Theory and Applications of Bidimensionality

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Bidimensionality Theory is a meta-algorithmic theory whose main ingredients are the *Grid Exclusion Theorem* of the Graph Minors series of Robertson and Seymour and the celebrated *Courcelle's Theorem.* The grid exclusion theorem states that if a graph excludes a bidimensional grid as a minor (a graph H is a *minor* of a graph G if H can be obtained by some subgraph of G by contracting edges) then its structure topologically resembles the structure of a tree (in technical terms, it has small treewidth). Intuitively, this result says that the absence of the bidimensional structure of a grid implies that a graph has the "mono-dimensional" structure of a tree. On the other side, Courcelle's theorem states that if a problem on graphs is expressible in Monadic Second Order Logic (MSOL), then it is possible to solve it in linear time when the treewidth of their input graphs is fixed. Intuitively, this theorem expresses the fact that the mono-dimensional structure of a tree (i.e., small treewidth) makes it possible to treat a graph as the input string of a finite-state tree automaton, where the finiteness of its states is guaranteed by the MSOL-expressibility of the problem. Combining these two results together, we derive that the *absence* of the bidimensional structure of a grid, enables the applicability of the "divide-and-conquer" technique for problems of certain descriptive complexity. It appears that for many graph theoretic problems the existence of a grid-minor (or other bidimensional strucures) on the input graph provides a certificate for an immediate negative (or positive) answer and, for the remaining instances, a dynamic programming approach on graphs of bounded treewidth may give an answer to the problem. This phenomenon reveals fruitful interleave between graph structure and logic in graph algorithms. Bidimensionality Theory aims at systematizing this idea and extending its applicability in diverse paradigms of algorithm design.

The notion of problem bidimensionality was proposed for the first time in [**?**]. Given some graph invariant $\mathbf{p}$, we denote by $\Pi_{\mathbf{p}}$ the problem of asking, for some pair (G, k), whether $\mathbf{p}(G) \leq k$ and we say that $\Pi_{\mathbf{p}}$ is *minor bidimensional* if

i) $\mathbf{p}$ is minor closed, i.e. if G_1 is a minor of G_2, then $\mathbf{p}(G_1) \leq \mathbf{p}(G_2)$.
ii) for every k, $\mathbf{p}(L_k) = \Omega(k^2)$ (here, L_k is the $(k \times k)$-grid).

Some of the main meta-algorithmic results of Bidimensionality Theory can be summarized as follows: Let $\Pi_{\mathbf{p}}$ be a minor bidimensional problem and let $\mathcal{G}$ be a class of graphs where

$$\forall_{G\in\mathcal{G}}\ \textit{treewidth}(G) = O(\max\{k \mid\ L_k \text{ is a minor of } G\}). \tag{1}$$

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Let $\Pi_{\mathbf{p}}^{\mathcal{G}}$ be the restriction of $\Pi_{\mathbf{p}}$ to the graphs in $\mathcal{G}$. Then the following hold:

1. If $\mathbf{p}(G)$ can be computed in $2^{O(treewidth(G))} \cdot n^{O(1)}$ steps, then there exists an algorithm that decides, given a graph G in $\mathcal{G}$ as input, whether $\mathbf{p}(G) \leq k$, in $2^{O(\sqrt{k})} \cdot n^{O(1)}$ steps (see [?]).
2. If $\mathbf{p}$ satisfies some separability property (see [?, ?] for the definition) and $\mathbf{p}(G) \leq k \iff \exists S \subseteq V(G) : |S| \leq k$ and $(G, S) \models \psi$ where ψ is a MSOL sentence, then the problem $\Pi_{\mathbf{p}}$ admits a *linear kernel*, i.e., there exists a polynomial algorithm reducing every instance (G, k) of Π_{p} to an equivalence instance (G', k') where $|V(G')| = O(k)$ and $k' \leq k$ (see [?]).
3. If $\mathbf{p}$ satisfies some separability property and is reducible (in the sense these notions are defined in [?]), then there is an EPTAS for computing $\mathbf{p}(G)$ on the graphs in $\mathcal{G}$ (see [?, ?]).

According to [?], every graph class excluding some fixed graph as a minor satisfies (**??**). Further extensions of the applicability of the above theory on geometric graphs have been given in [?] and [?].

All above results concern only minor-closed parameters. The counterpart of the above theory for contraction-closed parameters is based on the notion of *contraction bidimensionality*, uses slightly different versions of Conditions **i**, **ii**, and (**??**), and its algorithmic potential is investigated in [?]. Currently the combinatorial challenge of Bidimensionality Theory is to broaden its applicability by detecting graph classes where (**??**) holds or to suitable adapt/extend/modify Conditions **i**, **ii**, and (**??**) so that similar results can be derived for wider families of graph theoretic problems.

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