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Dimitrios M. Thilikos

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Algorithms and Combinatorics on the Erdős–Pósa property

Dimitrios M. Thilikos

AlGCo project team, CNRS, LIRMM

Department of Mathematics, National and Kapodistrian University of Athens

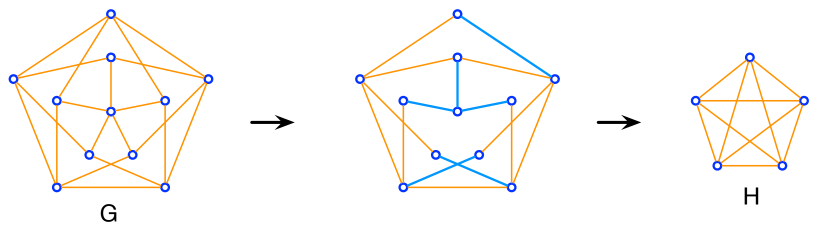
AGTAC 2015, June 18, 2015

Koper, Slovenia

Some (basic and necessary) definitions

Minors and models in graphs

H is a minor of G: H occurs from a subgraph of G by edge contractions

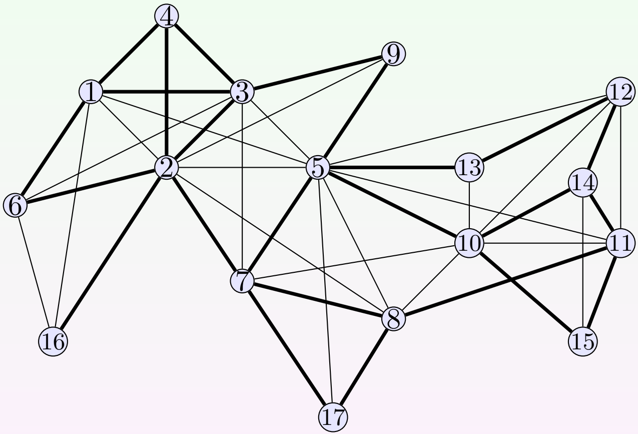


- H-model: any graph that contains H as a minor.
- $\mathcal{M}(H)$: the class of all minor models of H.
- H-minor free graphs: graphs that do not contain H as a minor.

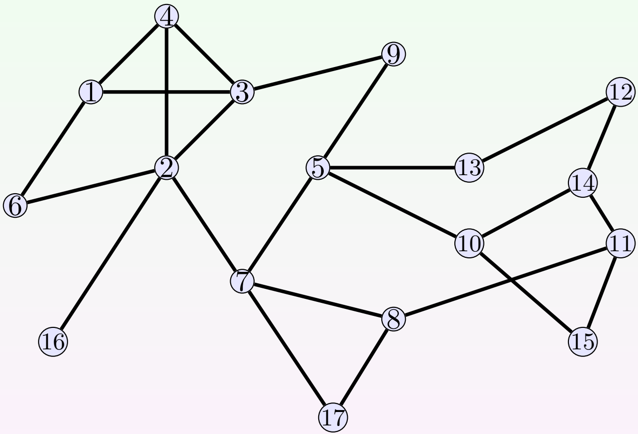
Treewidth

- A vertex in G is *simplicial* if its neighborhood induces a clique.
- A graph G is a k-tree if one of the following holds
  - $G = K_{k+1}$ or
  - the removal of G of a simplicial vertex creates a k-tree.
- The treewidth of a graph G is defined as follows

$$\textbf{tw}(G) = \min\{k \mid G \text{ is a subgraph of some } k\text{-tree}\}$$



A 3-tree



A subgraph of a 3-tree: a graph with treewidth at most 3

Minor exclusion of a planar graph:

Theorem (Robertson and Seymour – GM V)

For every planar graph H there is a constant c_H such that if a graph G is H-minor free, then $\textbf{tw}(G) \leq c_H$.

Erdős-Pósa Theorem

Theorem (Erdős & Pósa 1965)

There exists a function f such that For every k, every graph G has either k vertex disjoint cycles or $f(k)$ vertices that meet all of its cycles.

Facts:

- Gap: $f(k) = O(k \cdot \log k)$
- In the same paper they show that the gap $f(k) = O(k \log k)$ is *tight*

According to Diestel's monograph on graph theory:

- The same holds if we replace "vertices" by "edges".

[Graph Theory, 3rd Edition, Corollary 12.4.10 and Ex. 39 of Chapter 12]

Lemma

Cycles have the E&P property on planar graphs with linear gap

Proof.

Let G be a graph without any cycle packing of size $> k$

- Reduce: We can assume that G has no vertices of degree ≤ 2.
- Find: A planar graph has always a face (cycle) of length ≤ 5.

We build a *cycle covering* of G by setting $C = \emptyset$ and repetitively

1. **Reduce** G so that $\delta(G) \geq 3$.
2. **Find** a cycle of length ≤ 5 and add its vertices to C.

The above finish after $\leq k$ rounds and creates a cycle cover C of the input graph of at most $5k$ vertices.

Jones' Conjecture:

Cycles have the E&P property on **planar graphs** with gap $2k$.

- Wide Open (and famous)!

Why Jones'?

On October 29th, 2007 Anonymous says:

Does anyone know why this is called Jones' Conjecture?

reply

Reply: Why Jones'?

On November 16th, 2007 Anonymous says:

I am Jones. My Taiwanese name is Chuan-Min Lee. This conjecture came up when I was working on it with Ton Kloks and Jiping Liu. I used the name "Jones" instead of my Taiwanese name for ease of communication.

Fact: Linear gap extends to H-minor free graphs

We will derive the **Fact** by the following more general statement of Erdős-Pósa Theorem:

Theorem

For each graph H, cycles have the E&P property for H-minor free graphs with gap $O(k \cdot \log h)$, where $h = |V(H)|$.

E&P follows as a graphs with no k-cycle packings are K_{3k}-minor free.

We give a proof using the following results:

Theorem (Thomassen 1983)

Given an integer r, every graph G with **girth**$(G) \geq 8r+3$ *and* $\delta(G) \geq 3$ *has a minor J with* $\delta(J) \geq 2^r$.

- **girth**(G): minimum size of a cycle in G
- $\delta(G)$: minimum degree of G
- J is a minor of G: J occurs from a subgraph of G by edge contractions.

Theorem (Kostochka 1982 & Thomason 1984)

$\exists \alpha \; \forall h \;\; \delta(G) \geq \alpha h \sqrt{\log h} \Rightarrow G$ *contains* K_h *as a minor*

Proof.

Let G be a K_h-free graph with no k-cycle packing

- Reduce: $\delta(G) \geq 3$

As G is H-minor free, from 2nd theorem every minor F of G has

$\delta(F) \leq \alpha h \sqrt{\log h}$

Let r be such that $\alpha h \sqrt{\log h} < 2^r$

From 1st theorem contains a cycle of length $< 8r = O(\log h)$.

We build a *cycle covering* of G by setting $C = \emptyset$ and repetitively

1. **Reduce** G so that $\delta(G) \geq 3$.
2. **Find** a cycle of length $O(\log h)$ and add its vertices to C.

The above finish after $< k$ rounds and creates a cycle cover of the input graph of at most $O(k \log h)$ vertices.

Algorithmic Remarks:

- Both **Reduce** and **Find**, can be implemented in poly-time.

Therefore there is a polynomial algorithm that, for every k, returns one of the following

- a set of k disjoint cycles or
- a cycle cover of $O(k \cdot \log k)$ vertices.

Algorithmic Remarks:

- We just derived an $O(\log(OPT))$-approximation algorithm for both the maximum size of a vertex cycle packing and the minimum size of a vertex cycle covering.

Moreover:

All previous proofs, results, and algorithms extend directly to the edge variants of the above problems.

Algorithmic Remarks:

- We just derived an $O(\log(OPT))$-approximation algorithm for both the maximum size of a edge cycle packing and the minimum size of a edge cycle covering.

Moreover:

All previous proofs, results, and algorithms extend directly to the edge variants of the above problems.

Extensions on minor models

Let $\mathcal{G}$ and $\mathcal{C}$ be graph classes.

Question (About $\mathcal{G}$ and $\mathcal{H}$)

Is there a function f such that, for every k, every graph $G \in \mathcal{G}$ has either k vertex disjoint subgraphs in $\mathcal{C}$ or $f(k)$ vertices that meet all subgraphs in $\mathcal{C}$?

Question (Optimizing the gap f)

If the above question can be positively answered, what is the minimum f for which this holds?

- We say that $\mathcal{C}$ *has the Erdős & Pósa property on* $\mathcal{G}$ *with gap* f.
- **Task**: detect such $\mathcal{C}$ and $\mathcal{G}$ and optimize the corresponding gap f.
- **Erdős & Pósa Theorem**:

  Cycles have the E&P property on all graphs with gap $O(k \log k)$.

[Recall that $\mathcal{M}(H)$ is the graph class containing all H-models]

A vast generalziation of Erdős-Pósa Theorem:

Theorem (Robertson & Seymour)

Given a graph H, $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

- Original Erdős-Pósa theorem: $H =$ "double edge".
- "double edge" generalizes to any planar graph!!

We use f_H for the gap of $\mathcal{M}(H)$

Theorem (Robertson & Seymour)

Given a graph H, $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

The proof of the "only if" is a corollary of the planar exclusion theorem:

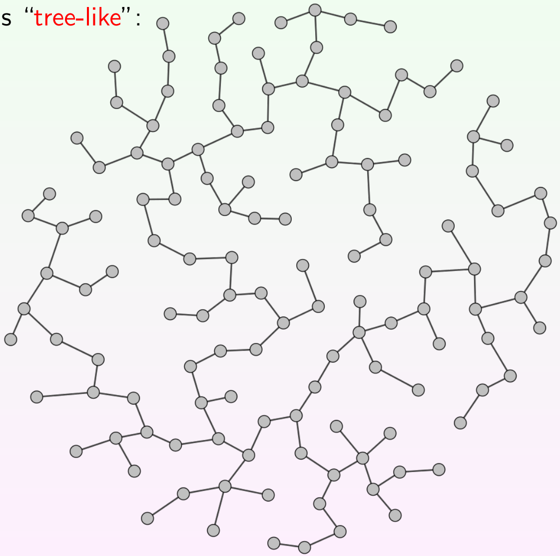
Theorem (Robertson and Seymour – GM V)

For every planar graph H there is a constant c_H such that if a graph G is H-minor free, then $\mathbf{tw}(G) \leq c_H$.

Ideas of proof:

- if a graph G does not contain any packing of k models of H, then it excludes their disjoint union as a minor (that is planar).
- Therefore, $\mathbf{tw}(G) \leq f(k, H) = w$.
- Let G be a subgraph of a w-tree R

The graph is "tree-like":



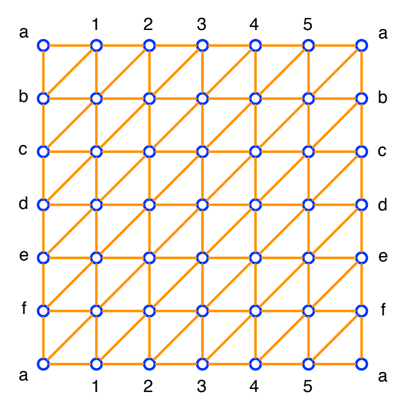
Theorem (Robertson and Seymour – GM V)

For every planar graph H there is a constant c_H such that if a graph G is H-minor free, then $\mathbf{tw}(G) \leq c_H$.

Ideas of the "if" proof: (we describe the case where $H = K_5$)

$\boxed{H = K_5}$ ✗

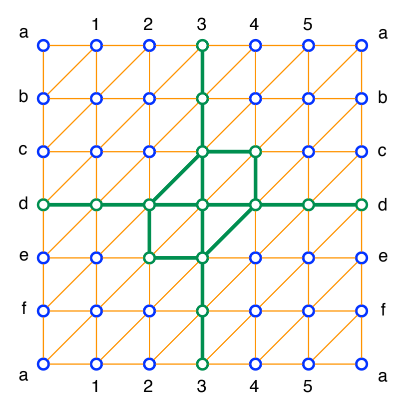
A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid Γ_n:



$$\texttt{pack}_H(G) = 1 \quad \text{but} \quad \texttt{cover}_H(G) = \Theta(\sqrt{n})$$

$\boxed{H = K_5}$ ✗

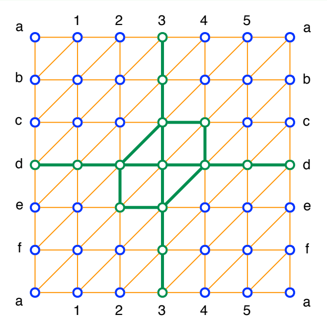
A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid Γ_n:



$$\texttt{pack}_H(G) = 1 \quad \text{but} \quad \texttt{cover}_H(G) = \Theta(\sqrt{n})$$

$H = K_5$ ✗

H not planar ✗



Therefore, the result of Robertson and Seymour is best possible.

Theorem (Robertson & Seymour)

Given a graph H, $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

- What about the "gap" f_H in the above theorem?

Lower bound:

If H is not acyclic, then $f_H(k) = \Omega_H(k \log(k))$

Proof:

Let G be an n-vertex cubic graph where

tw$(G) = \Omega(n)$ and

girth$(G) = \Omega(\log n)$

- Such graphs are well-known to exist: Ramanujan Graphs (expanders).

We use the fact that tw$(G) = \Omega(n)$:

- Assume that C covers all models of H in G.
- Then $G^{-} = G \setminus C$ is H-minor free.
- As H is planar, **tw**$(G^{-}) \leq c_H$
- A removal of a vertex reduces treewidth at most by one
- As **tw**$(G) = \Omega(n)$ and **tw**$(G^{-}) \leq c_H$, we have that $|C| = \Omega_h(n)$.

We use the fact that girth$(G) = \Omega(\log n)$:

- Let $\mathcal{P}$ be a packing of models of H in G
- As H contains a cycle and **girth**$(G) = \Omega(\log n)$,

  each graph in $\mathcal{P}$ contains at least $\Omega_h(\log n)$ vertices.
- Therefore $|\mathcal{P}| = O_h(n/\log n)$

Conclusion: for every packing $\mathcal{P}$ of models of H in G and every covering C of models of H in G it holds that $|C| = \Omega_h(|\mathcal{P}| \log |\mathcal{P}|)$

Therefore: $f_H(k) = \Omega_H(k \log(k))$

When can we do better than $O_h(k \log k)$?

- If H is acyclic, then the gap is linear, i.e., $f_H(k) = O_H(k)$

[Fiorini, Joret, & Wood, 2013]

- Let $\mathcal{R}$ be a non trivial minor-closed graph class.

Then for every planar graph H, $\mathcal{M}(H)$ has the E&P-property on $\mathcal{R}$ with linear gap $O_{\mathcal{R}}(k)$.

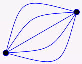
[Fomin, Saurabh, Thilikos 2011]

What about matching (or approaching) the lower bound?

- If H is not acyclic, then $f_H(k) = O_H(k \text{ polylog}(k))$

[Chekuri & Chuzhoy, 2013]

- **Most general existing tight bound:**

If $H = \theta_h =$  then $f_H(k) = O_h(k \ \log k)$ on all graphs.

[Fiorini, Joret, & Sau, 2013] and

[Chatzidimitriou, Florent, Sau, & Thilikos, 2015]

Open problem:

Prove or disprove:

- Given a planar graph H, $\mathcal{M}(H)$ has the vertex-Erdős–Pósa property on all graphs with (optimal) gap $f_H(k) = O_H(k \log k)$

Other variants of Erdős–Pósa properties

Edge variants:

- For every r, $\mathcal{M}(\theta_r)$ has the edge-Erdős-Pósa property with (optimal) gap $O(k \log k)$.

  $\langle$An $O(\log OPT)$-approximation also exists$\rangle$

  [Chatzidimitriou, Florent, Sau, & Thilikos, 2015]

Open problem:

Prove or disprove:

- Given a planar graph H, $\mathcal{M}(H)$ has the edge–Erdős–Pósa property on all graphs

and, if this is correct, prove that the gap is optimal $f_H(k) = O_H(k \log k)$

Minor models of cliques:

$\mathcal{M}(K_h)$ have the edge Erdős-Pósa property on $\Omega(k \cdot h)$-connected graphs

[Diestel, Kawarabayashi, Wollan JCTSB 2012]

Immersions:

$\mathcal{I}(H)$: Immersion models

$\forall H, \;\; \mathcal{I}(H)$ have the edge Erdős-Pósa property on 4-edge connected graphs

[Chun-Hung Liu, May 2015]

Topological Minors:

$\mathcal{T}(H)$: Topological Minor models

There is a class $\mathcal{C}$ (completely characterized) such that

$\mathcal{T}(H)$ has the vertex Erdős-Pósa property iff $H \in \mathcal{C}$.

[Chun-Hung Liu, 2015]

Odd cycles:

Odd cycles have vertex Erdős-Pósa property on 576-connected graphs with linear gap

[Rautenbach & Reed, 1999]

Odd cycles have vertex/edge Erdős-Pósa property on graphs embeddable in orientable surfaces

[Kawarabayashi, Nakamoto, 2007]

Odd cycles have edge Erdős-Pósa property on 4-edge connected graphs

[Kawarabayashi, Kobayashi, STACS 2012]

Long cycles:

$\mathcal{M}(C_r)$ has the vertex Erdős-Pósa property with gap

$f(k, l) = O(l \cdot k \cdot \log k)$.

[Fiorini & Herinckx, JGT 2013]

Cycles through a set of vertices:

We consider a graph G with terminals $T \subseteq V(G)$

T-cycle: a cycle intersecting T.

Cycles intersecting T have the vertex/edge Erdős-Pósa property

with (optimal) gap $f(k) = O(k \cdot \log k)$.

[Pontecorvia & Wollan, JCTSB 2012]

Directed cycles in directed graphs:

Directed cycles have the vertex Erdős-Pósa property.

[Reed, Robertson, Seymour, & Thomas, Combinatorica 1996]

Matroids:

[Geelen, Gerards, Whittle, JCTSB 2003]

[Geelen, Kabell JCTSB 2009]

Najlepša hvála

Thank you!

Diego Velázquez - El Triunfo de Baco o Los Borrachos

(Museo del Prado, 1628-29)