that we need to make the appropriate modifications to guarantee that both parts left out by the edge-cut are *connected*, which is not an issue in the MINIMUM BISECTION problem.

We will first rule out the existence of polynomial kernels for the generalization of **p**-REC where the edges have non-negative integer weights, and the objective is to decide whether the input graph can be partitioned into two connected subgraphs with at least k vertices each by removing a set of edges whose total weight does not exceed ℓ . We call this problem EDGE-WEIGHTED **p**-REC. Then it will just remain to get rid of the edge weights. This is done at the end of the proof of the theorem.

As shown by Bodlaender *et al.* [3], in order to prove that EDGE-WEIGHTED **p**-REC does not admit polynomial kernels when parameterized by ℓ (assuming that $\text{coNP} \subseteq \text{NP/poly}$), it is sufficient to define a *cross composition* from an NP-hard problem to EDGE-WEIGHTED **p**-REC. In our case, the NP-hard problem is MAXIMUM CUT (see [19]), which is defined as follows. Given a graph $G = (V, E)$ and an integer p , one has to decide whether V can be partitioned into two sets

A and B such that there are at least p edges with an endpoint in A and an endpoint in B .

A cross composition from MAXIMUM CUT to EDGE-WEIGHTED \mathbf{p} -REC parameterized by ℓ consists of a polynomial-time algorithm that, given t instances $(G_1, p_1), \dots, (G_t, p_t)$ of MAXIMUM CUT, constructs an instance (G^*, k, ℓ) of EDGE-WEIGHTED \mathbf{p} -REC such that (G^*, k, ℓ) is a YES-instance if and only if one of the t instances of MAXIMUM CUT is a YES-instance, and such that ℓ is polynomially bounded as a function of $\max_{1 \leq i \leq t} |V(G_i)|$. Similarly to [31], we may safely assume that for each $1 \leq i \leq t$ we have $|V(G_i)| =: n$ and $p_i =: p$ (by the arguments given in [3]), that $1 \leq p \leq n^2$ (as if $p = 0$ all instances are YES-instances, and if $p > n^2$ all instances are NO-instances), and that t is odd (otherwise, we can add a NO-instance consisting of an edgeless graph on n vertices).

Given $(G_1, p), \dots, (G_t, p)$, we create G^* as follows; see Fig. 2 for an illustration. Let $w_1 := 5n^2$ and $w_2 := 5$. For each graph $G_i = (V_i, E_i)$ add to G^* the vertices in V_i and a clique V'_i with $|V'_i| = n$ vertices whose edges have weight w_1 . Add an edge of weight w_1 between each vertex in V_i and each vertex in V'_i . For each pair of vertices $u, v \in V_i$, add the edge $\{u, v\}$ to G^* with weight $w_1 - w_2$ if $\{u, v\} \in E_i$, and with weight w_1 otherwise. Let s_i^1, s_i^2 be two arbitrary distinct vertices in V'_i , which we call *link* vertices. For $1 \leq i \leq t - 1$, add two edges with weight 1 between s_i^1 and s_{i+1}^1 and between s_i^2 and s_{i+1}^2 , and two edges with weight 1 between s_i^1 and s_1^1 and between s_i^2 and s_1^2 . This completes the construction of G^* . These $2t$ edges among distinct V'_i 's are called *chain* edges (cf. the thicker edges in Fig. 2). Finally, we set $k := |V(G^*)|/2$ and $\ell := w_1 n^2 - w_2 p + 4$. Note that k is *not* polynomially bounded in terms of n , but this is not a problem since the parameter we consider is ℓ , which is bounded by $5n^4$. This construction can be clearly performed in polynomial time in $t \cdot n$. We claim that (G^*, k, ℓ) is a YES-instance of EDGE-WEIGHTED \mathbf{p} -REC if and only if there exists $i \in \{1, \dots, t\}$ such that (G_i, p) is a YES-instance of MAXIMUM CUT.

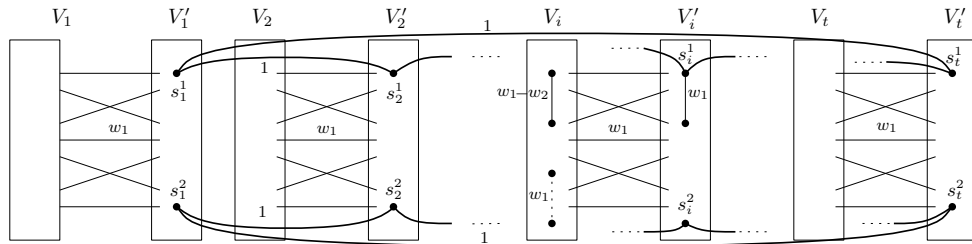


Fig. 2. Illustration of the graph G^* in the proof of Theorem 6.

Assume first that there exists $i \in \{1, \dots, t\}$ such that (G_i, p) is a YES-instance of MAXIMUM CUT. Assume without loss of generality that $i = 1$, and let $V_1 = A \uplus B$ such that there are at least p edges in E_1 between A and B . We proceed to partition $V(G^*)$ into two equally-sized sets A' and B' such that both $G^*[A']$ and $G^*[B']$ are connected, and such that the total weight of the edges in G^* with one endpoint in A' and one endpoint in B' is at most ℓ . The set A' contains $V_1 \cap A$, any set of $|B|$ vertices in V'_1 containing exactly one of s_1^1 and s_1^2 (this is possible since $1 \leq |B| \leq n - 1$), and $\bigcup_{i=2}^{\lceil t/2 \rceil} V_i \cup V'_i$. Then $B' = V(G^*) \setminus A'$. Since t is odd, $|A'| = |B'|$. Let us now see that $G^*[A']$ is connected. As $1 \leq |A| \leq n - 1$, the set $V_1 \cap A'$ is connected to $V'_1 \cap A'$, which is connected to V'_2 since A' contains exactly one of the link vertices s_1^1 and s_1^2 . The graph $G^*[\bigcup_{i=2}^{\lceil t/2 \rceil} V_i \cup V'_i]$ is clearly connected because of the chain edges, which implies that $G^*[A']$ is indeed connected. The proof for the connectivity of $G^*[B']$ is similar, using that $V_1 \cap B'$ is connected to $V'_1 \cap B'$ since $1 \leq |B| \leq n - 1$, which is in turn connected to V'_t since B' contains exactly one of the link vertices s_1^1 and s_1^2 . Finally, let us show that the total weight of the edges between A' and B' is at most ℓ . Note first that two chain edges incident to V'_1 and two chain edges incident to $V'_{\lceil t/2 \rceil}$ belong to the cut defined by A' and B' , and no other chain edge belongs to the cut. Beside the chain edges, only edges in the graph $G^*[V_1 \cup V'_1]$ are cut. Note that $G^*[V_1 \cup V'_1]$ is a clique on $2n$ vertices and each of $(V_1 \cup V'_1) \cap A'$ and $(V_1 \cup V'_1) \cap B'$ contains n vertices. Since $|[A, B]| \geq p$, at least p of the edges of weight $w_1 - w_2$ belong to the cut. Therefore, the total weight of the cut is at most $w_1 n^2 - w_2 p + 4 = \ell$.

Conversely, assume that for all $i \in \{1, \dots, t\}$, (G_i, p) is a NO-instance of MAXIMUM CUT, and we want to prove that (G^*, k, ℓ) is a NO-instance of EDGE-WEIGHTED **p**-REC. Let $A \uplus B$ be a partition of $V(G^*)$ such that $|A| = |B|$ and both $G^*[A]$ and $G^*[B]$ are connected, and such that the weight of the edges between A and B is minimized among all such partitions. For $1 \leq i \leq t$, we let $a_i := |(V_i \cup V'_i) \cap A|$. Since for $1 \leq i \leq t$, (G_i, p) is a NO-instance of MAXIMUM CUT, any bipartition of V_i cuts at most $p - 1$ edges. Therefore, the total weight of the edges between $(V_i \cup V'_i) \cap A$ and $(V_i \cup V'_i) \cap B$ is at least $w_1 a_i (2n - a_i) - (p - 1) w_2$.

Since t is odd, necessarily at least one of the graphs G_i is cut by $A \uplus B$. Assume first that exactly one graph G_i is cut by $A \uplus B$. Since $|A| = |B|$, we have that $a_i = n$, so the value of the cut is at least $w_1 n^2 - (p - 1) w_2 = w_1 n^2 - p w_2 + w_2 > w_1 n^2 - p w_2 + 4 = \ell$, and thus (G^*, k, ℓ) is a NO-instance of EDGE-WEIGHTED **p**-REC.

We claim that there is always exactly one graph G_i cut by $A \uplus B$. Assume for contradiction that it is not the case, that is, that there are two strictly positive values a_i, a_j for some $i \neq j$. By symmetry between A and B , we may assume that $a_i + a_j \leq 2n$. The total weight of the edges cut in $G^*[V_i \cup V'_i]$ and $G^*[V_j \cup V'_j]$ is at least

$$\begin{aligned} & w_1 a_i (2n - a_i) - (p - 1) w_2 + w_1 a_j (2n - a_j) - (p - 1) w_2 = \\ & 2n w_1 (a_i + a_j) - w_1 (a_i^2 + a_j^2) - 2w_2 (p - 1). \end{aligned}$$

Now we construct another solution $A' \uplus B'$ of EDGE-WEIGHTED **p**-REC in G^* where the $a_i + a_j$ vertices are cut in only one of $V_i \cup V'_i$ and $V_j \cup V'_j$, say $V_i \cup V'_i$ (note that this is possible since $a_i + a_j \leq 2n$). In order to do so, as far as there exists such a pair a_i, a_j , we proceed as follows. If $a_i + a_j = 2n$, then $V_i \cup V'_i$ is entirely contained in A' or B' . Otherwise, if $a_i + a_j < 2n$, then we choose $(V_i \cup V'_i) \cap A'$ such that it contains exactly one of s_i^1 and s_i^2 . At the end of this procedure, exactly one graph G_i is cut. Finally, we arrange the other G_ℓ 's, with $\ell \neq i$, consecutively into A' and B' . That is, we put the vertices of $\bigcup_{\ell=i+1}^{i+\lceil t/2 \rceil} V_\ell \cup V'_{\ell'}$ into A' , and the vertices of $\bigcup_{\ell=i+\lceil t/2 \rceil}^{i-1} V_\ell \cup V'_{\ell'}$ into B' , where the indices are counted cyclically from 1 to t . The connectivity of both $G^*[A']$ and $G^*[B']$ is guaranteed by the chain edges and the choice of the selector vertices s_i^1 and s_i^2 .

Let i, j be two indices for which the above procedure has been applied. Taking into account that each V'_i has four incident chain edges, the total weight of the edges cut in $G^*[V_i \cup V'_i]$ and $G^*[V_j \cup V'_j]$ by the new solution is at most

$$\begin{aligned} & w_1(a_i + a_j)(2n - a_i - a_j) + 8 = \\ & 2nw_1(a_i + a_j) - w_1(a_i^2 + a_j^2) - 2w_1a_ia_j + 8. \end{aligned}$$

That is, the weight of the cut defined by $A \uplus B$ minus the weight of the cut defined by $A' \uplus B'$ is at least

$$\begin{aligned} & -2w_2(p - 1) + 2w_1a_ia_j - 8 = \\ & 2(w_1a_ia_j - w_2(p - 1) - 4) \geq \\ & 2(w_1 - w_2(n^2 - 1) - 4) > 0, \end{aligned}$$

where we have used that $a_i, a_j \geq 1$, $p \leq n^2$, $w_1 = 5n^2$, and $w_2 = 5$. In other words, $A' \uplus B'$ defines a cut of strictly smaller weight, contradicting the definition of $A \uplus B$.

To conclude the proof of the theorem, it just remains to deal with the edge weights. As in [31], we show how to convert the instance (G^*, k, ℓ) of EDGE-WEIGHTED **p**-REC that we just constructed into an equivalent instance of **p**-REC such that the resulting parameter remains polynomial in n . Given (G^*, k, ℓ) , we define (\hat{G}, k, ℓ) as the instance of **p**-REC, where \hat{G} is an unweighted graph obtained from G^* as follows. We replace each vertex v of G^* with a clique C_v of size $w_1 + \ell + 1$, and for each edge $\{u, v\}$ of G^* with weight w , we add w pairwise disjoint edges between the cliques C_u and C_v . Since no cut of size at most ℓ in \hat{G} can separate a clique C_v introduced for a vertex v , it follows that (G^*, k, ℓ) is a YES-instance of EDGE-WEIGHTED **p**-REC if and only if (\hat{G}, k, ℓ) is a YES-instance of **p**-REC. Finally, it is clear that the desired cut size ℓ is still polynomial in n . \square

4 Considering the maximum degree as a parameter

Towards understanding the parameterized complexity of the REC problem, one may wonder whether considering the maximum degree of the input graph as

an extra parameter turns the problem easier (this is a classical approach in parameterized complexity, see for instance [25, 26]). We first prove that, from a classical complexity point of view, bounding the degree of the input graph does not turn the problem easier. Before stating the hardness result, we need to define the 3-DIMENSIONAL MATCHING problem, 3DM for short.

An instance of 3DM consists of a set $W = R \cup B \cup Y$, where R, B, Y are disjoint sets with $|R| = |B| = |Y| = m$, and a set of triples $T \subseteq R \times B \times Y$. The question is whether there exists a matching $M \subseteq T$ covering W , i.e., $|M| = m$ and each element of $W = R \cup B \cup Y$ occurs in exactly one triple of M .

An instance of 3DM can be represented by a bipartite graph $G_I = (W \cup T, E_I)$, where $E_I = \bigcup_{t=(r,b,y) \in T} \{\{r, t\}, \{b, t\}, \{y, t\}\}$; see Fig. 3.

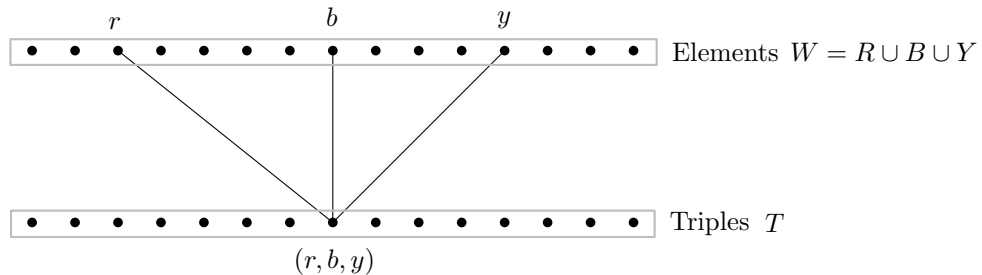


Fig. 3. Representation of an instance of 3DM.

It is known that 3DM is NP-complete even if each element of W appears in 2 or 3 triples only [14, 15]. In [14, Theorem 2.2] it is proved that partitioning a graph G into two connected subgraphs of equal size is NP-hard, using a reduction from 3DM. It is worth noting that the graph constructed in the NP-hardness reduction contains only two vertices of degree greater than five. In Theorem 7 we appropriately modify the reduction of [14, Theorem 2.2] so that the constructed graph has maximum degree at most 5.

Theorem 7. *The EREC problem is NP-complete even if the maximum degree of the input graph is 5.*

PROOF: Given an instance (W, T) of 3DM with $W = R \cup B \cup Y$, $|R| = |B| = |Y| = m$, and $T \subseteq R \times B \times Y$ such that each element of W appears in 2 or 3 triples only, we define an n -vertex graph $G = (V, E)$ with maximum degree 5 as follows (see Fig. 4 for an illustration).

The set of vertices of G is

$$V = W \cup T \cup T_a \cup T_b \cup \mathcal{P} \cup \{a\},$$

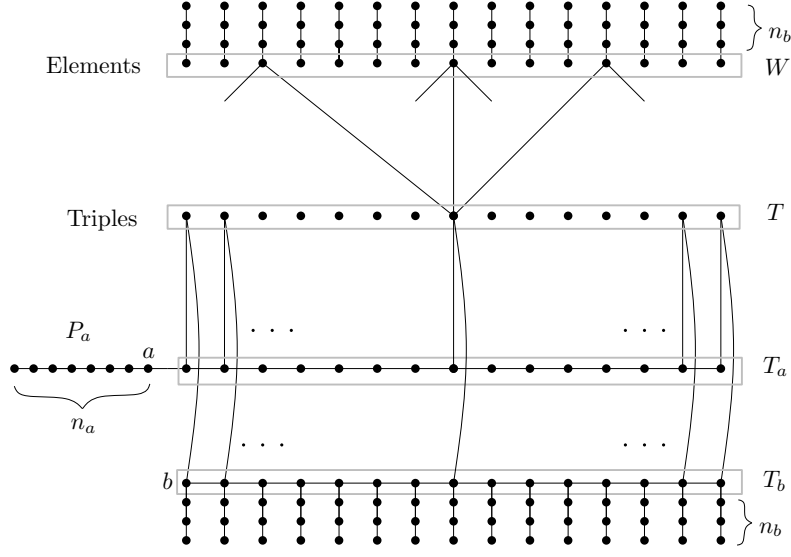


Fig. 4. Construction of the graph G in the proof of Theorem 7, with $\Delta(G) = 5$.

where $T_a = \{t_1^a, \dots, t_{|T|}^a\}$, $T_b = \{t_1^b, t_2^b, \dots, t_{|T|}^b\}$, $T = \{t_1, \dots, t_{|T|}\}$ is the set of triples, and $\mathcal{P} = \bigcup_{\sigma \in W \cup T_b \cup \{a\}} P_\sigma$, where $P_\sigma = \{(\sigma, t) : t = 1, \dots, n_\sigma\}$ with $n_a = (3m + |T|)n_b + 5m - |T| - 1$, $n_b = 2m^3$, and $n_\sigma = n_b$ for every $\sigma \in W \cup T_b$. The set of edges of G is

$$E = E_I \cup E_{T_a} \cup E_{T_b} \cup E_{T^+} \cup \bigcup_{\sigma \in W \cup T_b \cup \{a\}} E_\sigma,$$

where $E_{T_a} = \{\{t_i^a, t_{i+1}^a\} : 1 \leq i \leq |T| - 1\}$, $E_{T_b} = \{\{t_i^b, t_{i+1}^b\} : 1 \leq i \leq |T| - 1\}$, $E_{T^+} = \{\{t_i, t_i^a\}, \{t_i, t_i^b\} : 1 \leq i \leq |T|\}$, and $E_\sigma = \{\{\sigma, (\sigma, 1)\}\} \cup \{\{(\sigma, t), (\sigma, t + 1)\} : 1 \leq t \leq n_\sigma - 1\} \cup \{a, t_1^a\}$ for every $\sigma \in W \cup T_b \cup \{a\}$.

Note that the maximum degree of G is indeed 5 (only vertices of T could get degree 5, all other vertices have degree at most 4). Since $n = 1 + 3m + 3|T| + n_a + (3m + |T|)n_b$, we can observe that

$$n = 2(n_a + 1 + 2|T| - m).$$

Next, we show that for $k = n/2$, G is YES-instance of the REC problem if and only if T contains a matching covering W .

One direction is easy. Suppose first that T contains a matching M covering W . Let $S = \{a\} \cup P_a \cup T_a \cup (T \setminus M)$. It is straightforward to check that $|S| = n/2$ and that $G[S]$, $G[V \setminus S]$ are both connected.

Conversely, suppose that G can be partitioned into 2 connected subgraphs $G[S]$, $G[V \setminus S]$ with $|S| = n/2$. We can assume that $a \in S$, and then it follows that

$P_a \subseteq S$. Now $|S \setminus (P_a \cup \{a\})| = 2|T| - m < 2m^3 = n_b$ since $|T| \leq m^3$. As $P_\sigma \subseteq S$ if and only if $\sigma \in S \cap (W \cup T_b)$, then $S \cap (W \cup T_b) = \emptyset$ since $|S \setminus (P_a \cup \{a\})| < n_b$ and $|P_\sigma| = n_b$ for every $\sigma \in W \cup T_b$. Hence $S \setminus (P_a \cup \{a\}) \subseteq T \cup T_a$. Let $M = (V \setminus S) \cap T$. Then $|M| \leq m$ since $|S \setminus (P_a \cup \{a\})| = 2|T| - m$. Finally, as $G[V \setminus S]$ is connected and $W \cup T_b \subseteq V \setminus S$, it follows that $|M| \geq m$. Hence $|M| = m$ and M must be a matching covering W . \square

In order to understand to which extent the vertices of high degree make the complexity of computing the restricted edge-connectivity of a graph hard, we also consider the maximum degree of the input graph as a parameter for the **p-REC** problem.

Theorem 8. *The **p-REC** problem is FPT when parameterized by k and the maximum degree Δ of the input graph.*

PROOF: The algorithm is based on a simple exhaustive search. We use the property that, for any graph G and any two integers k, ℓ , $\lambda_k(G) \leq \ell$ if and only if G contains two vertex-disjoint trees T_1 and T_2 with $|V(T_1)| \geq k$ and $|V(T_2)| \geq k$, such that there exists an edge set S in G with $|S| \leq \ell$ such that in $G - S$ the trees T_1 and T_2 belong to different connected components. Hence, we just have to determine whether these trees exist in G or not. For doing so, for every pair of distinct vertices v_1 and v_2 of G , we exhaustively consider all trees T_1 and T_2 with k vertices containing v_1 and v_2 , respectively. Note that the number of such trees is at most Δ^{2k} . For every pair of vertex-disjoint trees T_1 and T_2 , we proceed as follows. We contract tree T_1 (resp. T_2) to a single vertex t_1 (resp. t_2), keeping edge multiplicities, and then we run in the resulting graph a polynomial-time MINIMUM CUT algorithm between t_1 and t_2 (cf. [30]). If the size of the returned edge-cut is at most ℓ , then T_1 and T_2 are the desired trees. Otherwise, we continue searching. It is clear that the overall running time of this algorithm is $O(\Delta^{2k} \cdot n^{O(1)})$. \square

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