Algorithms and Combinatorics on the Erdös–Pósa property
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### Some (basic and necessary) definitions
Minors and models in graphs

\( H \) is a minor of \( G \): \( H \) occurs from a subgraph of \( G \) by edge contractions

\begin{align*}
\text{\( H \)-model: any graph that contains \( H \) as a minor.} \\
\text{\( \mathcal{M}(H) \): the class of all minor models of \( H \).} \\
\text{\( H \)-minor free graphs: graphs that do not contain \( H \) as a minor.}
\end{align*}
Treewidth

- A vertex in $G$ is *simplicial* if its neighborhood induces a clique.
- A graph $G$ is a $k$-tree if one of the following holds
  - $G = K_{k+1}$ or
  - the removal of $G$ of a simplicial vertex creates a $k$-tree.
- The treewidth of a graph $G$ is defined as follows

$$\text{tw}(G) = \min\{k \mid G \text{ is a subgraph of some } k\text{-tree}\}$$
A 3-tree
A subgraph of a 3-tree: a graph with treewidth at most 3
Minor exclusion of a planar graph:

Theorem (Robertson and Seymour – GM V)

For every planar graph $H$ there is a constant $c_H$ such that if a graph $G$ is $H$-minor free, then $\text{tw}(G) \leq c_H$. 
Erdős-Pósa Theorem
Theorem (Erdős & Pósa 1965)

There exists a function $f$ such that For every $k$, every graph $G$ has either $k$ vertex disjoint cycles or $f(k)$ vertices that meet all of its cycles.

Facts:

- Gap: $f(k) = O(k \cdot \log k)$
- In the same paper they show that the gap $f(k) = O(k \log k)$ is tight

According to Diestel's monograph on graph theory:

- The same holds if we replace "vertices" by "edges".

[Graph Theory, 3rd Edition, Corollary 12.4.10 and Ex. 39 of Chapter 12]
Lemma

*Cycles have the E&P property on planar graphs with linear gap*

Proof.

Let $G$ be a graph without any cycle packing of size $\geq k$

- **Reduce**: We can assume that $G$ has no vertices of degree $\leq 2$.
- **Find**: A planar graph has always a face (cycle) of length $\leq 5$.

We build a *cycle covering* of $G$ by setting $C = \emptyset$ and repetitively

1. **Reduce** $G$ so that $\delta(G) \geq 3$.
2. **Find** a cycle of length $\leq 5$ and add its vertices to $C$.

The above finish after $\leq k$ rounds and creates a cycle cover $C$ of the input graph of at most $5k$ vertices.
Jones’ Conjecture:

Cycles have the E&P property on **planar graphs** with gap $2^k$.

► Wide Open (and **famous**)!
Fact: Linear gap extends to $H$-minor free graphs

We will derive the Fact by the following more general statement of Erdős-Pósa Theorem:

Theorem

*For each graph $H$, cycles have the E&P property for $H$-minor free graphs with gap $O(k \cdot \log h)$, where $h = |V(H)|$.*

E&P follows as a graphs with no $k$-cycle packings are $K_{3k}$-minor free.
We give a proof using the following results:

**Theorem (Thomassen 1983)**

*Given an integer $r$, every graph $G$ with $\text{girth}(G) \geq 8r + 3$ and $\delta(G) \geq 3$ has a minor $J$ with $\delta(J) \geq 2^r$.***

- $\text{girth}(G)$: minimum size of a cycle in $G$
- $\delta(G)$: minimum degree of $G$
- $J$ is a minor of $G$: $J$ occurs from a subgraph of $G$ by edge contractions.

**Theorem (Kostochka 1982 & Thomason 1984)**

$\exists \alpha \ \forall h \ \delta(G) \geq \alpha h \sqrt{\log h} \Rightarrow G$ contains $K_h$ as a minor
Proof.

Let $G$ be a $K_h$-free graph with no $k$-cycle packing

- **Reduce**: $\delta(G) \geq 3$

As $G$ is $H$-minor free, from 2nd theorem every minor $F$ of $G$ has

$\delta(F) \leq \alpha h \sqrt{\log h}$

Let $r$ be such that $\alpha h \sqrt{\log h} < 2^r$

From 1st theorem contains a cycle of length $< 8r = O(\log h)$.

We build a cycle covering of $G$ by setting $C = \emptyset$ and repetitively

1. **Reduce** $G$ so that $\delta(G) \geq 3$.

2. **Find** a cycle of length $O(\log h)$ and add its vertices to $C$.

The above finish after $< k$ rounds and creates a cycle cover of the input graph of at most $O(k \log h)$ vertices.
Algorithmic Remarks:

- Both **Reduce** and **Find**, can be implemented in poly-time.

Therefore there is a polynomial algorithm that, for every $k$, returns one of the following:

- a set of $k$ disjoint cycles or
- a cycle cover of $O(k \cdot \log k)$ vertices.
Algorithmic Remarks:

▶ We just derived an $O(\log(OPT))$-approximation algorithm for both the maximum size of a vertex cycle packing and the minimum size of a vertex cycle covering.

Moreover:

All previous proofs, results, and algorithms extend directly to the edge variants of the above problems.
Algorithmic Remarks:

- We just derived an $O(\log(OPT))$-approximation algorithm for both the maximum size of a edge cycle packing and the minimum size of a edge cycle covering.

Moreover:

All previous proofs, results, and algorithms extend directly to the edge variants of the above problems.
Extensions on minor models
Let $G$ and $C$ be graph classes.

**Question (About $G$ and $H$)**

Is there a function $f$ such that, for every $k$, every graph $G \in G$ has either $k$ vertex disjoint subgraphs in $C$ or $f(k)$ vertices that meet all subgraphs in $C$?

**Question (Optimizing the gap $f$)**

If the above question can be positively answered, what is the minimum $f$ for which this holds?

▶ We say that $C$ has the Erdős & Pósa property on $G$ with gap $f$.

▶ **Task**: detect such $C$ and $G$ and optimize the corresponding gap $f$.

▶ **Erdős & Pósa Theorem**:

Cycles have the E&P property on all graphs with gap $O(k \log k)$. 
[Recall that $\mathcal{M}(H)$ is the graph class containing all $H$-models]

**A vast generalization of Erdős-Pósa Theorem:**

**Theorem (Robertson & Seymour)**

*Given a graph $H$, $\mathcal{M}(H)$ has the E&P-property on all graphs iff $H$ is planar.*

- Original Erdős-Pósa theorem: $H = \text{“double edge”}.$
- “double edge” generalizes to any planar graph!!

We use $f_H$ for the gap of $\mathcal{M}(H)$
Theorem (Robertson & Seymour)

Given a graph $H$, $\mathcal{M}(H)$ has the E&P-property on all graphs iff $H$ is planar.

The proof of the “only if” is a corollary of the planar exclusion theorem:

Theorem (Robertson and Seymour – GM V)

For every planar graph $H$ there is a constant $c_H$ such that if a graph $G$ is $H$-minor free, then $\text{tw}(G) \leq c_H$.

Ideas of proof:

- if a graph $G$ does not contain any packing of $k$ models of $H$, then it excludes their disjoint union as a minor (that is planar).
- Therefore, $\text{tw}(G) \leq f(k, H) = w$.
- Let $G$ be a subgraph of a $w$-tree $R$
The graph is “tree-like”: 

[Graph image]
The proof of the general theorem

**Theorem (Robertson and Seymour – GM V)**

For every planar graph \( H \) there is a constant \( c_H \) such that if a graph \( G \) is \( H \)-minor free, then \( \text{tw}(G) \leq c_H \).

**Ideas of the “if” proof:** (we describe the case where \( H = K_5 \))
$H = K_5 \times 1$

A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $\Gamma_n$:

$\text{pack}_H(G) = 1$ but $\text{cover}_H(G) = \Theta(\sqrt{n})$
The proof of the general theorem

$H = K_5 \times \sqrt{n}$

A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid $\Gamma_n$:

$$\text{pack}_H(G) = 1 \quad \text{but} \quad \text{cover}_H(G) = \Theta(\sqrt{n})$$
Therefore, the result of Robertson and Seymour is best possible.
The proof of the general theorem

**Theorem (Robertson & Seymour)**

Given a graph $H$, $\mathcal{M}(H)$ has the E&P-property on all graphs iff $H$ is planar.

▶ What about the “gap” $f_H$ in the above theorem?

**Lower bound:**

If $H$ is not acyclic, then $f_H(k) = \Omega_H(k \log(k))$

**Proof:**

Let $G$ be an $n$-vertex cubic graph where

$\text{tw}(G) = \Omega(n)$ and

$\text{girth}(G) = \Omega(\log n)$

▶ Such graphs are well-known to exist: *Ramanujan Graphs* (expanders).
We use the fact that $\text{tw}(G) = \Omega(n)$:

- Assume that $C$ covers all models of $H$ in $G$.
- Then $G^- = G \setminus C$ is $H$-minor free.
- As $H$ is planar, $\text{tw}(G^-) \leq c_H$
- A removal of a vertex reduces treewidth at most by one
- As $\text{tw}(G) = \Omega(n)$ and $\text{tw}(G^-) \leq c_H$, we have that $|C| = \Omega_h(n)$. 
The proof of the general theorem

We use the fact that \( \text{girth}(G) = \Omega(\log n) \):

- Let \( \mathcal{P} \) be a packing of models of \( H \) in \( G \).

- As \( H \) contains a cycle and \( \text{girth}(G) = \Omega(\log n) \), each graph in \( \mathcal{P} \) contains at least \( \Omega_h(\log n) \) vertices.

Therefore \( |\mathcal{P}| = O_h(n / \log n) \).

Conclusion: for every packing \( \mathcal{P} \) of models of \( H \) in \( G \) and every covering \( \mathcal{C} \) of models of \( H \) in \( G \) it holds that \( |\mathcal{C}| = \Omega_h(|\mathcal{P}| \log |\mathcal{P}|) \).

Therefore: \( f_H(k) = \Omega_H(k \log(k)) \).
When can we do better than $O_h(k \log k)$?

- If $H$ is acyclic, then the gap is linear, i.e., $f_H(k) = O_H(k)$

  [Fiorini, Joret, & Wood, 2013]

- Let $\mathcal{R}$ be a non-trivial minor-closed graph class.

  Then for every planar graph $H$, $\mathcal{M}(H)$ has the E&P-property on $\mathcal{R}$ with linear gap $O_{\mathcal{R}}(k)$.

  [Fomin, Saurabh, Thilikos 2011]
What about matching (or approaching) the lower bound?

- If $H$ is not acyclic, then $f_H(k) = O_H(k \ polylog(k))$
  
  [Chekuri & Chuzhoy, 2013]

- **Most general existing tight bound:**
  
  If $H = \theta_h$ then $f_H(k) = O_h(k \ log k)$ on all graphs.
  
  [Fiorini, Joret, & Sau, 2013] and
  
  [Chatzidimitriou, Florent, Sau, & Thilikos, 2015]
Open problem:

Prove or disprove:

- Given a planar graph $H$, $\mathcal{M}(H)$ has the vertex-Erdős–Pósa property on all graphs with (optimal) gap $f_H(k) = O_H(k \log k)$.
Other variants of Erdős–Pósa properties
Edge variants:

For every $r$, $\mathcal{M}(\theta_r)$ has the edge-Erdős-Pósa property with (optimal) gap $O(k \log k)$.

(An $O(\log OPT)$-approximation also exists)

[Chatzidimitriou, Florent, Sau, & Thilikos, 2015]
Open problem:

Prove or disprove:

▶ Given a planar graph $H$, $\mathcal{M}(H)$ has the edge–Erdős–Pósa property on all graphs

and, if this is correct, prove that the gap is optimal $f_H(k) = O_H(k \log k)$
Minor models of cliques:

\[ \mathcal{M}(K_h) \text{ have the edge Erdős-Pósa property on } \Omega(k \cdot h)-\text{connected graphs} \]

[Diestel, Kawarabayashi, Wollan JCTSB 2012]
Immersions:

\( I(H) \): Immersion models

\( \forall H, \ I(H) \) have the edge Erdős-Pósa property on 4-edge connected graphs

[Chun-Hung Liu, May 2015]
Topological Minors:

$\mathcal{T}(H)$: Topological Minor models

There is a class $C$ (completely characterized) such that $\mathcal{T}(H)$ has the vertex Erdős-Pósa property iff $H \in C$.

[Chun-Hung Liu, 2015]
Odd cycles: 

Odd cycles have vertex Erdős-Pósa property on 576-connected graphs with linear gap

[Rautenbach & Reed, 1999]

Odd cycles have vertex/edge Erdős-Pósa property on graphs embeddable in orientable surfaces

[Kawarabayashi, Nakamoto, 2007]

Odd cycles have edge Erdős-Pósa property on 4-edge connected graphs

[Kawarabayashi, Kobayashi, STACS 2012]
Long cycles:

\( \mathcal{M}(C_r) \) has the vertex Erdős-Pósa property with gap

\[ f(k, l) = O(l \cdot k \cdot \log k). \]

[Fiorini & Herinckx, JGT 2013]
Cycles through a set of vertices:

We consider a graph $G$ with terminals $T \subseteq V(G)$.

$T$-cycle: a cycle intersecting $T$.

Cycles intersecting $T$ have the vertex/edge Erdős-Pósa property with (optimal) gap $f(k) = O(k \cdot \log k)$.

[Pontecorvia & Wollan, JCTSB 2012]
Directed cycles in directed graphs:

Directed cycles have the vertex Erdős-Pósa property.

[Reed, Robertson, Seymour, & Thomas, Combinatorica 1996]
Matroids:

[Geelen, Gerards, Whittle, JCTSB 2003]

[Geelen, Kabell JCTSB 2009]
Najlepša hvála

Thank you!
Diego Velázquez - El Triunfo de Baco o Los Borrachos
(Museo del Prado, 1628-29)