Algorithms and Combinatorics on the Erdős–Pósa property

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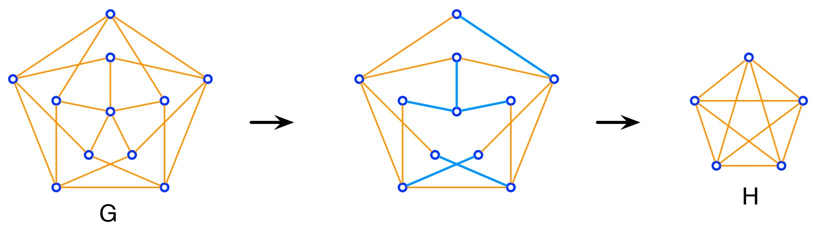
Some (basic and necessary) definitions

Minors and models in graphs

H is a minor of G: H occurs from a subgraph of G by edge contractions

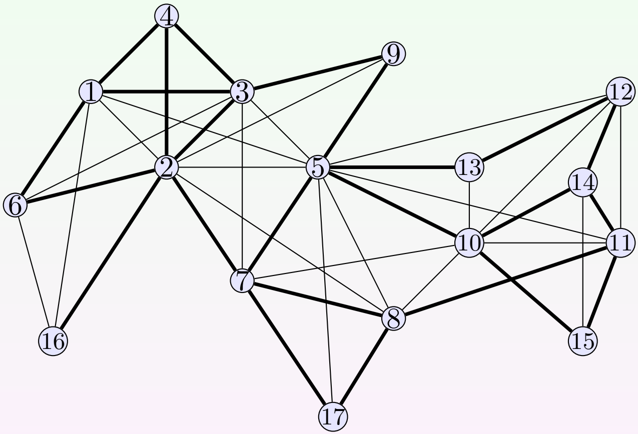


- H-model: any graph that contains H as a minor.
- $\mathcal{M}(H)$: the class of all minor models of H.
- H-minor free graphs: graphs that do not contain H as a minor.

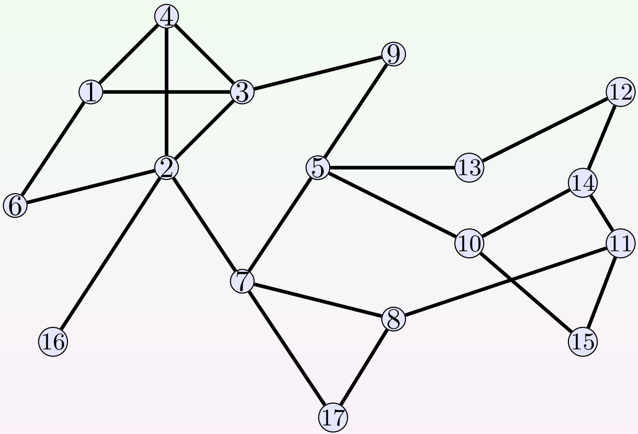
Treewidth

- A vertex in G is *simplicial* if its neighborhood induces a clique.
- A graph G is a k-tree if one of the following holds
  - $G = K_{k+1}$ or
  - the removal of G of a simplicial vertex creates a k-tree.
- The treewidth of a graph G is defined as follows

$$\mathbf{tw}(G) = \min\{k \mid G \text{ is a subgraph of some } k\text{-tree}\}$$



A 3-tree



A subgraph of a 3-tree: a graph with treewidth at most 3

Minor exclusion of a planar graph:

Theorem (Robertson and Seymour – GM V)

For every planar graph H there is a constant c_H such that if a graph G is H-minor free, then $\mathbf{tw}(G) \leq c_H$.

Erdős-Pósa Theorem

Theorem (Erdős & Pósa 1965)

There exists a function f such that For every k, every graph G has either k vertex disjoint cycles or $f(k)$ vertices that meet all of its cycles.

Facts:

- Gap: $f(k) = O(k \cdot \log k)$
- In the same paper they show that the gap $f(k) = O(k \log k)$ is *tight*

According to Diestel's monograph on graph theory:

- The same holds if we replace "vertices" by "edges".

[Graph Theory, 3rd Edition, Corollary 12.4.10 and Ex. 39 of Chapter 12]

Lemma

Cycles have the E&P property on planar graphs with linear gap

Proof.

Let G be a graph without any cycle packing of size $> k$

- Reduce: We can assume that G has no vertices of degree ≤ 2.
- Find: A planar graph has always a face (cycle) of length ≤ 5.

We build a *cycle covering* of G by setting $C = \emptyset$ and repetitively

1. **Reduce** G so that $\delta(G) \geq 3$.
2. **Find** a cycle of length ≤ 5 and add its vertices to C.

The above finish after $\leq k$ rounds and creates a cycle cover C of the input graph of at most $5k$ vertices.

Jones' Conjecture:

Cycles have the E&P property on **planar graphs** with gap $2k$.

- Wide Open (and famous)!

Why Jones'?

On October 29th, 2007 Anonymous says:

Does anyone know why this is called Jones' Conjecture?

reply

Reply: Why Jones'?

On November 16th, 2007 Anonymous says:

I am Jones. My Taiwanese name is Chuan-Min Lee. This conjecture came up when I was working on it with Ton Kloks and Jiping Liu. I used the name "Jones" instead of my Taiwanese name for ease of communication.

Fact: Linear gap extends to H-minor free graphs

We will derive the **Fact** by the following more general statement of Erdős-Pósa Theorem:

Theorem

For each graph H, cycles have the E&P property for H-minor free graphs with gap $O(k \cdot \log h)$, where $h = |V(H)|$.

E&P follows as a graphs with no k-cycle packings are K_{3k}-minor free.

We give a proof using the following results:

Theorem (Thomassen 1983)

Given an integer r, every graph G with **girth**$(G) \geq 8r+3$ *and* $\delta(G) \geq 3$ *has a minor J with $\delta(J) \geq 2^r$.*

- **girth**(G): minimum size of a cycle in G
- $\delta(G)$: minimum degree of G
- J is a minor of G: J occurs from a subgraph of G by edge contractions.

Theorem (Kostochka 1982 & Thomason 1984)

$\exists \alpha\ \forall h\ \ \delta(G) \geq \alpha h \sqrt{\log h} \Rightarrow G$ *contains K_h as a minor*

Proof.

Let G be a K_h-free graph with no k-cycle packing

- Reduce: $\delta(G) \geq 3$

As G is H-minor free, from 2nd theorem every minor F of G has

$\delta(F) \leq \alpha h \sqrt{\log h}$

Let r be such that $\alpha h \sqrt{\log h} < 2^r$

From 1st theorem contains a cycle of length $< 8r = O(\log h)$.

We build a *cycle covering* of G by setting $C = \emptyset$ and repetitively

1. **Reduce** G so that $\delta(G) \geq 3$.
2. **Find** a cycle of length $O(\log h)$ and add its vertices to C.

The above finish after $< k$ rounds and creates a cycle cover of the input graph of at most $O(k \log h)$ vertices.

Algorithmic Remarks:

- Both **Reduce** and **Find**, can be implemented in poly-time.

Therefore there is a polynomial algorithm that, for every k, returns one of the following

- a set of k disjoint cycles or
- a cycle cover of $O(k \cdot \log k)$ vertices.

Algorithmic Remarks:

- We just derived an $O(\log(OPT))$-approximation algorithm for both the maximum size of a vertex cycle packing and the minimum size of a vertex cycle covering.

Moreover:

All previous proofs, results, and algorithms extend directly to the edge variants of the above problems.

Algorithmic Remarks:

- We just derived an $O(\log(OPT))$-approximation algorithm for both the maximum size of a edge cycle packing and the minimum size of a edge cycle covering.

Moreover:

All previous proofs, results, and algorithms extend directly to the edge variants of the above problems.

Extensions on minor models

Let $\mathcal{G}$ and $\mathcal{C}$ be graph classes.

Question (About $\mathcal{G}$ and $\mathcal{H}$)

Is there a function f such that, for every k, every graph $G \in \mathcal{G}$ has either k vertex disjoint subgraphs in $\mathcal{C}$ or $f(k)$ vertices that meet all subgraphs in $\mathcal{C}$?

Question (Optimizing the gap f)

If the above question can be positively answered, what is the minimum f for which this holds?

- We say that $\mathcal{C}$ *has the Erdős & Pósa property on* $\mathcal{G}$ *with gap* f.
- **Task**: detect such $\mathcal{C}$ and $\mathcal{G}$ and optimize the corresponding gap f.
- **Erdős & Pósa Theorem**:

  Cycles have the E&P property on all graphs with gap $O(k \log k)$.

[Recall that $\mathcal{M}(H)$ is the graph class containing all H-models]

A vast generalziation of Erdős-Pósa Theorem:

Theorem (Robertson & Seymour)

Given a graph H, $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

- Original Erdős-Pósa theorem: $H =$ "double edge".
- "double edge" generalizes to any planar graph!!

We use f_H for the gap of $\mathcal{M}(H)$

Theorem (Robertson & Seymour)

Given a graph H, $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

The proof of the "only if" is a corollary of the planar exclusion theorem:

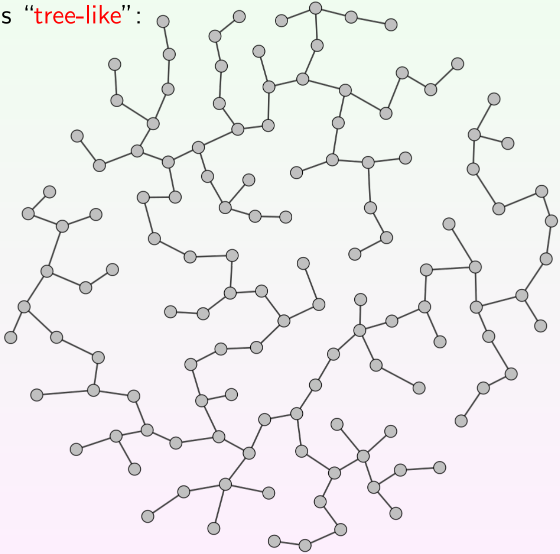
Theorem (Robertson and Seymour – GM V)

For every planar graph H there is a constant c_H such that if a graph G is H-minor free, then $\mathbf{tw}(G) \leq c_H$.

Ideas of proof:

- if a graph G does not contain any packing of k models of H, then it excludes their disjoint union as a minor (that is planar).
- Therefore, $\mathbf{tw}(G) \leq f(k, H) = w$.
- Let G be a subgraph of a w-tree R

The graph is "tree-like":



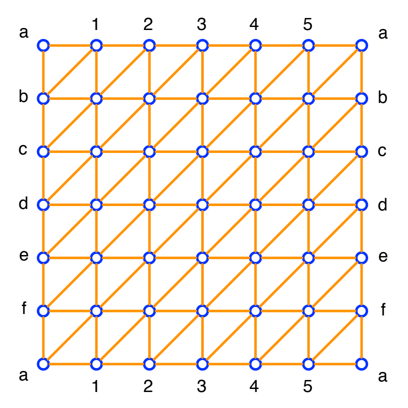
Theorem (Robertson and Seymour – GM V)

For every planar graph H there is a constant c_H such that if a graph G is H-minor free, then $\mathbf{tw}(G) \leq c_H$.

Ideas of the "if" proof: (we describe the case where $H = K_5$)

$H = K_5$ ✗

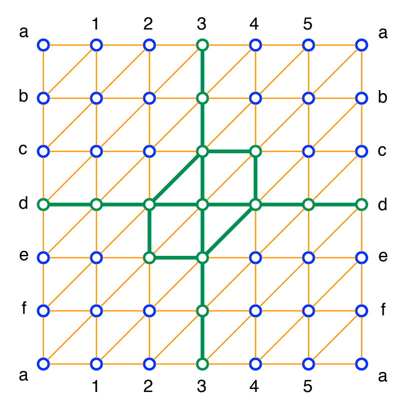
A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid Γ_n:



$\texttt{pack}_H(G) = 1$ but $\texttt{cover}_H(G) = \Theta(\sqrt{n})$

$H = K_5$ ✗

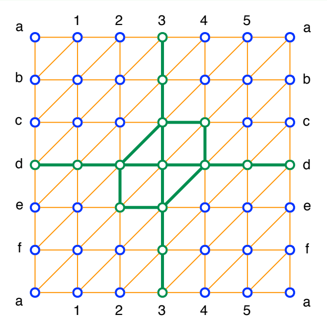
A $\sqrt{n} \times \sqrt{n}$ triangulated toroidal grid Γ_n:



$\texttt{pack}_H(G) = 1$ but $\texttt{cover}_H(G) = \Theta(\sqrt{n})$

$H = K_5$ ✗

H not planar ✗



Therefore, the result of Robertson and Seymour is best possible.

Theorem (Robertson & Seymour)

Given a graph H, $\mathcal{M}(H)$ has the E&P-property on all graphs iff H is planar.

- What about the "gap" f_H in the above theorem?

Lower bound:

If H is not acyclic, then $f_H(k) = \Omega_H(k \log(k))$

Proof:

Let G be an n-vertex cubic graph where

tw$(G) = \Omega(n)$ and

girth$(G) = \Omega(\log n)$

- Such graphs are well-known to exist: Ramanujan Graphs (expanders).

We use the fact that tw$(G) = \Omega(n)$:

- Assume that C covers all models of H in G.
- Then $G^{-} = G \setminus C$ is H-minor free.
- As H is planar, **tw**$(G^{-}) \leq c_H$
- A removal of a vertex reduces treewidth at most by one
- As **tw**$(G) = \Omega(n)$ and **tw**$(G^{-}) \leq c_H$, we have that $|C| = \Omega_h(n)$.

We use the fact that girth$(G) = \Omega(\log n)$:

- Let $\mathcal{P}$ be a packing of models of H in G
- As H contains a cycle and **girth**$(G) = \Omega(\log n)$, each graph in $\mathcal{P}$ contains at least $\Omega_h(\log n)$ vertices.
- Therefore $|\mathcal{P}| = O_h(n/\log n)$

Conclusion: for every packing $\mathcal{P}$ of models of H in G and every covering C of models of H in G it holds that $|C| = \Omega_h(|\mathcal{P}| \log |\mathcal{P}|)$

Therefore: $f_H(k) = \Omega_H(k \log(k))$

When can we do better than $O_h(k \log k)$?

- If H is acyclic, then the gap is <u>linear</u>, i.e., $f_H(k) = O_H(k)$

[Fiorini, Joret, & Wood, 2013]

- Let $\mathcal{R}$ be a non trivial minor-closed graph class.

Then for every planar graph H, $\mathcal{M}(H)$ has the E&P-property on $\mathcal{R}$ with linear gap $O_{\mathcal{R}}(k)$.

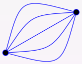
[Fomin, Saurabh, Thilikos 2011]

What about matching (or approaching) the lower bound?

- If H is not acyclic, then $f_H(k) = O_H(k \text{ polylog}(k))$

[Chekuri & Chuzhoy, 2013]

- **Most general existing tight bound:**

If $H = \theta_h =$  then $f_H(k) = O_h(k \ \log k)$ on all graphs.

[Fiorini, Joret, & Sau, 2013] and

[Chatzidimitriou, Florent, Sau, & Thilikos, 2015]

Open problem:

Prove or disprove:

- Given a planar graph H, $\mathcal{M}(H)$ has the vertex-Erdős–Pósa property on all graphs with (optimal) gap $f_H(k) = O_H(k \log k)$

Other variants of Erdős–Pósa properties

Edge variants:

- For every r, $\mathcal{M}(\theta_r)$ has the edge-Erdős-Pósa property with (optimal) gap $O(k \log k)$.

  $\langle$An $O(\log OPT)$-approximation also exists$\rangle$

  [Chatzidimitriou, Florent, Sau, & Thilikos, 2015]

Open problem:

Prove or disprove:

- Given a planar graph H, $\mathcal{M}(H)$ has the edge–Erdős–Pósa property on all graphs

and, if this is correct, prove that the gap is optimal $f_H(k) = O_H(k \log k)$

Minor models of cliques:

$\mathcal{M}(K_h)$ have the edge Erdős-Pósa property on $\Omega(k \cdot h)$-connected graphs

[Diestel, Kawarabayashi, Wollan JCTSB 2012]

Immersions:

$\mathcal{I}(H)$: Immersion models

$\forall H,\ \ \mathcal{I}(H)$ have the edge Erdős-Pósa property on 4-edge connected graphs

[Chun-Hung Liu, May 2015]

Topological Minors:

$\mathcal{T}(H)$: Topological Minor models

There is a class $\mathcal{C}$ (completely characterized) such that

$\mathcal{T}(H)$ has the vertex Erdős-Pósa property iff $H \in \mathcal{C}$.

[Chun-Hung Liu, 2015]

Odd cycles:

Odd cycles have vertex Erdős-Pósa property on 576-connected graphs with linear gap

[Rautenbach & Reed, 1999]

Odd cycles have vertex/edge Erdős-Pósa property on graphs embeddable in orientable surfaces

[Kawarabayashi, Nakamoto, 2007]

Odd cycles have edge Erdős-Pósa property on 4-edge connected graphs

[Kawarabayashi, Kobayashi, STACS 2012]

Long cycles:

$\mathcal{M}(C_r)$ has the vertex Erdős-Pósa property with gap

$f(k,l) = O(l \cdot k \cdot \log k)$.

[Fiorini & Herinckx, JGT 2013]

Cycles through a set of vertices:

We consider a graph G with terminals $T \subseteq V(G)$

T-cycle: a cycle intersecting T.

Cycles intersecting T have the vertex/edge Erdős-Pósa property

with (optimal) gap $f(k) = O(k \cdot \log k)$.

[Pontecorvia & Wollan, JCTSB 2012]

Directed cycles in directed graphs:

Directed cycles have the vertex Erdős-Pósa property.

[Reed, Robertson, Seymour, & Thomas, Combinatorica 1996]

Matroids:

[Geelen, Gerards, Whittle, JCTSB 2003]

[Geelen, Kabell JCTSB 2009]

Najlepša hvála

Thank you!

Diego Velázquez - El Triunfo de Baco o Los Borrachos

(Museo del Prado, 1628-29)