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Algorithms and Combinatorics

on the Erdős-Pósa property

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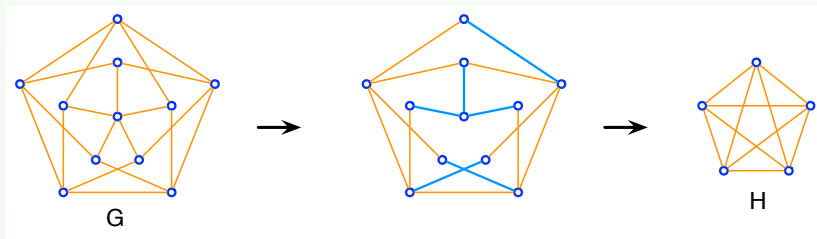
AGTAC 2015, June 18, 2015

Koper, Slovenia

Some (basic and necessary) definitions

Minors and models in graphs

H is a minor of G : H occurs from a subgraph of G by edge contractions

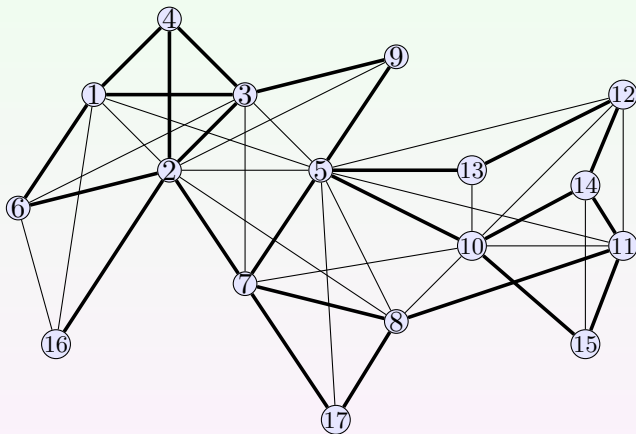


- ▶ H -model: any graph that contains H as a minor.
- ▶ $\mathcal{M}(H)$: the class of all minor models of H .
- ▶ H -minor free graphs: graphs that do **not** contain H as a minor.

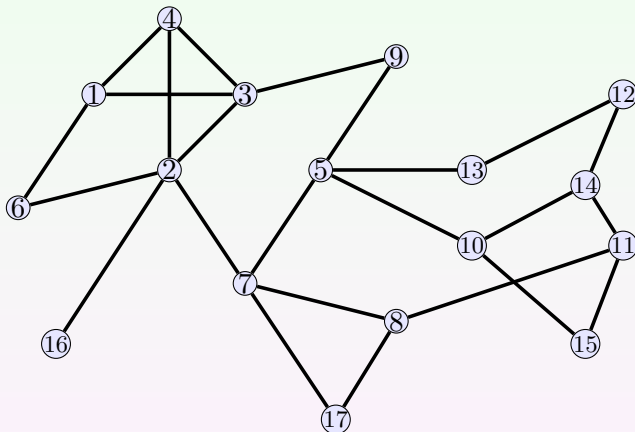
Treewidth

- ▶ A vertex in G is *simplicial* if its neighborhood induces a clique.
- ▶ A graph G is a k -tree if one of the following holds
 - $G = K_{k+1}$ or
 - the removal of G of a **simplicial** vertex creates a k -tree.
- ▶ The treewidth of a graph G is defined as follows

$$\text{tw}(G) = \min\{k \mid G \text{ is a subgraph of some } k\text{-tree}\}$$



A 3-tree



A subgraph of a **3-tree**: a graph with **treewidth** at most 3

Minor exclusion of a planar graph:

Theorem (Robertson and Seymour – GM V)

For every *planar* graph H there is a constant c_H such that if a graph G is H -minor free, then $\mathbf{tw}(G) \leq c_H$.

Erdős-Pósa Theorem

Theorem (Erdős & Pósa 1965)

There exists a function f such that For every k , every graph G has either k vertex disjoint cycles or $f(k)$ vertices that meet all of its cycles.

Facts:

- ▶ Gap: $f(k) = O(k \cdot \log k)$
- ▶ In the same paper they show that the gap $f(k) = O(k \log k)$ is *tight*

According to Diestel's monograph on graph theory:

- ▶ The same holds if we replace “vertices” by “edges”.

[Graph Theory, 3rd Edition, Corollary 12.4.10 and Ex. 39 of Chapter 12]

Lemma

Cycles have the E&P property on planar graphs with linear gap

Proof.

Let G be a graph without any cycle packing of size $> k$

- ▶ Reduce: We can assume that G has no vertices of degree ≤ 2 .
- ▶ Find: A planar graph has always a face (cycle) of length ≤ 5 .

We build a *cycle covering* of G by setting $C = \emptyset$ and repetitively

1. **Reduce** G so that $\delta(G) \geq 3$.
2. **Find** a cycle of length ≤ 5 and add its vertices to C .

The above finish after $\leq k$ rounds and creates a cycle cover C of the input graph of at most $5k$ vertices.

Jones' Conjecture:

Cycles have the E&P property on **planar graphs** with gap $2k$.

► Wide Open (and **famous**)!

Why Jones'?

On October 29th, 2007 Anonymous says:

Does anyone know why this is called Jones' Conjecture?

[reply](#)

Reply: Why Jones'?

On November 16th, 2007 Anonymous says:

I am Jones. My Taiwanese name is Chuan-Min Lee. This conjecture came up when I was working on it with Ton Kloks and Jiping Liu. I used the name "Jones" instead of my Taiwanese name for ease of communication.

Fact: Linear gap extends to H -minor free graphs

We will derive the **Fact** by the following **more general** statement of Erdős-Pósa Theorem:

Theorem

For each graph H , cycles have the E&P property for H -minor free graphs with gap $O(k \cdot \log h)$, where $h = |V(H)|$.

E&P follows as a graphs with no k -cycle packings are K_{3k} -minor free.

We give a proof using the following results:

Theorem (Thomassen 1983)

Given an integer r , every graph G with $\text{girth}(G) \geq 8r + 3$ and $\delta(G) \geq 3$ has a minor J with $\delta(J) \geq 2^r$.

- ▶ $\text{girth}(G)$: minimum size of a cycle in G
- ▶ $\delta(G)$: minimum degree of G
- ▶ J is a minor of G : J occurs from a subgraph of G by edge contractions.

Theorem (Kostochka 1982 & Thomason 1984)

$\exists \alpha \forall h \delta(G) \geq \alpha h \sqrt{\log h} \Rightarrow G$ contains K_h as a minor

Proof.

Let G be a K_h -free graph with no k -cycle packing

► Reduce: $\delta(G) \geq 3$

As G is H -minor free, from 2nd theorem every minor F of G has

$$\delta(F) \leq \alpha h \sqrt{\log h}$$

Let r be such that $\alpha h \sqrt{\log h} < 2^r$

From 1st theorem contains a cycle of length $< 8r = O(\log h)$.

We build a *cycle covering* of G by setting $C = \emptyset$ and repetitively

1. **Reduce** G so that $\delta(G) \geq 3$.
2. **Find** a cycle of length $O(\log h)$ and add its vertices to C .

The above finish after $< k$ rounds and creates a cycle cover of the input graph of at most $O(k \log h)$ vertices.

Algorithmic Remarks:

▶ Both **Reduce** and **Find**, can be implemented in poly-time.

Therefore there is a polynomial algorithm that, for every k , returns one of the following

- a set of k disjoint cycles or
- a cycle cover of $O(k \cdot \log k)$ vertices.

Algorithmic Remarks:

► We just derived an $O(\log(OPT))$ -approximation algorithm for both the **maximum** size of a **vertex cycle packing** and the **minimum** size of a **vertex cycle covering**.

Moreover:

All previous proofs, results, and algorithms extend directly to the **edge** variants of the above problems.

Algorithmic Remarks:

► We just derived an $O(\log(OPT))$ -approximation algorithm for both the **maximum** size of a **edge cycle packing** and the **minimum** size of a **edge cycle covering**.

Moreover:

All previous proofs, results, and algorithms extend directly to the **edge** variants of the above problems.

Extensions on minor models

Let \mathcal{G} and \mathcal{C} be graph classes.

Question (About \mathcal{G} and \mathcal{H})

Is there a function f such that, for every k , every graph $G \in \mathcal{G}$ has either k vertex disjoint subgraphs in \mathcal{C} or $f(k)$ vertices that meet all subgraphs in \mathcal{C} ?

Question (Optimizing the gap f)

If the above question can be positively answered, what is the minimum f for which this holds?

- ▶ We say that \mathcal{C} has the Erdős & Pósa property on \mathcal{G} with gap f .
- ▶ **Task:** detect such \mathcal{C} and \mathcal{G} and optimize the corresponding gap f .
- ▶ **Erdős & Pósa Theorem:**

Cycles have the E&P property on all graphs with gap $O(k \log k)$.

[Recall that $\mathcal{M}(H)$ is the graph class containing all H -models]

A vast generalization of Erdős-Pósa Theorem:

Theorem (Robertson & Seymour)

Given a graph H , $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

- ▶ Original Erdős-Pósa theorem: $H =$ “double edge”.
- ▶ “double edge” generalizes to **any planar graph!!**

We use f_H for the gap of $\mathcal{M}(H)$

Theorem (Robertson & Seymour)

Given a graph H , $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

The proof of the “only if” is a corollary of the planar exclusion theorem:

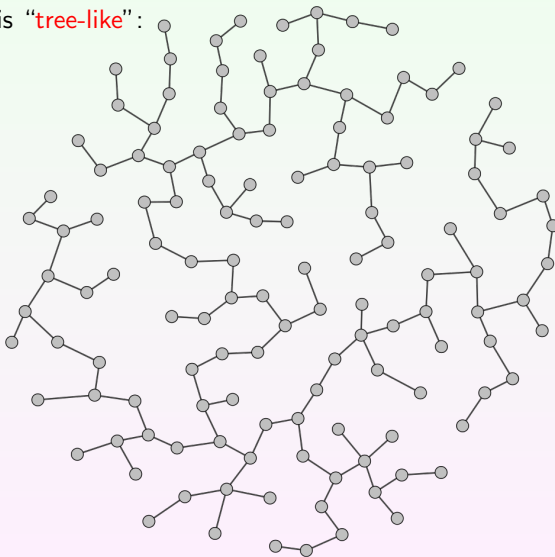
Theorem (Robertson and Seymour – GM V)

For every *planar* graph H there is a constant c_H such that if a graph G is H -minor free, then $\text{tw}(G) \leq c_H$.

Ideas of proof:

- ▶ if a graph G does not contain any packing of k models of H , then it excludes their disjoint union as a minor (that is planar).
- ▶ Therefore, $\text{tw}(G) \leq f(k, H) = w$.
- ▶ Let G be a subgraph of a w -tree R

The graph is “tree-like”:



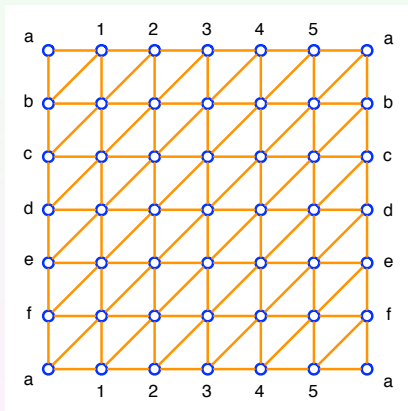
Theorem (Robertson and Seymour – GM V)

For every *planar* graph H there is a constant c_H such that if a graph G is H -minor free, then $\mathbf{tw}(G) \leq c_H$.

Ideas of the “if” proof: (we describe the case where $H = K_5$)

$$H = K_5 \quad \text{X}$$

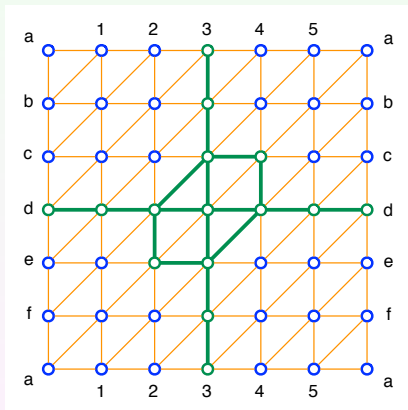
A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid Γ_n :



$$\text{pack}_H(G) = 1 \quad \text{but} \quad \text{cover}_H(G) = \Theta(\sqrt{n})$$

$$H = K_5 \quad \text{X}$$

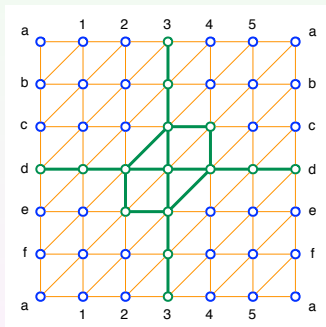
A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid Γ_n :



$$\text{pack}_H(G) = 1 \quad \text{but} \quad \text{cover}_H(G) = \Theta(\sqrt{n})$$

$$H = K_5 \quad \times$$

H not planar \times



Therefore, the result of Robertson and Seymour is **best possible**.

Theorem (Robertson & Seymour)

Given a graph H , $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

► What about the “gap” f_H in the above theorem?

Lower bound:

If H is not acyclic, then $f_H(k) = \Omega_H(k \log(k))$

Proof:

Let G be an n -vertex cubic graph where

$\text{tw}(G) = \Omega(n)$ and

$\text{girth}(G) = \Omega(\log n)$

► Such graphs are well-known to exist: [Ramanujan Graphs](#) (expanders).

We use the fact that $\text{tw}(G) = \Omega(n)$:

- ▶ Assume that C covers all models of H in G .
- ▶ Then $G^- = G \setminus C$ is H -minor free.
- ▶ As H is planar, $\text{tw}(G^-) \leq c_H$
- ▶ A removal of a vertex reduces treewidth at most by one
- ▶ As $\text{tw}(G) = \Omega(n)$ and $\text{tw}(G^-) \leq c_H$, we have that $|C| = \Omega_h(n)$.

We use the fact that $\text{girth}(G) = \Omega(\log n)$:

- ▶ Let \mathcal{P} be a packing of models of H in G
- ▶ As H contains a cycle and $\text{girth}(G) = \Omega(\log n)$,
each graph in \mathcal{P} contains at least $\Omega_h(\log n)$ vertices.
- ▶ Therefore $|\mathcal{P}| = O_h(n/\log n)$

Conclusion: for every packing \mathcal{P} of models of H in G and every covering \mathcal{C} of models of H in G it holds that $|\mathcal{C}| = \Omega_h(|\mathcal{P}| \log |\mathcal{P}|)$

Therefore: $f_H(k) = \Omega_H(k \log(k))$

When can we do better than $O_h(k \log k)$?

▶ If H is acyclic, then the gap is linear, i.e., $f_H(k) = O_H(k)$

[Fiorini, Joret, & Wood, 2013]

▶ Let \mathcal{R} be a non trivial minor-closed graph class.

Then for every planar graph H , $\mathcal{M}(H)$ has the E&P-property on \mathcal{R} with linear gap $O_{\mathcal{R}}(k)$.


[Fomin, Saurabh, Thilikos 2011]

What about matching (or approaching) the lower bound?

► If H is not acyclic, then $f_H(k) = O_H(k \text{ polylog}(k))$

[Chekuri & Chuzhoy, 2013]

► **Most general existing tight bound:**

If $H = \theta_h =$  then $f_H(k) = O_h(k \log k)$ on all graphs.

[Fiorini, Joret, & Sau, 2013] and

[Chatzidimitriou, Florent, Sau, & Thilikos, 2015]

Open problem:

Prove or disprove:

- ▶ Given a planar graph H , $\mathcal{M}(H)$ has the **vertex**-Erdős-Pósa property on all graphs with (optimal) gap $f_H(k) = O_H(k \log k)$

Other variants of Erdős–Pósa properties

Edge variants:

- ▶ For every r , $\mathcal{M}(\theta_r)$ has the **edge**-Erdős-Pósa property with (optimal) gap $O(k \log k)$.

⟨An $O(\log OPT)$ -approximation also exists⟩

[Chatzidimitriou, Florent, Sau, & Thilikos, 2015]

Open problem:

Prove or disprove:

▶ Given a planar graph H , $\mathcal{M}(H)$ has the edge–Erdős–Pósa property on all graphs

and, if this is correct, prove that the gap is optimal $f_H(k) = O_H(k \log k)$

Minor models of cliques:

$\mathcal{M}(K_h)$ have the **edge** Erdős-Pósa property on $\Omega(k \cdot h)$ -connected graphs

[Diestel, Kawarabayashi, Wollan JCTSB 2012]

Immersion:

$\mathcal{I}(H)$: Immersion models

$\forall H$, $\mathcal{I}(H)$ have the **edge** Erdős-Pósa property on **4-edge**
connected graphs

[Chun-Hung Liu, May 2015]

Topological Minors:

$\mathcal{T}(H)$: Topological Minor models

There is a class \mathcal{C} (completely characterized) such that

$\mathcal{T}(H)$ has the vertex Erdős-Pósa property iff $H \in \mathcal{C}$.

[Chun-Hung Liu, 2015]

Odd cycles:

Odd cycles have **vertex** Erdős-Pósa property on **576**-connected graphs with **linear** gap

[Rautenbach & Reed, 1999]

Odd cycles have **vertex/edge** Erdős-Pósa property on graphs embeddable in **orientable** surfaces

[Kawarabayashi, Nakamoto, 2007]

Odd cycles have **edge** Erdős-Pósa property on 4-edge connected graphs

[Kawarabayashi, Kobayashi, STACS 2012]

Long cycles:

$\mathcal{M}(C_r)$ has the **vertex** Erdős-Pósa property with gap

$$f(k, l) = O(l \cdot k \cdot \log k).$$

[Fiorini & Herinckx, JGT 2013]

Cycles through a set of vertices:

We consider a graph G with terminals $T \subseteq V(G)$

T -cycle: a cycle intersecting T .

Cycles intersecting T have the vertex/edge Erdős-Pósa property with (optimal) gap $f(k) = O(k \cdot \log k)$.

[Pontecorvia & Wollan, JCTSB 2012]

Directed cycles in directed graphs:

Directed cycles have the vertex Erdős-Pósa property.

[Reed, Robertson, Seymour, & Thomas, *Combinatorica* 1996]

Matroids:

[Geelen, Gerards, Whittle, JCTSB 2003]

[Geelen, Kabell JCTSB 2009]

Najlepša hvála

Thank you!

Diego Velázquez - El Triunfo de Baco o Los Borrachos

(Museo del Prado, 1628-29)

